The Number of Complete Subgraphs In Graphs with Non-Majorizable Degree Sequences

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ABSTRACT

Erdős proved that any graph that has no K_{r+1} as a subgraph can be degree-majorized by a complete r-partite graph. More generally, let $f_r(n, \Delta)$ be the smallest number of copies of K_{r+1} in a graph with n vertices and maximum vertex degree Δ that is not degree-majorizable by an r-partite graph. We show constructively that $f_r(n, \Delta) = 1$ when $\Delta \geq \lceil n - \frac{n-1}{r} + \frac{r-1}{2} \rceil$. Trivially, $f_r(n, \Delta) = \infty$ when $\Delta \leq \lfloor n - \frac{n}{r} \rfloor$. The construction generalizes to give $f_r(n, \Delta) \leq (n - \Delta)^t$, where t is the smallest integer such that $n - 1 \geq (n - \Delta)r + \binom{r-t}{2}$. We prove that this construction is optimal for $r = 2$ and for $r = 3$, $n \leq 7$. This yields $f_2(n, \frac{n+1}{2}) = \frac{n-1}{2}$, for which we characterize the extremal graphs.

1. Introduction

In 1941, Turán [17] proved that the graph on n vertices with most edges that has no K_{r+1} as a subgraph is the r-partite graph with the most edges, which occurs when the part-sizes differ by at most one. This has been proved and extended in many ways. To facilitate discussion, we call the complete graph K_r an r*−clique*, without requiring that an r-clique be a maximal complete subgraph of a graph. A number of papers have examined the number of $(r + 1)$-cliques that must arise as the number of edges increases, particularly for the case $r = 2$. Moon and Moser [15] proved that a graph on n vertices with m edges has at

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least $\frac{m}{3n}(4m - n^2)$ triangles. Bollobás [2] extended these results by obtaining lower bounds for the minimum number of r-cliques in a graph on n vertices that has at least x p-cliques. See also [4], [7], [8], [13], [14], [16]. A good discussion of many of these results appears in [3].

Erdős [11] gave a proof of Turán's Theorem by characterizing the maximal degree sequences among graphs with no $(r + 1)$-clique. In particular, he proved that any graph G containing no $(r + 1)$- clique is "degree-majorized" by some complete r-partite graph H. Letting the degree sequence $d(G)$ of a graph be the non-increasing order of its vertex degrees, we say that *H degree−majorizes G* if $d_i(G) \leq d_i(H)$ for all i. If a graph is degree-majorized by an r-partite graph, we say it (or its degree sequence) is *r−majorizable*. We wish to extend the theorem of Erdős to an extremal theorem analogous to those mentioned above. How many $(r + 1)$-cliques must appear in a graph on n vertices if it is not r-majorizable? It is easy to find a graph with maximum vertex degree $\Delta = n - 1$ that has only one K_{r+1} but is not r-majorizable. To make the problem non-trivial, we restrict the value of $\Delta(G)$. Bounds on vertex degrees have been examined before for this type of question. Zarankiewicz [18] first noted that graphs with no $(r + 1)$-clique must have some vertex with degree at most $\frac{r-1}{r}n$. This was improved slightly in [1] under the assumption that the graph is also known not to be r-colorable (i.e., not r-partite). See also [6], [12] for related extremal results on vertex degrees.

To facilitate discussion, we refer to a graph on n vertices with maximum degree Δ as an *(n, Δ)−graph*. Let $f_r(n, \Delta)$ be the minimum number of $(r + 1)$-cliques in an (n, Δ)-graph that is not r-majorizable. The purpose of this paper is to determine $f_r(n, \Delta)$ completely when $r = 2$ and to obtain a uniform upper bound for all cases, in terms of r, n, and Δ.

Any (n, Δ)-graph with $\Delta \leq \lfloor n - \frac{n}{r} \rfloor$ is r-majorized by the complete r-partite graph in which the part sizes differ by no more than 1. By convention, we say $f_r(n, \Delta) = \infty$ in that case. In Section 2 we provide a construction to show that $f_r(n, \Delta) = 1$ whenever $\Delta \geq \lceil n - \frac{n-1}{2} + \frac{r-1}{2} \rceil$. To settle the only remaining case on $r = 2$, we show in Section 3 that $f_2(n, \frac{n+1}{2}) = \frac{n-1}{2}$. The graph achieving this is unique except for ½$(n - 3)$ optional edges. In Section 4, we generalize these constructions to show that $f_r(n, \Delta) \leq (n - \Delta)^t$, where $t = t(r, n, \Delta)$ is the smallest integer such that $n - 1 \geq (n - \Delta)r + \binom{r - t}{2}$.

We conjecture that the bound of Section 4 is optimal. We have proved this for $r = 2$, for $r = 3$, $n \leq 7$, and for all (r, n) when Δ is sufficiently large. We note that this is similar to an old conjecture of Erdős and Stone [13] analogous to a theorem of Rademacher; Erdős [10] later proved it for $r = 3$. (See also [14].) They conjectured that any graph with rn vertices and at least $\binom{r}{2} n^2 + 1$ edges contains at least n^{r-1} $(r + 1)$-cliques. By the pigeonhole principle, such a graph always has a vertex of degree at least $(r - 1)n + 1$, but need not have one of higher degree. If it has none of higher degree, it cannot be r-majorizable. Since $t(r, rn, (r - 1)n + 1) = r - 2$, the present conjecture would say such a graph has at least $(n - 1)^{r-2}$ $(r + 1)$-cliques. Their lower bound is higher because they require more edges than needed to prevent a $(rn, (r - 1)n + 1)$-graph from being r-majorizable.

2. The Construction for Large Δ

THEOREM 1. If $\Delta \geq \lceil n - \frac{n-1}{r} + \frac{r-1}{2} \rceil$, then $f_r(n, \Delta) = 1$.

PROOF: Erdős' Theorem gives the lower bound. For the upper bound, we need only constuct a non-majorizable (n, Δ)- graph G having only one $(r + 1)$-clique. We begin with a complete r-partite graph on $n - 1$ vertices and then form G by adding an additional vertex z joined to one vertex in each part. Let the parts be A_i, and let the neighbor of z in A_i be x_i. The only $(r + 1)$-clique in G consists of z and $\{x_i\}$. The problem is to determine when (and how) the part-sizes $\{|A_i|\}$ of the initial r-partite graph can be chosen so that the resulting graph has maximum degree Δ and is not r-majorizable.

We claim it suffices to let the sizes of the parts be $\{k_i\}$ where $\{k_i\}$ are r integers such that $n - \Delta = k_1 < k_2 < \cdots < k_r$ and $\Sigma k_i = n - 1$. We also claim that this can be done precisely when $\Delta \geq \lceil n - \frac{n-1}{r} + \frac{r-1}{2} \rceil$ to get a degree sequence with the desired properties. An example of such a graph G with $(r, n, \Delta) = (3, 10, 8)$ and $\langle k_i \rangle = (2, 3, 4)$ is shown in Figure 1.

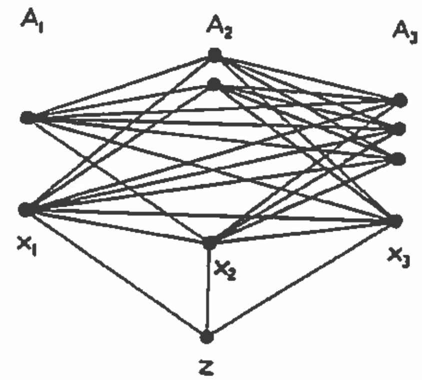


Figure 1

To show that a graph G with the k_i so chosen is not r-majorizable, we begin by computing the degree sequence of G. The degree of x_i is $n - k_i$. The degrees of the other vertices in A_i are $n - k_i - 1$. The degree of z is r. In the case $r = 1$ and $\Delta = n - 1$, the resulting sequence is $(n-1, 2, 2, 1, 1, 1, \ldots)$, which is not 2-majorizable. In all other cases, the requirement $\Sigma k_i = n - 1$ implies $n - k_i > r$. In addition, the k_i form a strictly increasing sequence, so $n - k_i - 1 \geq n - k_{i+1}$. Therefore, the degree sequence of G is

$$(n-k_1, n-k_1-1, \ldots | n-k_2, n-k_2-1, \ldots | \cdots | n-k_r, n-k_r-1, \ldots | r)$$

where the ith segment of this sequence has k_i terms, $1 \leq i \leq r$.

We claim that G is not r-majorizable. A graph is r-majorizable if and only if it has a vertex partition into r parts such that each vertex in a part of size q has degree at most $n - q$. This can be tested by the "greedy algorithm". Partition the vertices by repeating the following two steps r times.

1) Place the remaining vertex of highest degree d ($= n - k_1$ at first) in a new block.

2) Fill the rest of that block with the vertices having the next $n - d - 1$ highest degrees.

G is r-majorizable if and only if this procedure places *all* the vertices into the r blocks.

We have applied this procedure above to the degree sequence of

G, placing vertical bars as far to the right as possible to indicate when a part must end. Since the resulting r parts do not exhaust the vertices, the sequence is not r-majorizable.

It remains only to determine for what values of k_1 this construction can be performed. To make k_1 as large as possible and satisfy the constraints on $\{k_i\}$, these parameters should be a set of consecutive integers summing to $n - 1$. In particular, we have

$$n - 1 = \sum k_i \geq \sum_{i=1}^{r} k_1 + i - 1 = rk_1 + \frac{r(r-1)}{2} \qquad (*)$$

From the two ends of this we get

$$n - \Delta \leq \frac{n-1}{r} - \frac{r-1}{2} \quad \text{or} \quad \Delta \geq \left\lceil n - \frac{n-1}{r} + \frac{r-1}{2} \right\rceil$$

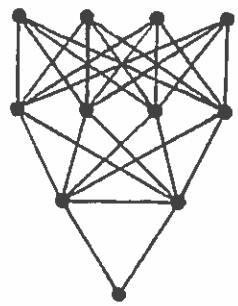
which gives the lower bound on Δ. To perform this construction for larger values of Δ, simply increase the value of $k_r, k_{r-1}, \ldots$, as needed to compensate for the decrease in k_1. This is always possible, except when $r = 2$ and $\Delta = n - 1$, which can be handled as described earlier. □

3. The Intermediate Case when $r = 2$

When $r = 2$, the only case remaining is $f_2(n, \frac{n+1}{2})$ for odd values of n. First we give a construction for an upper bound.

LEMMA 1. $f_2(n, \frac{n+1}{2}) \leq \frac{n-1}{2}$.

PROOF. Partition n vertices into four vertex sets, A, B, C, and D, having sizes $\frac{1}{2}(n-3)$, $\frac{1}{2}(n-3)$, 2, and 1, respectively. Form G by joining each vertex of A to each of B, each of B to each of C, and placing a triangle on $C \cup D$. The construction is illustrated in Figure 2.



class	multiplicity	degree
A	$\frac{n-3}{2}$	$\frac{n-3}{2}$
B	$\frac{n-3}{2}$	$\frac{n+1}{2}$
C	2	$\frac{n+1}{2}$
D	1	2

Figure 2

The resulting maximum degree is $\frac{1}{2}(n+1)$ and there are $\frac{1}{2}(n-1)$ triangles. The degree sequence is not 2-majorizable, since there are $\frac{1}{2}(n+1)$ vertices of degree $\frac{1}{2}(n+1)$ -- each would have to appear in a part with less than half the vertices, but no single such part can contain $\frac{1}{2}(n+1)$ vertices. With any fewer vertices of degree $\frac{1}{2}(n+1)$, no problem arises. □

The construction could also be modified by letting the vertex in D be joined to any subset of the $\frac{1}{2}(n-3)$ vertices in A. Let W be the collection of all such graphs. We will see that the graphs in W are the only graphs that achieve the minimum. In order to prove this, we need two numerical lemmas about the number of complete subgraphs that must be present when a graph contains other complete subgraphs of "high degree". Let the *degree* of a complete subgraph H of a graph be the sum of its vertex degrees. We write this as $d(H)$. Results have long been known about when high degree of an r-clique forces its inclusion in some $(r+1)$-clique. One of the most general appears in [9]. We phrase it for $r+1$ instead of the usual r.

LEMMA 2. If H is any $(r+1)$-clique in G and $d(H) > rn$, then there exist in G at least $d(H) - rn$ distinct $(r+2)$-cliques each containing H.

PROOF. We need only show that at least $d(H) - rn = s$ vertices not in H adjoin every vertex of H. There are $d(H) - r(r+1) = s + r(n-r-1)$ edges between H and the $n-r-1$ vertices of $G - H$. Since no vertex can be joined to more than $r+1$ vertices of

H, the pigeonhole principle implies that at least s must be joined to exactly $r + 1$. □

The most well-known case of this is $r = 1$; two adjacent points whose degrees sum to at least $n + k$ lie together in at least k triangles.

Since we are interested in the number of $(r + 1)$-cliques, we will be more interested in the following variation.

LEMMA 3. If H is an $(r + 1)$-clique in G and $(r - 1)n + r < d(H) \leq rn$, then G contains at least $d(H) - (r - 1)n - r$ $(r + 1)$-cliques that each contain at least r vertices of H. If $d(H) > rn$, then G contains at least $r[d(H) - rn] + n - r$ such $(r + 1)$-cliques.

PROOF. Like the previous lemma, this is a simple application of the pigeonhole principle. We may assume there are no edges among the verties of $G - H$, since they could only cause additional K_{r+1}'s. The number of $(r + 1)$-cliques involving vertices of H is minimized by adding the other edges greedily. In other words, add edges so that all the vertices in $G - H$ have $r - 1$ neighbors in H before any have r neighbors, and all have r neighbors in H before any have $r + 1$. To prove that this works, let t be the number of vertices in $G - H$ neighboring r vertices in H, and let s be the number neighboring $r + 1$ vertices of H. Then the number of $(r + 1)$-cliques in G is $t + (r + 1)s + 1$, counting one for H itself. There are $d(H) - r(r + 1)$ edges between H and $G - H$. In the lower range mentioned, after distributing $r - 1$ of these to each of the $n - r - 1$ vertices in $G - H$, we have $t = d(H) - (r - 1)n - r - 1$ and $s = 0$. In the upper range, $s = d(H) - rn$, as in Lemma 1, and $t = n - r - 1 - s$. In either range, computing $t + (r + 1)s + 1$ gives the lower bound claimed. □

Of course, both of these lemmas give best-possible bounds, achieved by letting $G - H$ be an independent set. Together, they give a short proof of the lower bound.

THEOREM 2. $f_2(n, \frac{n+1}{2}) = \frac{n-1}{2}$. Furthermore, the only $(n, \frac{n+1}{2})$-graphs that are not 2-majorizable yet have only $\frac{1}{2}(n - 1)$ triangles are those in the set W described in Lemma 1.

PROOF. Lemma 1 gave the upper bound. Let $I = I(G)$ be the set of vertices of degree $\frac{1}{2}(n + 1)$ in an $(n, \frac{n+1}{2})$-graph G. Note that G is

2-majorizable if and only if I has less than $\frac{1}{2}(n+1)$ vertices, as mentioned earlier. If G is non-majorizable, we will show that either there is a triangle among the vertices of I, or there are more than $\frac{1}{2}(n-1)$ triangles elsewhere in the graph. In the former case, the degree-sum in this triangle is $3(n+1)/2$. Since $3(n+1)/2 \geq n+3$ for $n \geq 3$, we can apply Lemma 3 with $r = 2$. It implies there will be at least $3(n+1)/2 - n - 2 = \frac{1}{2}(n-1)$ distinct triangles in the graph, each of which contains at least two of these three vertices in I.

For any edge in I, the degrees of the endpoints sum to $n+1$, so by Lemma 2 any such edge must appear at least one triangle. Also note that, since I consists of at least $\frac{n+1}{2}$ vertices of degree $\frac{n+1}{2}$, each must neighbor another; hence every vertex in I belongs to an edge in I and thus to a triangle in G.

Suppose there is no triangle in I. By the above, in this case it will suffice to show there must be more than $\frac{1}{2}(n-1)$ edges in I. Let e be the number of edges in I. The vertices not in I have degree at most $\frac{1}{2}(n-1)$, so they can absorb at most $\frac{1}{2}(n-1)(n-|I|)$ of the degree sum from I. The remainder is twice e. We have

$$2e \geq (\tfrac{1}{2}(n+1))|I| - (n-|I|)(\tfrac{1}{2}(n-1))$$
$$= n(|I| - (\tfrac{1}{2}(n-1))) \geq n,$$

where the last inequality follows from $|I| \geq \frac{1}{2}(n+1)$.

This means $e > \frac{1}{2}(n-1)$, which is the desired result.

This is also the first step toward uniqueness; not having a triangle in I forces *more* than $\frac{1}{2}(n-1)$ triangles in G. Therefore, for a graph achieving the minimum we can assume there is a triangle T in I. We have seen there are at least $\frac{1}{2}(n-1)$ triangles in G containing at least two vertices of T. If these $\frac{1}{2}(n-1)$ triangles are the only triangles in G, then each vertex of I belongs to one of them, since every vertex in I lies on a triangle. Any such triangle other than T is also a triangle in I and also has $\frac{1}{2}(n-1)$ triangles involving at least two of its vertices. If there are at most $\frac{1}{2}(n-1)$ triangles altogether, these must be the same triangles, and hence they all share an edge.

Thus, minimality requires that all $\frac{1}{2}(n-1)$ triangles share a common edge in I, and that all the other vertices in I appear as apexes of these triangles. To show G is in W, let the set C be the common vertices of the triangles. The apexes of the triangles are a set of $\frac{1}{2}(n-1)$ vertices, of which at least $\frac{1}{2}(n-3)$ have degree $\frac{1}{2}(n+1)$; let this be the set $B \cup D$. There can be no edges between these vertices without creating more triangles, so if the vertices of B are to have degree

½$(n + 1)$, they must all be joined to all the remaining vertices, which form the set A. Now no further edges can be added to the graph without creating more triangles (except between D and A), so G lies in W. □

4. The Uniform Upper Bound

The above constructions generalize to give an upper bound valid for all cases. The bound appears to be optimal, but we have not yet succeeded in showing this.

THEOREM 3. $f_r(n, \Delta) \le (n - \Delta)^t$, where t is the smallest integer such that

$$r(n - \Delta) + \binom{r - t}{2} \le n - 1.$$

If there is no such integer, we take $t = \infty$.

PROOF. Note that there is no such integer if and only if $\Delta \le \lfloor n - \frac{n}{r} \rfloor$. On the other hand, we get $t = 0$ if and only if $\Delta \ge \lceil n - \frac{r-1}{r} + \frac{r-1}{2} \rceil$. This agrees with the result of Section 2.

We proceed as in the proof of Theorem 1. Again, we begin with an r-partite graph on $n - 1$ vertices. However, it no longer suffices to join the remaining vertex to only one vertex in each part, because there will not be enough vertices of sufficiently high degree to make the degree sequence non-majorizable.

Again, let the parts be A_i, with sizes k_i. This time we make the small parts the same size. We choose the k_i so that $n - \Delta = k_1 = \cdots = k_t = k_{t+1} < \cdots < k_r$ and $\Sigma k_i = n - 1$. Note that by the definition of t we have $t < r$ when it is finite. To construct the graph, we join the last vertex z to every vertex in A_i when $i \le t$ and to a single vertex x_i from each part A_i with $i > t$. The graph is schematically illustrated in Figure 3 for the case $t = 2$, $r = 5$.

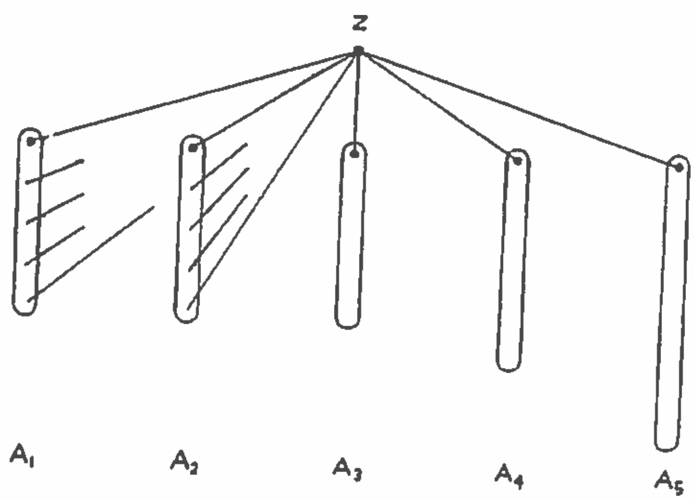


Figure 3

The equality among part-sizes and joining of z to all vertices in A_i "propagates" the needed high degree. Each neighbor of z in A_i has degree $n - k_i$; each non-neighbor has degree $n - k_i - 1$. Thus the degree sequence is

$$(n-k_1,\ldots|n-k_1,\ldots|\cdots|n-k_{t+1},\ n-k_{t+1}-1,\ldots|\cdots$$
$$\cdots|n-k_r,n-k_r-1,\ldots|tk_1+r-t)$$

where each of the first $t+1$ segments of this sequence has k_1 terms, and the ith segment has k_i terms, for $t+2 \le i \le r$.

All $(r+1)$-cliques in this graph consist of z and one neighbor of z from each A_i. The number of these is $(n-\Delta)^t$.

As in section 2, this degree sequence is non-majorizable by construction, even if the degree of z is large enough to require a different ordering of the degrees. That reordering will be needed, for example, if $\Delta = n - (n-1)/r$ and $t = r - 1$, in which case the degree of z is also Δ. For $r = 2$, this is precisely the case discussed in section 3, and

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the graph we have constructed is the graph in W with the most edges. The edges that can be deleted without losing non-majorizability are those joining the vertices of A_r other than x_r to some single vertex not in $\{z\} \cup A_r$.

To get the best upper bound from this construction, we need only determine how small t can be made when k_1 is as large as required and Σk_i is as small as required. To make Σk_i small, we again let the part-sizes increase singly after k_{t+1}. We get

$$n - 1 = \Sigma k_i \geq tk_1 + \sum_{i=0}^{r-t-1} k_t + i = (n - \Delta)r + \binom{r-t}{2},$$

which completes the proof.

5. Further Remarks

To motivate the conjecture, we present a short proof that $f_3(7,5) = 4$. This is the next undetermined case.

THEOREM 4. $f_3(7,5) = 4$. Furthermore, there are only two (7,5)-graphs with four 4-cliques that are not 3-majorizable, namely those constructed in the proof of Theorem 3.

PROOF. A (7,5)-graph G is not 3-majorizable only if it has at least five vertices of degree 5 and at most two vertices of smaller degree. Let H be the subgraph of G induced by vertices of degree 5. We may assume by discarding edges that H has exactly five vertices and $G - H$ consists of isolated vertices. Otherwise G would be K_6 plus an isolated vertex, in which case it would have twenty triangles.

Each vertex in H has degree at least three in H. Every edge in H has degree 10 in G and thus belongs to at least three triangles in G. Since there are only two vertices in $G - H$, at least one of these triangles belongs entirely to H. Every triangle in H has degree 15 in G; by Lemma 2, it belongs to 4-clique in G.

There are at least eight edges in H, since each vertex of H neighbors at least three others. By the theorem of Moon and Moser, there are at least $\lceil \frac{8}{15}(32 - 25) \rceil = 4$ triangles in H. If there is no 4-clique in H, we are done, since the 4-cliques containing each of the triangles in H must then be different. So, assume H has a 4-clique. The remaining vertex of H neighbors at least three of those vertices, creating another 4-clique in H.

Let I be the three vertices shared by the two 4-cliques in H. The vertices of $H - I$ cannot be joined, else G contains K_5 with five 4-cliques. Thus, they have degree 3 in H and must each be joined to the two vertices in $G - H$. The vertices of I must each be joined to one vertex in $G - H$. Whenever two of them are joined to the same vertex in $G - H$, which must happen at least once, two additional 4-cliques are created, by including either of the vertices in $H - I$.

The restrictions obtained in proving this imply that the only (7,5)-graphs with four 4-cliques that are not 3-majorizable are those constructed in the proof of Theorem 3. □

This type of argument can be pushed farther; we have done so to obtain the appropriate lower bound for $f_3(3k+2, 2k+2)$. However, the argument becomes increasingly tedious as r increases. Therefore, we omit the proof of the lower bound for $f_3(3k+2, 2k+2)$ in the expectation that a more elegant inductive proof exists.

Acknowledgement

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