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Networks and Chain Coverings in Partial Orders and their Products

CRAIG A. TOVEY
Georgia Institute of Technology, Atlanta, GA 30332, U.S.A.

and

DOUGLAS B. WEST
University of Illinois, Urbana, IL 61801, U.S.A.

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Abstract. For an arbitrary poset **P**, subposets $\{P_i : 1 \leqslant i \leqslant k\}$ form a *transitive basis* of **P** if **P** is the transitive closure of their union. Let u be the minimum size of a covering of **P** by chains within posets of the basis, s the maximum size of a family of elements with no pair comparable in any basis poset, and a the maximum size of an antichain in **P**. Define a *dense covering* to be a collection D of chains within basis posets such that each element belongs to a chain in D within each basis poset and is the top of at least $k - 1$ chains and the bottom of at least $k - 1$ chains in D. Dense coverings generalize ordinary chain coverings of poset. Let $d = \min\{|D| - (k - 1)|\mathbf{P}|\}$. For an arbitrary poset and transitive basis, a convenient network model for dense coverings yields the following: Theorem 1: $d \geqslant a$, with equality iff **P** has a minimum chain decomposition in which every pair of consecutive elements on each chain are comparable in some basis poset. Theorem 2: $u \geqslant s \geqslant d \geqslant a$. Theorem 3: $s = d$ iff $s = a$. The most interesting special case is where the transitive basis expresses **P** as the product of two posets, in which case u and s measure the minimum and maximum sizes of unichain coverings and semiantichains.

AMS (MOS) subject classifications.

Key words. Partial order, direct product, semiantichain, network, integral min-max relation, chain covering, Sperner property.

1. Introduction

In this paper we consider various pairs of dual integer programs arising from products of partial orders (henceforth 'posets'). We use a network flow model to obtain inequalities relating them. These integer programs seek optimal packings and coverings using objects related to chains and antichains in posets. A *chain* is a totally ordered subset of a poset, an *antichain* is a totally unordered subset. Dilworth [1] proved that the largest antichain in a poset has the same size as the smallest covering of its elements by chains. The best-known generalization of Dilworth's Theorem is that of Greene and Kleitman [2]. West and Saks [8] conjectured a further generalization to direct products of posets, which we now describe.

The *direct product* $\mathbf{P} = Q_1 \times Q_2$ of two posets Q_1 and Q_2 is a poset defined on the

Cartesian product of their underlying sets by setting $(a, b) \leqslant (a', b')$ in $\mathbf{P}$ if and only if $a \leqslant a'$ in Q_1 and $b \leqslant b'$ in Q_2. A *unichain* in $Q_1 \times Q_2$ is a chain throughout which one of the coordinates is fixed; equivalently, it is the product of an element in one order with a chain in the other. A *semiantichain* in $Q_1 \times Q_2$ is a collection of elements that are mutually incomparable if they agree in either coordinate; note that any antichain is a semiantichain. Since no semiantichain contains more than one element from a single unichain, any covering of the poset by unichains requires at least as many unichains as the size of any semiantichain. The conjecture of West and Saks [8] is that the size of the largest semiantichain always equals the size of the smallest unichain covering. Dilworth's Theorem follows when one of the component orders is a single point. Partial results on the conjecture, generalizing various earlier results, appear in [9]. That paper also points out that the Greene–Kleitman generalization of Dilworth's Theorem follows from the conjecture when one of the orders is a chain of k elements. Another special case is proved in [10]. In [7] the easy conjugate result appears: the size of the largest unichain in a direct product always equals the size of the smallest covering of the elements by semiantichains.

In this paper, we discuss a related covering problem that has some implications for the problems discussed above. Its interest arises from having a very natural network flow model, and from the fact that it generalizes easily to arbitrary posets without a product structure. Since this optimization problem and associated network originally arose from an attempt to model the unichain covering problem, we first describe the result in the direct product case.

Define a *Q_i-unichain* in $Q_1 \times Q_2$ to be a unichain whose Q_{3-i}-coordinate is fixed. A *double covering* of $\mathbf{P} = Q_1 \times Q_2$ is a collection of unichains such that every element of $\mathbf{P}$ is covered by some Q_1-unichain and by some Q_2-unichain, with the additional requirement that every element is the top of at least one unichain in the collection and also the bottom of at least one unichain in the collection. A single element can be considered a Q_1-unichain or a Q_2-unichain, but not both in a given covering.

Let U, S, D, A be optimum collections of unichains or elements for the minimum unichain covering, maximum semiantichain, minimum double covering, and maximum antichain problems in $\mathbf{P}$. To relate these, let $u = |U|$, $s = |S|$, $a = |A|$, but for the double covering let $d = |D| - |\mathbf{P}|$. Since any antichain in $\mathbf{P}$ is a semiantichain, and since semiantichains and unichains intersect at most once, we have $u \geqslant s \geqslant a$. Our main result fits d into this framework; $u \geqslant s \geqslant d \geqslant a$.

We prove a generalization of this. Let $\mathbf{P}$ be an arbitrary poset, and let $\{P_i : 1 \leqslant i \leqslant k\}$ be subposets of $\mathbf{P}$ on the same elements, viewed as collections of relations. If $\mathbf{P}$ is the transitive closure of $\cup P_i$, then we say that $\{P_i\}$ is a *transitive basis* of $\mathbf{P}$. Expressed graphically, what is required is that all the arcs of the (directed) Hasse diagram of $\mathbf{P}$ appear among those of the P_i. The number k of posets in a basis is its size.

Given a transitive basis $\{P_i\}$ of $\mathbf{P}$, denote the order relation in $\mathbf{P}$ by $<, \leqslant, >, \geqslant$, the order relation in P_i by $\overset{i}{<}, \overset{i}{\leqslant}, \overset{i}{>}, \overset{i}{\geqslant}$. A pair of elements is *P_i-comparable* if they are related in P_i. They are *unicomparable* if they are P_i-comparable for some i. A *P_i-chain* or *unichain in P_i* is a chain in $\mathbf{P}$ in which every pair of elements is P_i-comparable. An *S-family* or *semiantichain* has no pair of unicomparable elements. A *dense covering* of

$\mathbf{P}$ is a collection of unichains such that every element is covered by a P_i-unichain for each i, and every element is the top of at least $k - 1$ of the chains and is the bottom of at least $k - 1$ of the chains. Define U, S, D, A, u, s, a as before, but let $d = |D| - (k - 1)|\mathbf{P}|$. The results, proved by obtaining canonical forms for minimum dense coverings and for optimum cuts in a network flow model for dense coverings, are

THEOREM 1. *For a poset* $\mathbf{P}$ *with transitive basis* $\{P_i\}$, $d \geqslant a$, *with equality if and only if* $\mathbf{P}$ *has a minimum chain decomposition in which every pair of consecutive elements on a chain are unicomparable.*

THEOREM 2. *For a poset* $\mathbf{P}$ *with transitive basis* $\{P_i\}$, $u \geqslant s \geqslant d \geqslant a$.

THEOREM 3. *For a poset* $\mathbf{P}$ *with transitive basis* $\{P_i\}$, $s = d$ *iff* $s = a$.

To make the subtraction of $(k - 1)|\mathbf{P}|$ in computing d seem more natural, it is worth noting that dense coverings generalize ordinary chain coverings. In particular, if $k = 1$, then the dense covering condition is the same as the ordinary chain covering condition, and $d = a$ always. At the other end of the spectrum, there is a poset in the transitive basis for every edge of the Hasse diagram. It is not surprising that the spectrum between these is well-behaved. A transitive basis W of $\mathbf{P}$ is a *refinement* of a transitive basis V of $\mathbf{P}$ if every poset in V is the transitive closure of a union of posets in W. It is clear that u and s cannot decrease under refinement, and the same is true for d. In fact, Theorem 1 is a special case of the following more general statement, since for a transitive basis of size 1 the dense coverings are precisely the chain coverings, and $d = a$. A dense covering is in *standard form* if every element appears on exactly one unichain in each poset of the transitive basis.

THEOREM 1′. *If* W *and* V *are transitive bases of* $\mathbf{P}$, *with* W *a refinement of* V, *then* $d_W \geqslant d_V$, *with equality if and only if* $\mathbf{P}$ *has a minimum dense covering in standard form under* V *in which every pair of consecutive elements on each unichain are unicomparable under* W.

Theorem 1 immediately suggests examples with $d > a$ even when $k = 2$. In the example below, every decomposition of $\mathbf{P}$ into two chains consists of a 3-chain in which consecutive elements are unicomparable and a 2-chain in which they are not, so Theorem 1 implies $d > a$ (in fact, $d = 3$: see Section 3).

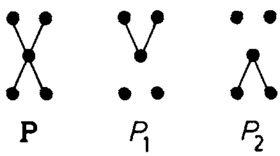

Fig. 1. A poset and transitive basis with $d > a$.

In the special case where $\mathbf{P} = Q_1 \times Q_2$, let P_i consist of all copies of Q_i, i.e., one for each element of the other factor. Then the dense coverings reduce to what were called double coverings above, and Theorems 1 and 2 reduce to the desired results. Theorem

2 is of particular interest in this setting. A ranked poset has the *Sperner property* if the largest antichain consists of a single rank. The product of ranked posets P and Q has the *2-part Sperner property* if the largest semiantichain in $Q_1 \times Q_2$ consists of a single rank. See [4] for a recent survey of results on the Sperner property.

COROLLARY. *The product of ranked posets P and Q has the 2-part Sperner property if and only if it has the Sperner property and has* $s = d$.

We can compute d using network flow methods, but the complexity of computing s is open, so this condition is only of theoretical interest at present. In addition, it may turn out to be nothing but a restatement of the definition; at present we know of no example of a direct product for which $d > a$. Indeed, there is even a stronger question: it may be true that every product poset has a minimum chain decomposition in which every chain is the union of two unichains, with the top of one being the bottom of the other. For transitive bases in general, one could ask for a characterization of when **P** has a minimum chain decomposition in which every chain consists of (at most) one unichain in each direction.

Another way to weaken the statement of Theorem 2 is to say that $u = d$ implies $s = a$. The converse of this, which we have not been able to prove, would make the 2-part Sperner property a sufficient condition for the semiantichain conjecture $u = s$ to hold. Put another way, it would strengthen Dilworth's Theorem in the case where the largest semiantichain in a direct product is an antichain, by saying that a Dilworth decomposition (minimum chain covering) can be found using only unichains.

Between the 2-fold product version and the general result for transitive bases, we can consider k-fold products of posets. Let $\mathbf{P} = \Pi Q_i = Q_1 \times \cdots \times Q_k$. **P** consists of k-tuples in which the ith coordinate is an element of Q_i, and $x \leqslant y$ in **P** if and only if $x_i \leqslant y_i$ in Q_i for all i. As in the product of two posets, the subposet P_i used in the transitive basis consists of all copies of Q_i. Unicomparable pairs are thus pairs of elements related in one coordinate and equal in all others. The resulting semiantichains have been studied in the case where all Q_i are Boolean algebras [3,5]. The largest S-families are larger than the largest rank; [5] obtains asymptotically optimal bounds on their size.

Finally, a product of k posets yields a spectrum of problems between the maximum semiantichain and maximum antichain. Let x and y be called *r-comparable* if they are comparable and differ in at most r components. An *S^r-family* is a collection of elements in which no pair are r-comparable. An *r-chain* is a chain whose elements agree on a set of at least $k - r$ coordinates. Let U with size u be a minimum r-chain covering, and let S with size s be a maximum S^r-family. Theorems 1 and 2 apply by choosing a transitive basis with $\binom{k}{r}$ subposets, each of which consists of all copies of the product of r particular factors. The S^1-families are the semiantichains, and the S^k-families are the antichains.

2. Description of the Network

Consider a poset **P** with transitive basis $\{P_i\}$ of size k. We construct a network with $(2k + 2)|\mathbf{P}| + 2$ nodes in which feasible flows correspond to dense coverings of **P**. All arcs have infinite capacity, and some have a lower bound of 1 or $k-1$ on the feasible flow. The nodes consist of a source s, a sink t, and $2k+2$ copies of each element $x \in \mathbf{P}$. We call these copies $W(x) = \{x_0, x_{k+1}\} \cup \{x_i^-, x_i^+ : 1 \leqslant i \leqslant k\}$.

The network has arcs to encode the basis subposets $\{P_i\}$ and also utility arcs for each $x \in \mathbf{P}$ to enforce the dense covering. The arcs encoding P_i are (x_i^+, y_i^-) for all x, y such that $x >^i y$. The arcs defined for each $x \in \mathbf{P}$ are (s, x_0), (x_{k+1}, t), and $\{(x_0, x_i^-), (x_i^-, x_i^+), (x_i^+, x_{k+1}) : 1 \leqslant i \leqslant k\}$. All arcs of the network have infinite capacity. For three types of arcs we place lower bound requirements on the feasible flow. The lower bound is $k-1$ for arcs of the form (s, x_0) and (x_{k+1}, t), and it is 1 for arcs of the form (x_i^-, x_i^+). These lower bounds enforce the dense covering requirements. The network has no cycles. The network flow problem is to find the minimum value of a feasible flow.

For any minimum feasible network flow problem there is a dual problem involving cuts. A *cut* $[S \mid T]$ is a partition of the network's nodes into a *source set* S containing the source s and a *terminal set* T containing the sink t. The *value* of the cut $[S \mid T]$ is the sum of the lower bounds for arcs passing from S to T minus the sum of the capacities for arcs passing from T to S. Simply put, any cut requires a net flow from source to sink of at least its value. Duality theory for network flows states that the minimum feasible flow value in a network equals the maximum cut value (see [6] for a discussion of this). Thus we can view d as arising from either problem, and indeed most of the work of this paper involves characterizing the maximum cuts. The *size* of a dense covering, like the size of a unichain covering, is the number of unichains in it, i.e., $d + (k-1)|\mathbf{P}|$ for the smallest one.

LEMMA 1. *Any integral feasible flow in the network formed from* $\mathbf{P}$ *corresponds naturally to a dense covering of* $\mathbf{P}$ *whose size equals the value of the flow.*

Proof. Any integral feasible flow has a partition into paths of unit flow. By the construction of the network, each path of unit flow uses a sequence of nodes like $s, x_0, x_i^-, x_i^+, y_i^-, y_i^+, \ldots, z_i^-, z_i^+, z_{k+1}, t$, for some value of i. This corresponds to a P_i-unichain with x at the top and z at the bottom. Taking all unichains so generated, the lower bounds on the flows in sx_0 and $x_{k+1}t$ ensure that x is the top of $k-1$ unichains and is the bottom of $k-1$ unichains, and the lower bounds on $x_i^- x_i^+$ ensure that every element belongs to a P_i-unichain for all $1 \leqslant i \leqslant k$. □

LEMMA 2. *For a poset* $\mathbf{P}$ *with transitive basis* $\{P_i\}, u \geqslant d$.

Proof. A minimum unichain covering U of $\mathbf{P}$ transforms immediately into a dense covering of size $|U| + (k-1)|\mathbf{P}|$ by adding up to $k-1$ single-point unichains for each $x \in \mathbf{P}$, corresponding to each component P_i such that x is not covered by a P_i-unichain in U. The minimum dense covering is no bigger than this. □

LEMMA 3. *For a poset* $\mathbf{P}$ *with transitive basis* $\{P_i\}, d \geqslant a$.

Proof. Given any maximal antichain M in $\mathbf{P}$, we construct a cut in the network whose value is $|M| + (k-1)|\mathbf{P}|$. Since the maximum cut value equals the minimum feasible flow value in the network, this and Lemma 1 yield the result.

Since M is maximal, every other element of $\mathbf{P}$ is related to some element of M. Let

$M^+ = \{x : x > y \text{ for some } y \in M\}$, and let $M^- = \{x : x < y \text{ for some } y \in M\}$. M, M^+, M^- are disjoint, and their union is **P**. Construct the cut as follows. Let the (source) set S consist of s, all copies of elements in M^+, no copies of elements in M^-, and the $k + 1$ copies $\{x_0\} \cup \{x_i^-: 1 \leqslant i \leqslant k\}$ for all $x \in M$. The complementary set T has a complementary description.

By construction, no arcs point from T to S. Thus the value of the cut is the sum of lower bounds on arcs from S to T. These arcs are $\{(x_i^-, x_i^+): 1 \leqslant i \leqslant k\}$ for $x \in M$, (s, x_0) for $x \in M^-$, and (x_{k+1}', t) for $x \in M^+$. The arcs of the latter two types contribute $k - 1$ each, but those of the first type come in sets of k. Hence the value of the cut is $k\,|M| + (k-1)|M^-| + (k-1)|M^+| = |M| + (k-1)|\mathbf{P}|$. □

3. Structure of Minimum Dense Coverings

In this section we characterize the posets and transitive bases for which $d = a$. We use the idea of *standard form*, which we defined earlier to describe a dense covering in which every element appears on exactly one unichain in each poset of the transitive basis. The next lemma shows we always have a minimum dense covering in this form, which generalizes the fact that minimum chain coverings of a poset can be assumed to be chain decompositions. In that setting, transitivity allows an element appearing on more than one chain to be dropped from one of them.

LEMMA 4. *For any poset* **P** *with transitive basis* $\{P_i\}$, *there is a minimum dense covering in standard form.*

Proof. Consider a minimum dense covering, and suppose that $x \in \mathbf{P}$ appears on two P_i-chains. In every case, we alter the dense covering to reduce the multiplicity of x on P_i-chains, without violating any of the dense covering requirements. We consider four cases, which exhaust all possibilities. (1) If one of the chains is a singleton, drop x from the other chain. (2) If x is an interior element (not the top or bottom) on one of these chains, delete it from that chain. (3) If x is the bottom of both chains (or the top of both), delete it from one of the chains. (4) If x is the bottom of one chain and the top of the other, combine the two chains (P_i-comparability is transitive), omitting x, and establish x as a singleton chain. In each case, x is still the top of a P_i-chain and the bottom of a P_i-chain, and there are no more chains than before. □

For a dense covering in standard form, the ways in which x can appear in its various chains are limited. A 1-element chain is a *trivial* chain.

LEMMA 5. *For a minimum dense covering in standard form, every* $x \in \mathbf{P}$ *belongs to one of the following classes:*

$x \in X_0$ if x appears only on trivial unichains.
$x \in X_1$ if x is an interior element on one unichain and appears on $k - 1$ trivial unichains.
$x \in X_2$ if x is the bottom of a nontrivial unichain and appears on $k - 1$ trivial unichains.
$x \in X_3$ if x is the top of a nontrivial unichain and appears on $k - 1$ trivial unichains.
$x \in X_4$ if x is the bottom of one nontrivial unichain, the top of another, and appears on $k - 1$ trivial unichains.

Proof. By the preceding lemma, x appears on only one chain in each direction. Thus x can be a nonbottom or nontop in only one of the k unichains. □

The standard form for dense coverings makes the characterization of $d = a$ easy.

THEOREM 1′. *If W and V are transitive bases of* **P**, *with W a refinement of V, then $d_W \geqslant d_V$, with equality if and only if* **P** *has a minimum dense covering in standard form under V in which every pair of consecutive elements on each unichain are unicomparable under W.*

Proof. First consider the inequality. Let D be a minimum dense covering under W in standard form. We need only construct an appropriate dense covering D' of size $|D| - (|W| - |V|)|\mathbf{P}|$. Every unichain under W is a unichain under V. Construct D' by combining unichains of D as follows. For each $x \in \mathbf{P}$ and $P_i \in V$, combine into a single chain the unichains containing x for posets in W that belong to P_i. The result is a unichain in P_i. For each x, this results in the loss of $|W| - |V|$ trivial unichains, except when $x \in X_4$ and the two nontrivial unichains containing x belong to the same P_i, in which case $|W| - |V| - 1$ trivial unichains and one nontrivial unichain are lost. This proves $d_V \leqslant d_W$.

Note that in this construction, since D is in standard form, nontrivial chains get combined only at endpoints. Hence D' is a dense covering under V in which every pair of consecutive elements on each unichain are unicomparable under W. It is also in standard form, since D was in standard form. If $d_V = d_W$, then D' is minimal and the first half of the characterization is proved.

For the converse, suppose that D is a minimum dense covering under V of the desired type. Since we have proved $d_W \geqslant d_V$, it suffices to construct a dense covering under W of size $|D| + (|W| - |V|)|\mathbf{P}|$; call it D'. Let $C_1, \ldots, C_r$ be the nontrivial unichains in D. Since each pair of consecutive elements on C_i is unicomparable under W, each chain C_i breaks into some number of nontrivial unichains $C_{i,j}$ under W whose cover relations partition the cover relations in the chain C_i. To construct D', begin with these unichains on the points of C_i. For each breakpoint belonging to $C_{i,j}$ and $C_{i,j+1}$ for some j, add $|W| - 2$ trivial chains for the other directions. Note that these points contribute $|V| - 1$ trivial chains in D. Other points of C_i get $|V| - 1$ trivial chains in D and $|W| - 1$ trivial chains in D', except the top or bottom of C_i if it is the bottom or top of some other $C_{i'}$, in which case the number of trivial unichains are $|V| - 2$ and $|W| - 2$. If x appears on no nontrivial unichain in D, it appears on $|V|$ trivial unichains in D and on $|W|$ in D'.

To verify that we have added $|W| - |V|$ chains for each point, follow each C_i up from the bottom. For both the breakpoints and non-breakpoints, exactly $|W| - |V|$ chains have been added, since D is in standard form. This also holds for the points on no nontrivial unichains in D. □

As mentioned in the introduction, this immediately yields $d \geqslant a$ and a characterization of when $d = a$. This argument does not use the network at all, although the network provided a short proof that $d \geqslant a$. We know of no example violating the condition for equality when the transitive basis for **P** expresses it as the direct product of two posets; perhaps equality always holds. Figure 2 illustrates the network for the example given earlier, omitting some of the edges. The flow and cut of value 8 are indicated, showing that $d = 3$ and $a = 2$.

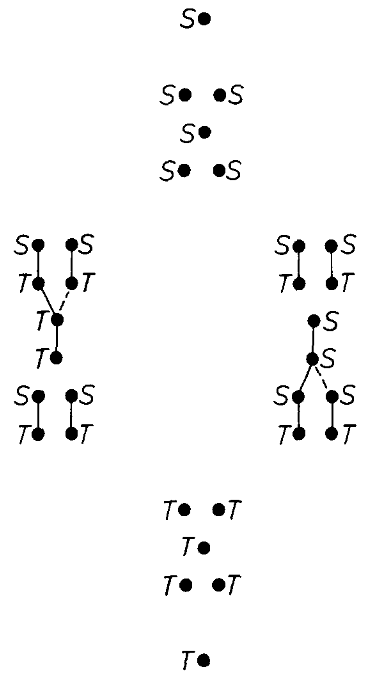

Fig. 2. The network for a small example with $d > a$.

4. Evaluating Cuts

We have shown $u \geqslant s \geqslant a$ and $u \geqslant d \geqslant a$ by turning a minimal unichain covering of **P** into a dense covering and by turning a maximal antichain into a cut having the appropriate value. We will establish $s \geqslant d$ by obtaining an S-family of size d from a maximum cut. We prepare for this by breaking the value of the cut into contributions due to the various elements of **P**. It turns out that elements contribute at most k, those contributing k must form an S-family, and there are at least d of them. 'Finite value' in this context means $\neq -\infty$.

LEMMA 6. *Every cut* $[S \mid T]$ *having finite value satisfies* (a) *from every path of the form* $(s, x_0, x_i^-, x_i^+, x_{k+1}, t)$, *S contains an initial segment and T contains the complementary final segment;* (b) $x >^i y$ *implies* $y_i^- \notin S$ *or* $x_i^+ \notin T$.

Proof. In a cut with finite value, no node of S can follow a node of T along an arc of infinite capacity. □

A cut is completely determined by specifying, for each $x \in \mathbf{P}$, which copies of x belong to S and which belong to T. Lemma 6a restricts the ways that the nodes $W(x)$ corresponding to $x \in \mathbf{P}$ can be partitioned by S and T. There are still many ways this partition can occur, but we will see that only a few of the possibilities can occur in a maximum cut.

Meanwhile, we need additional notation to describe these possibilities. Let

$$S(x) = W(x) \cap S \quad \text{and} \quad T(x) = W(x) \cap T.$$

Let

$$W^-(x) = \{x_0\} \cup \{x_i^- : 1 \leqslant i \leqslant k\} \quad \text{and} \quad W^+(x) = \{x_{k+1}\} \cup \{x_i^+ : 1 \leqslant i \leqslant k\}.$$

Given a fixed cut $[S \mid T]$ with finite value, we define various classes for the elements $x \in \mathbf{P}$ by various possibilities for the partition $[S(x) \mid T(x)]$.

$x \in C_0^+$ if $[S(x) \mid T(x)] = [W(x) \mid \emptyset]$.

$x \in C_0^-$ if $[S(x) \mid T(x)] = [\emptyset \mid W(x)]$.

$x \in C_0$ if $[S(x) \mid T(x)] = [W^-(x) \mid W^+(x)]$.

$x \in C_i^+$ if $[S(x) \mid T(x)] = [W^-(x) \cup \{x_i^+\} \mid W^+(x) - \{x_i^+\}]$.

$x \in C_i^-$ if $[S(x) \mid T(x)] = [W^-(x) - \{x_i^-\} \mid W^+(x) \cup \{x_i^-\}]$.

Note that these classes must be disjoint, although they need not be exhaustive. On the other hand, specifying a partition of the nodes into these classes immediately yields a cut. With this notation, the cut constructed in the proof of Lemma 3 can be defined by specifying $C_0 = M$, $C_0^- = M^-$, $C_0^+ = M^+$. The next lemma shows that the value of such a cut is easy to compute; indeed, we could have obtained Lemma 3 from Lemma 7.

Note that no arc with nonzero lower bound connects copies of different elements of $\mathbf{P}$. Thus, the value of any cut with finite value is the sum of the values obtained in the subnetworks induced by the sets $W(x)$. In order to combine symmetric cases, we denote $C^\sim = C_0^- \cup C_0^+$ and $C^* = \cup_{i \geqslant 1}(C_i^- \cup C_i^+)$. Also let $C^- = \cup_{i \geqslant 0} C_i^-$ and $C^+ = \cup_{i \geqslant 0} C_i^+$; note that $C^- \cup C^+ = C^* \cup C^\sim$.

LEMMA 7. *Given a cut $[S \mid T]$ with finite value, the contribution by x to that value is k if $x \in C_0$, $k-1$ if $x \in C^+ \cup C^-$, and less than $k-1$ otherwise.*

Proof. The contribution by x is the sum of lower bounds on arcs from $S(x)$ to $T(x)$. There are $k+2$ lower-bounded arcs in $W(x)$ to consider. By inspection, x contributes k arcs of value 1 if $x \in C_0$, $k-1$ arcs of value 1 if $x \in C^*$, and 1 arc of value $k-1$ if $x \in C^\sim$. Conversely, Lemma 6a implies that in each path of the form $(s, x_0, x_i^-, x_i^+, x_{k+1}, t)$, which together exhaust the lower-bounded arcs for x, at most one arc can contribute. Thus if either of the arcs (s, x_0) and (x_{k+1}, t) contribute to the cut, then x contributes value $k-1$ and belongs to $C^\sim$. This leaves the arcs $\{(x_i^-, x_i^+) : 1 \leqslant i \leqslant k\}$. If at least $k-1$ of them point from S to T, then by definition x belongs to C_0 or C^*. □

LEMMA 8. *For any cut $[S \mid T]$ with finite value, C_0 is an S-family in* $\mathbf{P}$.

Proof. By Lemma 6b, $x >^i y$ implies $y \notin C_0$ or $x \notin C_0$. □

Given a maximum cut, we can pull out a large enough S-family (simply the set C_0) to prove the theorem.

THEOREM 2. *For any poset* $\mathbf{P}$ *with transitive basis $\{P_i\}$, $u \geqslant s \geqslant d \geqslant a$.*

Proof. We have shown everything except $s \geqslant d$. Given a maximum cut, let $S = C_0$ (an

S-family, by Lemma 8). By Lemma 7, the value of the cut is at most $k|C_0| + (k-1)(|\mathbf{P}| - |C_0|) = (k-1)|\mathbf{P}| + |C_0|$. Hence $s \geq d$. □

5. Structure of Maximum Cuts

In this section we prove $s = d$ iff $s = a$. Our approach is as follows. In the construction for Theorem 2, the S-family C_0 obtained from the maximum cut is strictly larger than d unless every element of $\mathbf{P}$ not in C_0 contributes exactly $k - 1$ (no less) to the cut. By Lemma 7, this means they all belong to $C^+ \cup C^-$. We will show that C_0 is an antichain when $s = d$, by showing that if $x, y \in C_0$ and $x > y$, then there exists a z with $x > z > y$ that does not contribute $k - 1$ to the cut. First we need a lemma.

LEMMA 9. *Suppose z contributes $k - 1$ to a cut $[S|T]$. If $x >^i z$ and $x_i^+ \in T$, then $z \in C^-$. Similarly, $z >^i y$ and $y_i^- \in S$ implies $z \in C^+$.*

Proof. In the first case, Lemmas 6b, 6a, and 7 successively imply $z_i^- \in T$, $z_i^+ \in T$, and $z \in C^-$. The other case follows by the symmetric argument. □

THEOREM 3. *For any poset $\mathbf{P}$ with transitive basis $\{P_i\}$, $s = d$ iff $s = a$.*

Proof. It suffices to look at what happens when $s = d$. Choose a maximum cut. By Lemma 8, C_0 is an S-family of size d; since $s = d$, it is a maximum S-family. We need to show it is an antichain.

As suggested above, we need only show that not all elements of $\mathbf{P} - C_0$ can contribute $k - 1$ to the cut unless C_0 is an antichain. If C_0 is an S-family but not an antichain, then it contains related elements $x > y$ that are not unicomparable. We may assume no other element of C_0 lies between them. Since $\{P_i\}$ is a transitive basis, there is a chain $x = z_0 > z_1 > \cdots > z_t = y$ in $\mathbf{P}$ such that for each $1 \leq j < t$, z_{j-1} and z_j are $P_{i(j)}$-comparable for some $i(j)$.

Suppose $z_1, \ldots, z_s$ all contribute $k - 1$ to the cut, and $z_0 >^{i(1)} z_1$. Since $z_0 \in C_0$, we have $(z_0)_{i(1)}^+ \in T$. Lemma 9 then implies $z_1 \in C^-$. This means that all the +-copies of z_1 belong to T, including $(z_1)_{i(2)}$. Hence, we can apply Lemma 9 again to obtain $z_2 \in C^-$, and so on through $z_t \in C^-$, contradicting $z_t = y \in C_0$. □

The minimum dense covering is a combinatorial interpretation of the minimum feasible flow in the network we have constructed. We would like also to have a combinatorial interpretation of the maximum cut. For arbitrary k, there are too many ways that an element of $\mathbf{P}$ can contribute to the maximum cut. However, in the special case where $\mathbf{P}$ is the direct product of two posets, the structure is simple enough to formulate conjectures.

For the remainder of this section, let $k = 2$ and $\mathbf{P} = Q_1 \times Q_2$. Given a finite cut $[S|T]$, let $C^{\|}$ be the class of elements in $\mathbf{P}$ such that $[S(x)|T(x)] = [\{x_0, x_1^-, x_1^+\}|\{x_2^-, x_2^+, x_3\}]$ or $[S(x)|T(x)] = [\{x_0, x_2^-, x_2^+\}|\{x_1^-, x_1^+, x_3\}]$. Note that an element in $C^{\|}$ contributes 0 to the value of the cut. When looking for maximum cuts, the only classes we need are $C^{\|}$ and those defined earlier:

LEMMA 10. *For* $\mathbf{P} = Q_1 \times Q_2$, *the network has a maximum cut such that every element* $x \in \mathbf{P}$ *belongs to* $C_0 \cup C^{\sim} \cup C^{\|}$.

Proof. Consider the other possibilities for x; in each case we alter $[S(x) \mid T(x)]$ without reducing the cut value. If $S(x) = \{x_0\}$, then moving x_0 to T (i.e., placing x in C_0^+) increases the cut value by 1 ($= k - 1$). If $S(x) = \{x_0, x_i^-\}$, i.e., $x \in C_{3-i}^+$, again we move x into C_0^+. The cut value loses the unit from (x_i^-, x_i^+), but it gains the unit from (s, x_0). No other arcs are affected and Lemma 6 is still satisfied, so the new cut has the same value. The cases $T(x) = \{x_3\}$ and $T(x) = \{x_i^+, x_3\}$ are symmetric to these. Other cases allowed by Lemma 6 are those already in $C_0 \cup C^{\sim} \cup C^{\|}$. □

A cut for which all of $\mathbf{P}$ falls into these classes is a cut in *standard form*. Given a maximum cut in standard form, we want to know which classes contain elements related to the elements of the special semiantichain C_0 in various ways. Given a semiantichain S, an element $x \in \mathbf{P}$ satisfying $x >^i y$ and $x <^{3-i} z$ for some $y, z \in S$ is called an *elbow* of S (the term comes from the pictorial representation in the Hasse diagram). Let $E(S)$ be the set of elbows of S. Let $F^+(S) = \{x \in \mathbf{P} : x >^1 y$ and $x >^2 z$ for some $y, z \in S\}$. Similarly define $F^-(S)$ by replacing $>$ with $<$. The sets $E(S)$, $F^-(S)$, $F^+(S)$ are disjoint. They need not exhaust P, but they satisfy the following.

LEMMA 11. *For any maximum cut in standard form,* $E(C_0) \subseteq C^{\|}$, $F^+(C_0) \subseteq C_0^+$, *and* $F^-(C_0) \subseteq C_0^-$.

Proof. These statements follow immediately from Lemma 6 by avoiding infinite capacity. For example, if $x >^1 y$, $x <^2 z$, $y, z \in C_0$, then $y_1^- \in S(y)$ implies x_1^+, x_1^-, $x_0 \in S$, and $z_2^+ \in T(z)$ implies $x_2^-, x_2^+, x_3 \in T$. □

By Lemma 7 and the fact that elements of $C^{\|}$ contribute 0, the value of a maximum cut in standard form is $2|C_0| + |C^{\sim}| = |C_0| + |\mathbf{P}| - |C^{\|}|$, so $d = |C_0| - |C^{\|}|$. Let $e = \max\{|S| - |E(S)| : S$ is a semiantichain$\}$. Since $E(C_0) \in C^{\|}$ for a maximum cut in standard form, d is bounded above by an element of the set being maximized. Hence, $s \geqslant e \geqslant d$. It is possible that a further restriction of the 'standard form' of a maximum cut will show $e = d$ always. We only know now that elements unicomparable in only one direction to elements of C_0 belong to $C^{\|}$ or $C^{\sim}$ (actually a bit more by looking at Q_1 vs. Q_2 in more detail, but nothing helpful).

We would like to put $x \in C_0^+$ if $x >^i y$ for some $i \in \{1, 2\}$ and $y \in C_0$ and similarly $x \in C_0^-$ if $x <^i y$ for some $i \in \{1, 2\}$ and $y \in C_0$. Unfortunately, this cannot be done consistently for arbitrary semiantichains. For example, let Q_1 be a 2-chain and Q_2 a 3-chain. Let $C_0 = \{00, 12\}$. Then $E(C_0) = \{10, 02\}$, which forces $\{10, 02\} \subseteq C^{\|}$. The remaining elements 01 and 11 do not belong to any of E, F^+, F^-. Although $01 >^2 00$ and $11 <^2 12$, we cannot put both $01 \in C_0^+$ and $11 \in C_0^+$ because of the arc $(11_1^+, 01_1^-)$. On the other hand, this is not the C_0 for a maximum cut.

This suggests the idea of a 'compressed' semiantichain. Let x be *inside* S if $x > y$ and $x < z$ for some $y, z \in S$. Let $I(S)$ be the set of elements inside S. Note that $E(S) \subseteq I(S)$, since the definition here allows any relations in the poset to be used. We say that S is *compressed* if $I(S) = E(S)$. It may be possible to show that C_0 for the maximum cut can be taken to be a compressed semiantichain, and move on from there to get $e = d$, but we have not succeeded in specifying local changes to compress C_0 without reducing the cut value.

Finally, there seems to be no natural way to obtain a unichain covering from an optimal flow in the network, even if $s = d = a$ and $k = 2$, so the question of $s = d = a \Rightarrow u = s$ remains open.

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