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UNICHAIN COVERINGS IN PARTIAL ORDERS WITH THE NESTED SATURATION PROPERTY

Douglas B. WEST*
University of Illinois, Urbana, IL 61801, U.S.A.

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The theory of saturated chain partitions of partial orders is applied to the minimum unichain covering problem in the product of partial orders (posets). Define the *nested saturation property* for a poset to be the existence of a sequence of chain partitions $\mathscr{C}_1, \mathscr{C}_2, \ldots$ such that $\mathscr{C}_k$ is k- and $k+1$-saturated and the elements on chains of size at most k in $\mathscr{C}_k$ contain the elements on chains of size at most $(k-1)$ in $\mathscr{C}_{k-1}$. For the product of two posets P and Q with the nested saturation property, a unichain covering is constructed of size $\sum \Delta_k^P \Delta_k^Q$, where d_k^P is the size of the largest k-family in P and $\Delta_k = d_k - d_{k-1}$. This is applied to prove that the largest semiantichain and smallest unichain covering have the same size for products of a special class of posets.

1. Introduction

In a direct product of partial orders (posets), *unichains* are chains that vary in only one coordinate. We are interested in the minimum number of unichains required to cover the elements of a product poset. The natural dual problem is to maximize the size of a collection taking no pair of elements from a single unichain; such a set is called a *semiantichain*. These optimization problems are dual integer programs, with the size of any unichain covering bounded below by the size of any semiantichain. The question of whether the largest semiantichain and smallest unichain covering have the same size for the product of every pair of posets is open. In this paper, we use the theory of saturated chain partitions of posets to construct efficient unichain coverings when the component orders satisfy a property we call "nested saturation".

A *k-family* in a poset P is the union of k antichains; the largest size of a k-family is denoted d_k. A chain partition $\mathscr{C} = \{C_i\}$ of P is *k-saturated* if the trivial bound $d_k \leq \sum \min\{k, |C_i|\}$ holds with equality. The Greene–Kleitman Theorem [1] asserts the existence, for every P and k, of a chain partition that is both k- and $k+1$-saturated (see [3] for a summary of various proofs of this theorem). A chain partition that is k-saturated for all k is *completely saturated*. Let $S_k(\mathscr{C})$ denote the set of elements on chains of size at most k in $\mathscr{C}$. A partial order has the *nested saturation property* if it has a sequence $\boldsymbol{C} = \mathscr{C}_1, \mathscr{C}_2, \ldots$ of chain partitions such

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that for all k: (1) $\mathscr{C}_k$ is k, $k+1$-saturated, and (2) $S_{k-1}(\mathscr{C}_{k-1}) \subseteq S_k(\mathscr{C}_k)$. Such a sequence of partitions is a *nested saturation sequence.* In discussing k-families, it is helpful to define $\Delta_k = d_k - d_{k-1}$ and use superscripts when discussing more than one poset.

Our main result is a constructive proof (also obtained by Michael Saks [2]) of the following.

Theorem 1. *If P and Q have the nested saturation property, then $P \times Q$ has a unichain covering with $\sum \Delta_k^P \Delta_k^Q$ chains.*

This result does not always give the minimum unichain composition, even when P and Q have completely saturated partitions. Figure 1 (from [5]) exhibits a unichain covering of size 9 for a poset product with $\sum \Delta_k^P \Delta_k^Q = 10$, even though both posets have completely saturated partitions. Nevertheless, Theorem 1 yields optimal unichain coverings for some classes of posets where they were not known before.

Given chain partitions $\mathscr{C} = \{C_i\}$ and $\mathscr{D} = \{D_j\}$ of P and q, West and Tovey [5] built "decomposable" unichain coverings of $P \times Q$ by covering the product of a k-chain from $\mathscr{C}$ and an l-chain from $\mathscr{D}$ with $\min\{k, l\}$ copies of the $\max\{k, l\}$-chain. The number of unichains in the resulting covering is $\sum_{i,j} \min\{|C_i|, |D_j|\}$. They showed that the number of unichains in such a covering is bounded below by $\sum \Delta_k^P \Delta_k^Q$ and determined when equality holds. They also considered fixing a partition of only one poset, and associating with each k-chain in that partition a k-saturated partition of the other poset. This always does as least as well as the best decomposable covering, and is still bounded below by $\sum \Delta_k^P \Delta_k^Q$. The construction of the present paper achieves $\sum \Delta_k^P \Delta_k^Q$ for some products in which the methods above do not. When the posets have completely saturated chain partitions, the covering we construct is the same as the decomposable covering constructed by West and Tovey from the completely saturated partitions. In other words, we again cover copies of an element on a 'short chain' with copies of 'long chains' from the other order, in a manner made precise in the next section.

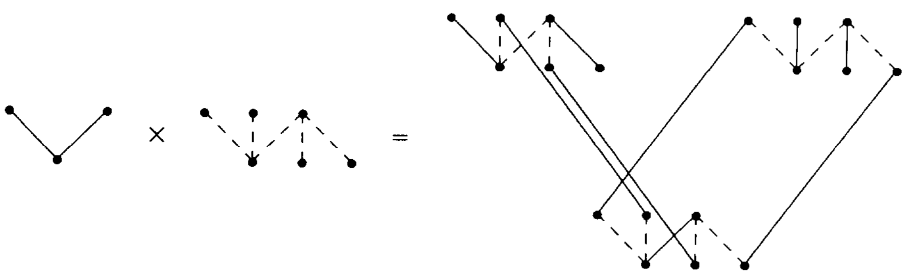

Fig. 1. A product covered by fewer than $\sum \Delta_k^P \Delta_k^Q$ unichains.
(Solid lines indicate unichains in the covering.)

We should note that the posets with nested saturation sequences properly include those with completely saturated partitions, but that not every poset has this new property. An easy way to show that a poset Q fails nested saturation, based on Theorem 1, is to present a semiantichain of size larger than $\sum \Delta_k^P \Delta_k^Q$ in the product of Q with some P having the nested saturation property. For example, let Q be the poset of Fig. 2, and let P be the poset P_3 of Fig. 3. Then $\sum \Delta_k^Q = 2\cdot 3 + 2\cdot 2 + 1\cdot 2 + 1\cdot 1 = 13$, but $P \times Q$ has a semiantichain of size 14 consisting of the pairs $\{a1, a6, b3, b8, c2, c7, d4, d9, e3, e5, e6, f4, f5, f7\}$. P has the nested saturation property, so Q cannot. Motivated by this, we ask for a characterization of posets with the nested saturation property.

As a sufficient condition for a unichain covering of size $\sum \Delta_k^P \Delta_k^Q$, the nested saturation property is independent of the condition in [5]. If P and Q are each taken to be the poset of Fig. 2, then neither has the nested saturation property, but the product $P \times Q$ has a decomposable unichain covering of size $\sum \Delta_k^P \Delta_k^Q = 3^2 + 2^2 + 2^2 + 1^2 + 1^2 = 19$, given by the chain partitions with size sequences (441) and (522).

On the other hand, in Section 3 we discuss a class of posets that have the nested saturation property, but whose products do not in general have decomposable unichain coverings of size $\sum \Delta_k^P \Delta_k^Q$. These are the posets P_n constructed in [4]; P_3 and P_4 are illustrated in Fig. 3. These posets do have 'decomposable' semiantichains of size $\sum \Delta_k^P \Delta_k^Q$, so the unichain covering of size $\sum \Delta_k^P \Delta_k^Q$ constructed in Theorem 1 demonstrates equality between maximum semiantichain and minimum unichain covering for these posets. P_n is a poset of height n that is shown in [4] to have no chain partition that is k-saturated for any non-consecutive values of n. Hence it is not very surprising that $P_n \times P_m$ does not have decomposable unichain coverings of size $\sum \Delta_k^{P_n} \Delta_k^{P_m}$. The lack of more highly saturated partitions than k, $k+1$ means that the any nested saturation sequence uses a different chain partition for each value of k. In Fig. 3, the solid edges in the diagram of P_4 indicate the long chains in its 3,4-saturated partition.

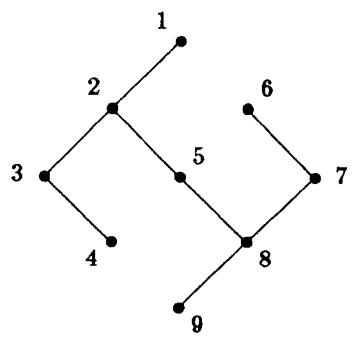

Fig. 2. A poset Q not having the nested saturation property.

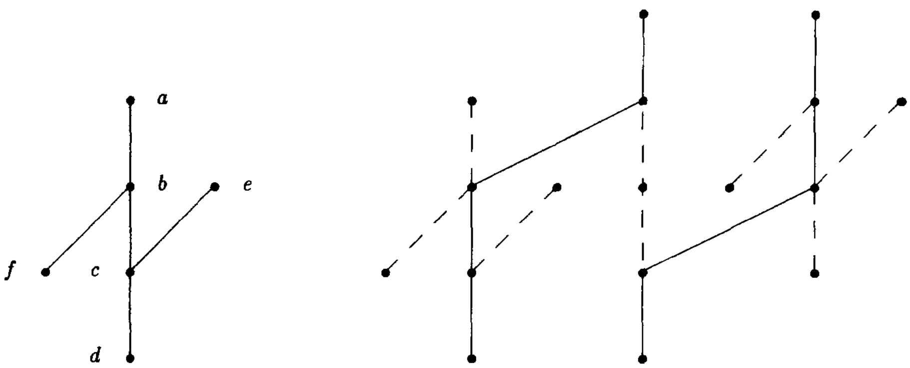

Fig. 3. P_3, P_4.

2. The nested saturation theorem

We give a constructive proof that products of posets with the nested saturation property have unichain coverings of size $\sum \Delta_k^P \Delta_k^Q$. First we construct the desired covering. Let $\boldsymbol{C} = \mathscr{C}_1, \mathscr{C}_2, \ldots$ and $\boldsymbol{D} = \mathscr{D}_1, \mathscr{D}_2, \ldots$ be nested saturation sequences for P and Q. Let $T_k(\boldsymbol{C}) = S_k(\mathscr{C}_k) - S_{k-1}(\mathscr{C}_{k-1})$, and similarly for $T_k(\boldsymbol{D})$. Nested saturation implies that the sets $T_k(\boldsymbol{C})$ partition P, so let $t(x) = k$ if $x \in T_k(\boldsymbol{C})$. Similarly let $t(y) = k$ if $y \in T_k(\boldsymbol{D})$. For each point $(x, y) \in P \times Q$, we specify the unichain containing (x, y). If $t(x) \leq t(y)$, then the unichain containing (x, y) is the x-copy of the chain containing y in $\mathscr{D}_{t(x)-1}$ (i.e., pair each element of this chain in Q with x to obtain the unichain in $P \times Q$). If $t(x) > t(y)$, then the unichain containing (x, y) is the y-copy of the chain containing x in $\mathscr{C}_{t(y)}$. Note that this covering is in fact a unichain partition: Let (w, z) be another point on the unichain defined for (x, y). If $w = x$, then $z \notin T_{k-1}(\mathscr{D})$. Hence $t(w) \leq t(z)$, and the same unchain is the unichain defined for (w, z). The proof is analogous if $y = z$.

To prove the theorem, it suffices to count these unichains, so we need to know how many are constructed fixed at any given point of P or Q. These numbers are obtained from the following well-known lemma [1].

Lemma 1. *If $\mathscr{C}$ is a $(k-1)$, k-saturated partition of a poset P, then the number of chains of size at least k in $\mathscr{C}$ is exactly Δ_k^P.*

Proof. Let α_k be the number of these chains. The bounds placed on the size of the largest $(k-1)$- and k-families by this partition are tight, so we have $d_{k-1} = (k-1)\alpha_k + |S|$ and $d_k = k\alpha_k + |S|$, where S is the set of elements on chains of size at most $k-1$. Subtracting these two equations yields $\Delta_k = \alpha_k$. □

Theorem 1. *If P and Q have the nested saturation property, then the unichain covering of $P \times Q$ defined above has $\sum \Delta_k^P \Delta_k^Q$ chains.*

Proof. We count the unichains used in the construction according to their fixed element in P or Q. (A one-element unichain is fixed in each coordinate, but it arises in the construction as being fixed in a particular one of those.) In particular, if $x \in T_k(\boldsymbol{C})$, the unichains fixed at x in their first coordinate correspond to the chains of $\mathscr{D}_{k-1}$ of size at least k, since they contain points (x, y) with $t(y) \geq t(x)$. By the lemma, there are Δ_k^Q of these. Similarly, if $y \in T_k(\boldsymbol{D})$, the unichains fixed at y in their second coordinate are the chains of $\mathscr{C}_k$ with size at least $k+1$, and there are Δ_{k+1}^P of them. Hence the construction uses $\sum |T_k(\boldsymbol{C})|\, \Delta_k^Q + \sum |T_k(\boldsymbol{D})|\, \Delta_{k+1}^P$ unichains.

To compute the sizes of $T_k(\boldsymbol{D})$ and $T_k(\boldsymbol{C})$, apply the lemma to $S_k(\mathscr{C}_k)$ and $S_{k-1}(\mathscr{C}_{k-1})$. This yields $|S_k(\mathscr{C}_k)| = d_k = k\Delta_{k+1}^P$ and $|S_{k-1}(\mathscr{C}_{k-1})| = d_{k-1} - (k-1)\Delta_k^P$, so $|T_k(\boldsymbol{C})| = d_k^P - d_{k-1}^P + (k-1)\Delta_k^P - k\Delta_{k+1}^P = k(\Delta_k^P - \Delta_{k+1}^P)$. Similarly, $|T_k(\boldsymbol{D}) = k(\Delta_k^Q - \Delta_{k+1}^Q)$.

Now it is easy to count the unichains in the construction. The total is

$$\sum k(\Delta_k^P - \Delta_{k+1}^P)\Delta_k^Q + \sum k(\Delta_k^Q - \Delta_{k+1}^Q)\Delta_{k+1}^P$$
$$= \sum [k\Delta_k^P\Delta_k^Q - (k-1)\Delta_k^Q\Delta_k^P] = \sum \Delta_k^P\Delta_k^Q. \quad \square$$

3. The polyunsaturated posets

Now we apply Theorem 1 to the class of 'polyunsaturated posets' constructed in [4]. P_3 and P_4 are pictured in Fig. 3. To construct P_n for $n > 3$, begin by taking two copies of P_{n-1} and a 'central chain' of size $(n+1)$. Regard the two copies of P_{n-1} as the "left" or "lower" copy P_n^- and the "right" or "higher" copy P_n^+. This makes it natural to call the 'central chain' P_n^0. Denote the top two and bottom two elements of P_n^0 by a_n, b_n, c_n, d_n, in descending order. Denote the similarly defined elements in P_n^- and P_n^+ by a_n^-, b_n^-, c_n^-, d_n^- and a_n^+, b_n^+, c_n^+, d_n^+. To complete the construction of P_n, add two more relations $b_n > b_n^-$ and $c_n < c_n^+$, and those implied by transitivity.

For P_3 the central chain is the unique chain of size 4. For $n \geq 4$ the decomposition of P_n into P_{n-1}, P_n^0, P_n^+ is unique, since although there are $(n+1)$-chains other than P_n^0, there is no other whose deletion leaves two copies of P_{n-1}.

For convenience, we write $d_k^n = d_k^{P_n}$ and $\Delta_k^n = \Delta_k^{P_n}$. The maximum k-family sizes in P_4, for example, are 5, 10, 13, 15, 17, and its (1,2)-, (2,3)-, and (3,4)-saturated partitions have chain size sequences (53333), (5441111), and (5522111), the last of which is illustrated in Fig. 3.

We consider products $P_n \times P_m$. It is shown by induction in [4] that P_n has a partition into antichains of sizes $\Delta_1^n, \ldots, \Delta_{n+1}^n$; hence $P_n \times P_m$ has a 'decomposable' semi-antichain of size $\sum \Delta_k^n \Delta_k^m$ obtained by pairing up the largest antichains from each decomposition, then the next largest, etc. Theorem 1 yields the desired min-max relation for products $P_n \times P_m$ as soon as we show that P_n has the nested saturation property. We will construct the nested saturation sequence and then verify that it works.

For $n \geq 4$, k-families in P_n can be formed for $k \leq n-1$ by taking copies of k-families in P_n^- and P_n^+ together with k elements of $P_n^0 - \{a_n, b_n, c_n, d_n\}$. Furthermore, $P_n - \{a_n, b_n\}$ is an n-family and $P_n - \{a_n, b_n, c_n, d_n\} - \{a_n^-, d_n^+\}$ is an $n-1$-family. (For $k = n-2$, $n-1$, verifying that this is valid requires throwing into the induction hypothesis the fact that a maximum n-family contains exactly one of $\{a_n, b_n\}$ and one of $\{c_n, d_n\}$. The straightforward details appear in [4].)

These are in fact maximum-sized k-families, which we show by constructing saturated partitions to match them. These will form the nested saturation sequence $\boldsymbol{C}^n = \mathscr{C}_1^n, \mathscr{C}_2^n, \ldots, \mathscr{C}_n^n$ for P_n. The partitions $\mathscr{C}_k^n$ are constructed inductively; for P_3 we have $\mathscr{C}_1^3$ and $\mathscr{C}_2^3$ being the well-known partitions with chain-size sequences (33) and (411). For $n \geq 4$ and $k < n-1$, construct $\mathscr{C}_k^n$ by taking the chain P_n^0 and copies of $\mathscr{C}_k^{n-1}$ from each of P_n^- and P_n^+. Note that in this range $\sum_{C_i \in \mathscr{C}^n} \min\{j, |C_i|\}$ obeys the same recurrence as the size of the j-families constructed in the preceding paragraph. The construction and proof that it works are completed by defining $\mathscr{C}_{n-1}^n$ appropriately. To match the k-families constructed above for $k = n-1$, n when $n > 4$, we need only construct a chain partition of P_n with two $(n+1)$-chains and two n-chains.

For $n = 4$, let $\mathscr{C}_3^4$ be the unique partition of P_4 with size sequence (5522111), illustrated in Fig. 1. For $n > 4$, we construct $\mathscr{C}_{n-1}^n$ inductively so that one of the $(n+1)$-chains contains $\{a_n, b_n\}$ and the other contains $\{c_n, d_n\}$. $\mathscr{C}_3^4$ satisfies this. Assuming that $\mathscr{C}_{n-2}^n$ has been so constructed, construct $\mathscr{C}_{n-1}^n$ by taking copies of $\mathscr{C}_{n-2}^n$ from P_n^- and P_n^+ and modifying them as follows; on the n-chains where they appear, replace a_n^- by $\{a_n, b_n\}$ and d_n^+ by $\{c_n, d_n\}$, thus turning two n-chains into $(n+1)$-chains. The replaced element $a_n^-[d_n^+]$ can be added to the end of any available chain; it is convenient to add it to the end of the chain of $n-4$ central elements of the central chain of $P_n^-[P_n^+]$. The other chains from the copies of P_{n-1} remain as they were, including the other n-chain from each, and the $n-3$ elements of $P_n^0 - \{a_n, b_n, c_n, d_n\}$ are placed in a single chain. Now $\mathscr{C}_{n-1}^n$ has exactly two $(n+1)$-chains and exactly two n-chains (hence is $n-1$-and n-saturated), and the hypothesis about membership on the longest chains has been preserved.

Equipped with $\boldsymbol{C}^n$, we can prove

Theorem 2. *$P_n \times P_m$ has a largest semi-antichain and smallest unichain covering of size $\sum \Delta_k^n \Delta_k^m$.*

Proof. As remarked, it suffices to show that $\boldsymbol{C}^n$ is a nested saturation sequence for P_n. This is very easy to do by induction on n. For $n = 3$, 4, it is easy to verify by inspection. For $n > 4$, Let $\boldsymbol{C}^- = \mathscr{C}_1^-, \mathscr{C}_2^-, \ldots$ and $\mathscr{C}^+ = \mathscr{C}_1^+, \mathscr{C}_2^+, \ldots$ be the nested saturation sequences for P_n^- and P_n^+. For $k < n-1$, $S_{k-1}(\mathscr{C}_{k-1}^n) \subset S_k(\mathscr{C}_k^n)$

by induction, since this holds for C^- and C^+ and $\mathscr{C}^n_{k-1}$, $\mathscr{C}^n_k$ are built from those by including P^0_n. Since $\mathscr{C}^n_n = \mathscr{C}^n_{n-1}$, it remains only to verify the inclusion for $k = n$. Recall the alteration of $\mathscr{C}^n_{n-2}$ performed to construct $\mathscr{C}^n_{n-1}$. The two $(n+1)$-chains and two n-chains of $\mathscr{C}^n_{n-1}$ comprise all of $P_n - S_{n-1}(\mathscr{C}^n_{n-1})$. They consist of everything that was on the two n-chains of $S_k(\mathscr{C}^-_{n-2})$ and of $S_k(\mathscr{C}^+_{n-2})$, except with a^-_n and d^+_n replaced by $\{a_n, b_n, c_n, d_n\}$. Hence $S_{n-1}(\mathscr{C}^n_{n-1})$ (properly) contains $S_{n-2}(\mathscr{C}^-_{n-2})$ and $S_{n-2}(\mathscr{C}^+_{n-2})$. However, $S_{n+2}(\mathscr{C}^n_{n-2})$ equals precisely $S_{n-2}(\mathscr{C}^-_{n-2}) \cup S_{n-2}(\mathscr{C}^+_{n-2})$, so the required inclusion holds. □

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