Decomposition of Sparse Graphs: Nine Dragon Tree Conjecture for $k \le 2$

Douglas B. West

Department of Mathematics
Zhejiang Normal University and
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

Joint work with
Min Chen, Seog-Jin Kim,
Alexandr V. Kostochka, Xuding Zhu

preprint & slides at DBW homepage

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Nine Dragon Tree (NDT) Conjecture:
(Montassier, Ossona de Mendez, Raspaud, Zhu [2010])
$\mathrm{Arb}(G) \leq k + \frac{d}{k+d+1}$ $\Rightarrow$ G decomposes into $k+1$ forests, with the last being d-bounded.

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Motivation: Applies to "game coloring number", a bound on game chromatic number χ_g (always $\chi_g(G) \le \mathrm{col}_g(G)$).

- Zhu [1999]: If G decomposes into G_1 and G_2, then $\mathrm{col}_g(G) \le \mathrm{col}_g(G_1) + \Delta(G_2)$.
- Faigle-Kern-Kierstead-Trotter [1993]: If G is a forest, then $\mathrm{col}_g(G) \le 4$.

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NDT Conj. for $k = 2$ cannot imply this ($\text{Arb}(G) \to 3$).

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Def. coloring game: Alice and Bob alternately color vertices (properly) from k available colors.
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- Arboricity 2 places no bound on $\chi_g(G)$.

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$\mathrm{Mad}(G) < 2b$: too weak but works well with discharging.

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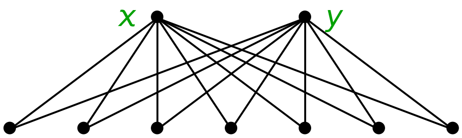

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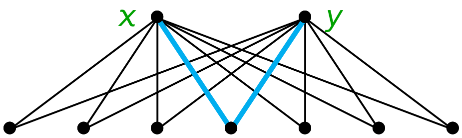

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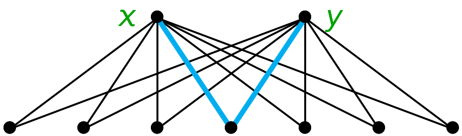

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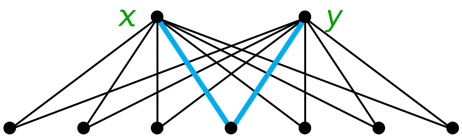

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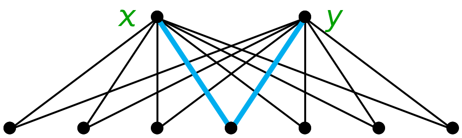

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(Same for any subdivided $(2d+2)$-regular multigraph.)

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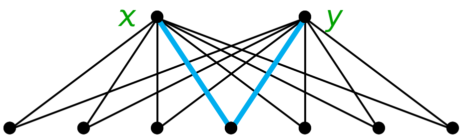

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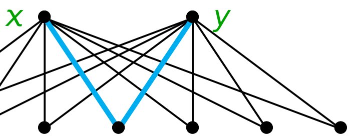

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(implies the earlier results for planar with large girth.)

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(k,d)-sparse	$(k+1)(k+d)\,|A| - (k+d+1)\,\|A\| \ge k^2$
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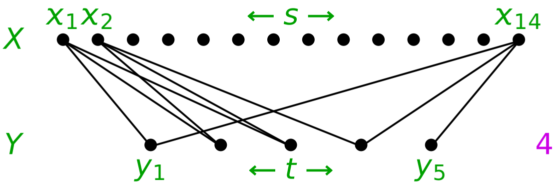
Ex. An X, Y-bigraph G with $|X| = s$ and $|Y| = t$, where $s = t(k + d) - k + 1$. Put $x_i \longleftrightarrow \{y_i, \ldots, y_{i+k}\}$ (modulo t). $|E(G)| = (k + 1)s$.



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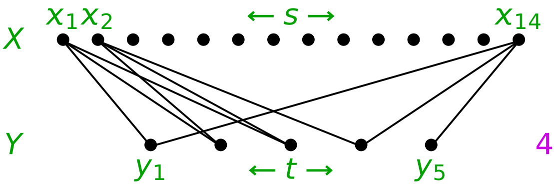

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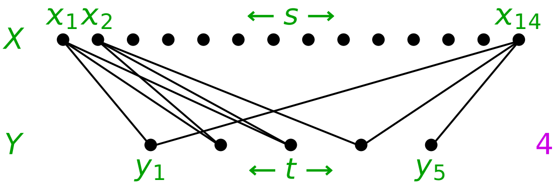

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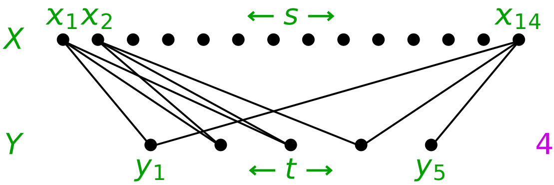

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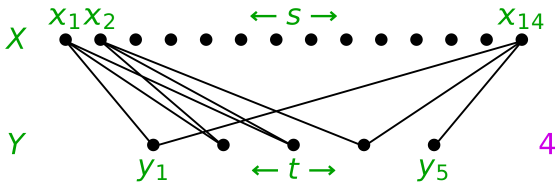

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Under additional constraints true for $k = 1$ and for $k = 2$ when $d > 1$, all v and e end with nonpositive charge.

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This contradicts the feasibility of f. ■

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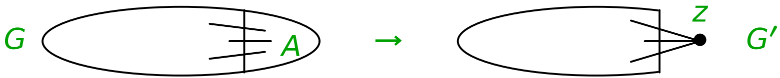

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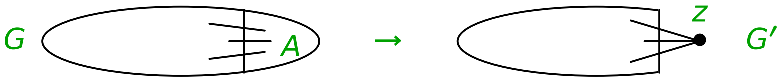

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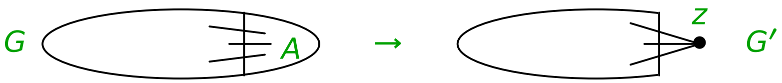

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No cycles arise, and D' increases no degrees in $V(D^*)$. ∎

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Lem. G has no full set A with $|A| \geq 2$.

Pf. (sketch) No overfull set $\Rightarrow$ $\|A\| = (k+1)(|A|-1)$.
Lemma $\Rightarrow A_0 = \varnothing$, so $\rho(e) = -(k+1+d)$ for $e \in E(G[A])$.

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Case 2: $A \neq V(G)$. In this case, contracting A into a vertex z with appropriate nonzero capacity again yields a smaller instance (G', f') whose (k, f')-decomp merges with one for $G[A]$ to form a (k, f)-decomp of G. ■

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Add one copy of ux to D and the other k edges at u to distinct forests in F. ■

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(That is, the endpoint in A always has capacity 0.)

The Discharging Argument

Thm. Let (G, f) be a smallest counterexample, and put $h(v) = \left|[v, V_0]\right|$. If $d \geq k$, then some $v \in V(G)$ satisfies

(1) $f(v) = 0$ and $h(v) > 2(d_G(v) - k - 1)\frac{k}{k-1}$, or

(2) $f(v) = d$ and $h(v) < \frac{(2k+2-d_G(v))(k+1+d)-2(k+1)}{d+1-k}$.

Pf. Initial charge = potential (verts & edges), total $\geq k^2$.

Each edge xy takes charge from its ends, reaching 0.

R1: For $f(x)=0, f(y)>0$, get k from x and $d+1$ from y.
R2: For $f(x), f(y) > 0$, get $(k+1+d)/2$ from each end.
R3: For $f(x), f(y) = 0$, get $(k+1)/2$ from each end.

With the earlier reductions, all vertices lose all their charge when no (1) or (2) exists: total potential ≤ 0.

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Since $d + 1 \geq k$, v loses at least $\frac{k+1+d}{2}$ to each edge.

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For $k = 2$, we may assume $d_G(v) \in \{4, 5\}$.
We need neighbors of v with capacity 0:
one when $d_G(v) = 5$, three when $d_G(v) = 4$ and $d \geq 3$,
all four when $d_G(v) = 4$ and $d = 2$.

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We omit the case $(k, d) = (2, 1)$, where some of our later arguments do not apply.