

# Decomposition of Sparse Graphs into Forests: The Nine Dragon Tree Conjecture for $k \leq 2$

Min Chen<sup>\*</sup>, Seog-Jin Kim<sup>†</sup>, Alexandr Kostochka<sup>‡</sup>,  
Douglas B. West<sup>§</sup>, Xuding Zhu<sup>¶</sup>

Revised November 2014 and April 2016

## Abstract

For a loopless multigraph  $G$ , the *fractional arboricity*  $\text{Arb}(G)$  is the maximum of  $\frac{|E(H)|}{|V(H)|-1}$  over all subgraphs  $H$  with at least two vertices. Generalizing the Nash-Williams Arboricity Theorem, the Nine Dragon Tree Conjecture asserts that if  $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ , then  $G$  decomposes into  $k + 1$  forests with one having maximum degree at most  $d$ . The conjecture was previously proved for  $d = k + 1$  and for  $k = 1$  when  $d \leq 6$ . We prove it for all  $d$  when  $k \leq 2$ , except for  $(k, d) = (2, 1)$ .

## 1 Introduction

Throughout this paper, we consider loopless multigraphs; this is the model we mean when we say “graph”. A *decomposition* of a graph  $G$  consists of edge-disjoint subgraphs with union  $G$ . The *arboricity* of  $G$ , written  $\Upsilon(G)$ , is the minimum number of forests needed to decompose it. The famous Nash-Williams Arboricity Theorem [14] states that  $\Upsilon(G) \leq k$  if and only if no subgraph  $H$  has more than  $k(|V(H)| - 1)$  edges.

The *fractional arboricity* of  $G$ , written  $\text{Arb}(G)$ , is  $\max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$  (Payan [15]). Nash-Williams’ Theorem states  $\Upsilon(G) = \lceil \text{Arb}(G) \rceil$ . If  $\text{Arb}(G) = k + \epsilon$  (with  $k \in \mathbb{N}$  and  $0 < \epsilon \leq 1$ ), then  $k + 1$  forests are needed. When  $\epsilon$  is small, one may hope to restrict the form of the last

---

<sup>\*</sup>Department of Mathematics, Zhejiang Normal University, chenmin@zjnu.edu.cn. Research supported by NSFC grant 11471293 and ZJNSFC grant LY14A010014.

<sup>†</sup>Department of Mathematics Education, Konkuk University, skim12@konkuk.ac.kr. Research supported by Basic Science Research Program through the National Research Foundation of Korea (NRF), funded by the Ministry of Education, grant NRF-2015R1D1A1A01057008.

<sup>‡</sup>Departments of Mathematics, University of Illinois and Zhejiang Normal University, kostochk@math.uiuc.edu. Research supported in part by NSF grants DMS-1266016 and DMS-1600592.

<sup>§</sup>Departments of Mathematics, Zhejiang Normal University and University of Illinois, west@math.uiuc.edu. Research supported by Recruitment Program of Foreign Experts, 1000 Talent Plan, State Administration of Foreign Experts Affairs, China.

<sup>¶</sup>Department of Mathematics, Zhejiang Normal University, xdzhuzh@zjnu.edu.cn. Research supported by CNSF grant 11571319.

forest, since  $k$  forests are “almost” enough to decompose  $G$ . The *Nine Dragon Tree (NDT) Conjecture* asserts that one can bound the maximum degree of the last forest in terms of  $\epsilon$  and  $k$ . Call a graph  $d$ -bounded if its maximum degree is at most  $d$ . We call a decomposition into  $k + 1$  forests, one of which is  $d$ -bounded, a  $(k, d)$ -decomposition. A graph having a  $(k, d)$ -decomposition is  $(k, d)$ -decomposable.

**Conjecture 1.1** (NDT Conjecture [13]). *If  $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ , then  $G$  has a  $(k, d)$ -decomposition.*

Montassier, Ossona de Mendez, Raspaud, and Zhu [13] posed the NDT Conjecture, showed that the hypothesis cannot be relaxed, and proved the conjecture for  $(k, d) \in \{(1, 1), (1, 2)\}$ . Kim, Kostochka, Wu, West, and Zhu [10] proved the conjecture for  $d = k + 1$  and for the case  $k = 1$  when  $d \leq 6$ .

Our main result implies the NDT Conjecture for  $k \leq 2$ , except for the case  $(k, d) = (2, 1)$ . That case (indeed all cases with  $d = 1$ ), was proved by D. Yang [17]. Our paper thus completes the proof for  $k \leq 2$ . After the completion of our paper, a proof of the NDT Conjecture for all  $k$  was obtained by Jiang, Yang, and Zhang [9] by a completely different method. Our proof remains of interest, partly due to the difference in methods, but also because it proves special cases of a stronger statement than the NDT Conjecture. We state this stronger conjecture as Conjecture 1.5.

Another stronger version of the NDT conjecture, different from ours, also remains open. A *strong  $(k, d)$ -decomposition* is a decomposition into  $k + 1$  forests, in one of which every component has at most  $d$  edges. Such a decomposition is more restricted than a  $(k, d)$ -decomposition.

**Conjecture 1.2** (Strong NDT Conjecture [13]). *If  $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ , then  $G$  has a strong  $(k, d)$ -decomposition.*

The Strong NDT Conjecture was proved for  $(k, d) = (1, 2)$  in [10]. Montassier et al. [13] also posed a Weak NDT Conjecture, stating that the same hypothesis guarantees a *weak  $(k, d)$ -decomposition*, which is a decomposition into  $k$  forests plus one  $d$ -bounded graph (not necessarily a forest). The Weak NDT Conjecture was proved for  $d > k$  in [10] and follows in full from the proof of the NDT Conjecture in [9]. Note that for  $d = 1$  the notions of weak  $(k, d)$ -decomposition,  $(k, d)$ -decomposition, and strong  $(k, d)$ -decomposition are the same.

Study of the NDT Conjecture was motivated by problems about weak  $(k, d)$ -decomposition and  $(k, d)$ -decomposition of planar graphs discussed in [1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 16]. These results, some of which are successive reductions of the girth needed to guarantee  $(1, 1)$ - or  $(1, 2)$ -decompositions of planar graphs, are summarized in [10]. Our result implies all of these except the results about  $(2, d)$ -decomposition of planar graphs in [1] and [7].

In the computation of  $\text{Arb}(G)$ , it suffices to maximize over induced subgraphs. Letting  $G[A]$  denote the subgraph of  $G$  induced by a vertex set  $A$ , and letting  $\|A\|$  denote  $|E(G[A])|$ ,

we can rewrite the condition bounding  $\text{Arb}(G)$  as an integer inequality. In the same format we define a weaker condition of being  $(k, d)$ -sparse:

Condition	Equivalent constraint (when imposed for $\emptyset \neq A \subseteq V(G)$ )
$\text{Arb}(G) \leq k + \frac{d}{k+d+1}$	$(k+1)(k+d) A  - (k+d+1)\ A\  \geq (k+1)(k+d)$
$(k, d)$ -sparse	$(k+1)(k+d) A  - (k+d+1)\ A\  \geq k^2$

In [10], it was proved that every  $(k, d)$ -sparse graph is weakly  $(k, d)$ -decomposable when  $d > k$ . Graphs without weak  $(k, d)$ -decompositions were given where the  $(k, d)$ -sparseness inequality holds for every nonempty proper vertex set but fails by 1 on the entire set. For sufficiency, a more general model involving ‘‘capacities’’ was used to control vertex degrees.

**Definition 1.3.** Fix positive integers  $k$  and  $d$ . A *capacity function* on a graph  $G$  is a function  $f: V(G) \rightarrow \{0, \dots, d\}$ . A  $(k, f)$ -*decomposition* of  $G$  is a decomposition into graphs  $F$  and  $D$ , where  $F$  decomposes into  $k$  forests and  $D$  is a forest having degree at most  $f(v)$  at each vertex  $v$ . The *uniform case* is  $f(v) = d$  for all  $v \in V(G)$ .

In this setting, we define a *potential function*  $\rho$  on the vertices, edges, and vertex subsets of  $G$ . For a vertex  $v$ , let  $\rho(v) = (k+1)(k+f(v))$ . For an edge  $e$ , let  $\rho(e) = -(k+1)$  if the endpoints of  $e$  both have capacity 0, and otherwise  $\rho(e) = -(k+1+d)$ . For  $A \subseteq V(G)$ , let  $\rho(A) = \sum_{v \in A} \rho(v) + \sum_{e \in E(G[A])} \rho(e)$ . A capacity function  $f$  is *feasible* if the corresponding potential function  $\rho$  satisfies  $\rho(A) \geq k^2$  for every nonempty vertex subset  $A$ .

In the uniform case, every vertex has capacity  $d$ , so every vertex has potential  $(k+1)(k+d)$  and every edge has potential  $-(k+1+d)$ . The feasibility inequality then reduces to precisely the definition of  $(k, d)$ -sparseness. However, in order to obtain  $(k, d)$ -decomposability, the hypothesis must be strengthened, because feasibility leaves the possibility of subgraphs having too many edges to decompose into  $k+1$  forests.

**Definition 1.4.** A vertex subset  $A \subseteq V(G)$  is *overfull* if  $\|A\| > (k+1)(|A| - 1)$ .

Large overfull sets are forbidden by  $(k, d)$ -sparseness, but small ones are not. An overfull set  $A$  of size  $r$  with  $\|A\| = (k+1)(r-1) + 1$  satisfies the  $(k, d)$ -sparseness inequality when  $r \leq \frac{k(d+1)}{k+1}$ . For example,  $(k, d)$ -sparse graphs may have edges with multiplicity  $k+2$ , but  $(k, d)$ -decomposable graphs cannot. Since  $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$  implies that  $G$  is  $(k, d)$ -sparse and has no overfull set, the conjecture below strengthens the NDT Conjecture in a different way from the Strong NDT Conjecture.

**Conjecture 1.5.** Fix  $k, d \in \mathbb{N}$ . If  $G$  is  $(k, d)$ -sparse and has no overfull set, then  $G$  is  $(k, d)$ -decomposable.

Our main result proves a more general statement in the case  $k \leq 2$  and  $d \geq k$ ; capacity functions facilitate the proof.

**Theorem 1.6.** *For  $k \leq 2$  and  $d \geq k$ , if  $f$  is a feasible capacity function defined on  $G$ , and  $G$  has no overfull set, then  $G$  is  $(k, f)$ -decomposable.*

**Remark 1.7.** The statement for general capacity functions contains the statement for the uniform case, but the uniform case also implies the general case. This was shown in [10] for a similar potential function; we explain it here more simply.

Capacity  $f(v)$  on  $v$  can be modeled in the uniform case by adding  $d - f(v)$  neighbors of  $v$ , each with  $k + 1$  edges to  $v$  and having no other neighbors. Each such neighbor forces an edge at  $v$  into  $D$ . When  $u$  with capacity  $d$  is added as such a neighbor of  $v$  and  $f(v)$  is increased by 1 to allow the added edge in  $D$ , the old potential of a set  $A$  containing  $v$  equals the new potential of  $A \cup \{u\}$  (we gain and lose  $(k + 1 + d)(k + 1)$ ).

Thus feasibility in the general case is equivalent to feasibility of the corresponding augmented sets in the uniform case, and the existence of the desired decompositions is also equivalent. Nevertheless, the general result is easier to prove: capacity functions facilitate reserving an edge for the  $d$ -bounded forest  $D$  while reducing the number of edges, by deleting the edge and reducing the capacity of its endpoints.

Under the potential function used in [10], every edge has potential  $-(k + 1 + d)$ . In our definition, most edges still have potential  $-(k + 1 + d)$ , but edges whose endpoints both have capacity 0 have potential  $-(k + 1)$ . This makes more capacity functions feasible and hence applies more generally.

Our approach to proving Theorem 1.6 is to restrict the form of a smallest counterexample. The restrictions we prove are valid for general  $(k, d)$ . For example, in a smallest counterexample the only nonempty proper vertex subsets with potential at most  $k(k + 1)$  consist of single vertices with capacity 0 (Lemma 2.9). Furthermore, when  $A$  is a proper vertex subset with  $\rho(A) \leq k(k + 1) + d$  and  $|A| \geq 2$ , every vertex of  $A$  having a neighbor outside  $A$  must have capacity 0 (Lemma 3.2).

We then use discharging, restricting the argument to  $d \geq k$ . The initial charge of each vertex or edge is its potential. Hence the total charge is at least  $k^2$ , but after vertices give their charge to incident edges, all vertices and edges in an instance satisfying the reductions have nonpositive charge, if additional constraints on vertices of capacity 0 or  $d$  hold. Those constraints hold automatically when  $k = 1$ , and additional lemmas in Section 4 show that they also hold when  $k = 2$  if  $d > 1$ .

## 2 General Reduction Lemmas

Given fixed  $(k, d)$ , an *instance* of our problem is a pair  $(G, f)$  such that  $G$  has no overfull set and  $f$  is feasible on  $G$ . We speak of  $\rho$  in Definition 1.3 as the potential function for the pair  $(G, f)$ . We need to place the instances  $(G, f)$  in order (actually a partial order).

**Definition 2.1.** An instance  $(G', f')$  is *smaller than* an instance  $(G, f)$  if (1)  $|E(G')| < |E(G)|$ , or if (2)  $G' = G$  and  $\sum_v f'(v) > \sum_v f(v)$ , or if (3)  $|E(G')| = |E(G)|$  and  $|V(G')| < |V(G)|$ .

A counterexample is an instance  $(G, f)$  such that  $G$  has no  $(k, f)$ -decomposition. Throughout our discussion,  $(G, f)$  is assumed to be a *smallest counterexample* in the sense of Definition 2.1. Since isolated vertices can be added to any forest, the presence of isolated vertices is irrelevant for our decomposition problems, so the definition of “smaller than” allows us to assume that  $G$  has no isolated vertices. To further restrict the form of  $(G, f)$ , we construct a smaller instance  $(G', f')$  and use the guaranteed  $(k, f')$ -decomposition of  $G'$  to obtain a  $(k, f)$ -decomposition of  $G$ . Showing that  $(G', f')$  is a smaller instance includes showing that it has no overfull set and that its potential function  $\rho'$  is feasible.

Given an instance  $(G, f)$ , we write  $(G[A], f)$  for the instance where  $f$  is restricted to  $A$ .

**Lemma 2.2.** *If  $A$  is a proper subset of  $V(G)$ , then  $G[A]$  is  $(k, f)$ -decomposable.*

*Proof.* Vertex deletion does not change edge count or the potential of vertex sets that remain. Hence  $(G[A], f)$  is a smaller instance and is  $(k, f)$ -decomposable.  $\square$

When  $S$  and  $T$  are disjoint vertex subsets of a graph, let  $[S, T]$  denote the set of edges with endpoints in both  $S$  and  $T$ . When  $S \cup T = V(G)$ , the set  $[S, T]$  is an *edge cut* of  $G$ .

**Lemma 2.3.**  *$G$  is  $(k + 1)$ -edge-connected (and hence has minimum degree at least  $k + 1$ ).*

*Proof.* Let  $[S, T]$  be an edge cut of  $G$ . By Lemma 2.2,  $G[S]$  and  $G[T]$  have  $(k, f)$ -decompositions  $(F_S, D_S)$  and  $(F_T, D_T)$ . If  $|[S, T]| \leq k$ , then these can be combined by adding one edge of  $[S, T]$  to the union of the  $i$ th forests in  $F_S$  and  $F_T$ , for  $1 \leq i \leq |[S, T]|$ .  $\square$

Let  $N_G(v)$  denote the neighborhood of  $v$  in  $G$ . Let  $V_i = \{v \in V(G) : f(v) = i\}$ .

**Lemma 2.4.** *If  $f(v) < d$ , then  $d_G(v) \geq k + 1 + f(v)$ . If  $f(v) > 0$  and  $N_G(v) \subseteq V_0$ , then  $f(v) = d$ .*

*Proof.* If either statement fails, then  $f(v) < d$ . Raising the capacity of a vertex with positive capacity does not change the potential of edges incident to it, though it raises the potential of the vertex. Hence it does not lower the potential of any vertex subset, and it changes no edges. We conclude that  $(G, f')$  is an instance, where  $f'(v) = f(v) + 1$  and  $f'(u) = f(u)$  for  $u \in V(G) - \{v\}$ . By criterion (2),  $(G, f')$  is smaller than  $(G, f)$  and hence has a  $(k, f')$ -decomposition  $(F', D')$ .

Note that  $(F', D')$  is a  $(k, f)$ -decomposition of  $G$  unless  $d_{D'}(v) = f(v) + 1$ . If also  $d_G(v) \leq k + f(v)$ , then  $v$  is isolated in some forest in the decomposition of  $F'$ . Moving one edge of  $D'$  incident to  $v$  into that forest yields a  $(k, f)$ -decomposition of  $G$ .

If  $N_G(v) \subseteq V_0$ , then  $v$  is isolated in  $D'$  and  $(F', D')$  is a  $(k, f)$ -decomposition of  $G$ .  $\square$

A vertex subset  $A$  is *nontrivial* if  $2 \leq |A| \leq |V(G)| - 1$ . In most applications of the next lemma we take  $f'(z) = 0$  in the statement, in which case the resulting capacity function  $f^*$  is just the restriction of  $f$  to  $A$ .

**Lemma 2.5.** *Let  $A$  be a nontrivial vertex set in a graph  $H$ , with  $H'$  obtained by contracting  $A$  to a vertex  $z$ . Let  $f'(v) = f(v)$  for  $v \in V(H') - \{z\}$ , with  $f'(z)$  arbitrary. Let  $(F', D')$  be a  $(k, f')$ -decomposition of  $H'$ . Let  $d'(x)$  for  $x \in A$  denote the number of edges incident to  $x$  that become edges of  $D'$  incident to  $z$ . If also  $H[A]$  has a  $(k, f^*)$ -decomposition  $(F^*, D^*)$ , where  $f^*(x) = f(x) - d'(x)$  for  $x \in A$ , then  $H$  is  $(k, f)$ -decomposable.*

*Proof.* Viewing edges at  $z$  in  $H'$  as the corresponding edges in  $H$ , define  $(F, D)$  by letting  $F$  consist of  $k$  subgraphs, where the  $i$ th subgraph is the union of the  $i$ th subgraphs in the decompositions of  $F'$  and  $F^*$  into forests. Similarly let  $D = D' \cup D^*$ .

The  $k$  subgraphs in the decomposition of  $F$  are forests, as is  $D$ , because any cycle would contract to a cycle in the corresponding forest in  $F'$  or  $D'$ . Also,  $d_D(v) \leq f(v)$  for all  $v$ , because for  $v \in A$  the number of edges incident to  $v$  in  $D^*$  is at most  $f(v) - d'(v)$ .  $\square$

When counting the edges induced by a vertex set  $A$  in a graph  $H$  other than  $G$ , we use  $\|A\|_H$  to avoid confusion. We say that a vertex set  $A$  is *full* if  $\|A\|_H \geq (k+1)(|A| - 1)$  (nearly overfull). Our next lemma shows that full sets cannot have vertices with capacity 0.

**Lemma 2.6.** *Let  $f$  be a feasible capacity function on some graph  $H$ , and consider  $A \subseteq V(H)$  with  $|A| \geq 2$ . Let  $A_0 = \{v \in A : f(v) = 0\}$ . If  $\|A\|_H \geq (k+1)(|A| - 1)$ , then  $A_0 = \emptyset$ .*

*Proof.* If  $|A_0| \geq 1$ , then  $\rho(A_0) \geq k^2$  is equivalent to  $\|A_0\|_H \leq k(|A_0| - 1) + \frac{k}{k+1}$ . Since  $\|A_0\|_H$  and  $|A_0|$  are integers, we can drop the last term. We then compute

$$\begin{aligned} \rho(A) &\leq (k+1)(k+d)|A| - (k+1+d)\|A\|_H - (k+1)d|A_0| + d\|A_0\|_H \\ &\leq -(k+1)|A| + (k+1+d)(k+1) - d|A_0| - dk \leq k^2 - 1, \end{aligned}$$

where the last inequality uses  $|A| \geq 2$ . This contradicts the feasibility of  $f$ .  $\square$

**Definition 2.7.** For a nontrivial vertex set  $A \subseteq V(G)$ , the *A-contraction* of an instance  $(G, f)$  is the pair  $(G', f')$ , where  $G'$  is obtained from  $G$  by shrinking  $A$  to a single vertex  $z$ , and  $f'$  is defined on  $G'$  by  $f'(z) = 0$  and  $f'(v) = f(v)$  for  $v \in V(G') - \{z\}$ . Edges of  $G$  induced by  $A$  disappear in  $G'$ , and an edge joining  $x \in A$  and  $y \notin A$  in  $G$  becomes an edge joining  $z$  and  $y$  in  $G'$ .

**Lemma 2.8.** *Let  $(G', f')$  be the A-contraction of an instance  $(G, f)$ , where  $A$  is a nontrivial subset of  $V(G)$ . If  $f'$  is feasible, then  $(G, f)$  cannot be a smallest counterexample.*

*Proof.* If  $f'$  is feasible, then  $G'$  has no overfull set containing a vertex of capacity 0, by Lemma 2.6. Hence  $G'$  has no overfull set containing  $z$ . Also  $G'$  has no overfull set not containing  $z$ , since  $G$  has no overfull set. Hence  $(G', f')$  is a smaller instance than  $(G, f)$ . If  $(G, f)$  is a smallest counterexample, then  $G'$  has a  $(k, f')$ -decomposition  $(F', D')$ .

Since  $f'(z) = 0$ , the capacity function  $f^*$  defined on  $G[A]$  for the application of Lemma 2.5 is just the restriction of  $f$  to  $A$ . Hence  $G[A]$  is  $(k, f)$ -decomposable, since  $(G, f)$  is a smallest counterexample. By Lemma 2.5, we conclude that  $G$  is  $(k, f)$ -decomposable and is not a counterexample.  $\square$

Proving feasibility for the potential function  $\rho'$  of the smaller instance  $(G', f')$  means proving  $\rho'(A') \geq k^2$  for  $A' \subseteq V(G')$ . We do not need to comment on subsets  $A'$  such that  $G'[A'] = G[A']$  and  $f'(v) = f(v)$  for all  $v \in A'$ .

**Lemma 2.9.** *If  $\emptyset \neq A \subset V(G)$ , then  $\rho(A) > k(k+1)$  unless  $A$  consists of a single vertex with capacity 0.*

*Proof.* Suppose  $\rho(A) \leq k(k+1)$ . If  $A$  is not a single vertex with capacity 0, then  $|A| > 1$ . Let  $(G', f')$  be the  $A$ -contraction of  $(G, f)$ , and recall that  $V_i = \{v \in V(G) : f(v) = i\}$ .

To prove that  $f'$  is feasible, consider  $A' \subseteq V(G')$  with  $z \in A'$ . Let  $A^* = (A' - \{z\}) \cup A$ . Let  $E' = [A - V_0, A' - \{z\} \cap V_0]$ . In moving from  $G$  to  $G'$ , the potential of each edge in  $E'$  changes from  $-(k+1+d)$  to  $-(k+1)$ ; other edges keep the same potential. Thus

$$\rho'(A') = \rho(A^*) - \rho(A) + \rho(\{z\}) + d|E'| \geq \rho(A^*) \geq k^2$$

under the assumption  $\rho(A) \leq k(k+1)$ , since  $\rho(\{z\}) = (k+1)k$ .

Since  $f'$  is feasible, Lemma 2.8 applies, and  $G$  is  $(k, f)$ -decomposable.  $\square$

Recall that a set  $A$  is *full* when  $\|A\| \geq (k+1)(|A|-1)$ . A full set of size 2 is an edge with multiplicity at least  $k+1$ . The exclusion of vertices with capacity 0 from full sets (Lemma 2.6) helps us to exclude all full sets when  $(G, f)$  is a smallest counterexample.

**Lemma 2.10.**  *$G$  has no full set  $A$  with  $|A| \geq 2$ .*

*Proof.* Since  $G$  has no overfull set, we may assume  $\|A\| = (k+1)(|A|-1)$ . By Lemma 2.6,  $A \cap V_0 = \emptyset$ . Hence all edges in  $G[A]$  have potential  $-(k+1+d)$ . We compute

$$\rho(A) \leq (k+1)(k+d)|A| - (k+1+d)(k+1)(|A|-1) = (k+1)(k+1+d-|A|). \quad (*)$$

If  $A = V(G)$ , then let  $i = \min\{f(v) : v \in V(G)\}$ . Having a vertex with capacity  $i$  reduces the bound in  $(*)$  by  $(k+1)(d-i)$  to  $\rho(V(G)) \leq (k+1)(k+1+i-|V(G)|)$ . Now  $k^2 \leq \rho(V(G)) \leq k^2 - 1 + (k+1)(2+i-|V(G)|)$  yields  $1 \leq (k+1)(2+i-|V(G)|)$  and hence  $i > |V(G)| - 2$ . This requires  $d \geq |V(G)| - 1$ , so the degrees of vertices in the last forest are not restricted, and having no overfull sets ensures that  $G$  decomposes into  $k+1$  forests.

Hence we may assume  $A \neq V(G)$ . Continuing the computation in (\*) and using  $|A| \geq 2$ ,

$$\rho(A) \leq (k+1)(k+1+d-|A|) \leq (k+1)(k-1+d) = k^2 + d(k+1) - 1.$$

Now let  $\ell = \rho(A) - k^2$  and  $m = \lfloor \frac{\ell}{k+1} \rfloor$ ; the bound on  $\rho(A)$  yields  $m < d$ . If  $f(x) \leq m$  for some  $x \in A$ , then adjusting the potential for this vertex (again using  $|A| \geq 2$ ) yields

$$\rho(A) \leq (k+1)(k+1+d-|A|) - (k+1)(d-m) \leq (k+1)(k-1+m) \leq k^2 - 1 + \ell.$$

This contradicts  $\rho(A) = k^2 + \ell$ , and hence  $f(x) > m$  for all  $x \in A$ .

Form  $G'$  by contracting  $A$  to a new vertex  $z$ . Let  $f'(z) = m$  and  $f'(v) = f(v)$  for  $v \in V(G) - A$ . For any set  $A'$  with  $z \in A' \subseteq V(G')$ , replacing  $z$  with  $A$  adds  $|A| - 1$  vertices and  $(k+1)(|A| - 1)$  edges. Hence  $A'$  is overfull in  $G'$  if and only if  $A \cup A'$  is overfull in  $G$ . Since  $G$  has no overfull set,  $G'$  has no overfull set.

For  $z \in A' \subseteq V(G')$ , let  $A^* = (A' - \{z\}) \cup A$ . Since  $A \cap V_0 = \emptyset$ , the potential of any edge incident to  $z$  in  $G'$  is at least as large as that of the corresponding edge in  $G$ . Using this and  $m \geq \frac{\ell}{k+1} - 1$ , we compute

$$\begin{aligned} \rho'(A') - \rho(A^*) &\geq \rho'(z) - \rho(A) = (k+1)(k+m) - k^2 - \ell \\ &\geq k + (k+1) \left( \frac{\ell}{k+1} - 1 \right) - \ell = -1 \end{aligned}$$

However, Lemma 2.9 yields  $\rho(A^*) > k(k+1)$ , so  $\rho'(A') \geq k(k+1)$ . Hence  $(G', f')$  is feasible.

Since  $(G', f')$  is smaller than  $(G, F)$ , there is a  $(k, f')$ -decomposition  $(F', D')$  of  $G'$ , with  $d_{D'}(z) \leq m$ . Let  $f^*(x) = f(x) - g(x)$  for  $x \in A$ , where  $g(x)$  is the number of edges joining  $x$  to  $V(G) - A$  that become edges of  $D'$  when  $A$  is contracted (an edge in  $D'$  may have several choices for which vertex it is assigned to).

Since  $f'(z) = m$  and  $f(x) > m$  for  $x \in A$ , we have  $f^*(x) > 0$  for  $x \in A$ , so  $f^*$  is a capacity function on  $G[A]$ . Since  $G$  has no overfull sets, subsets of  $A$  are not overfull, and hence  $\|A\| - \|X\| \geq (k+1)(|A| - |X|)$  for  $X \subseteq A$ . The potential of  $X$  is smallest in comparison to that of  $A$  when all vertices of  $A - X$  have capacity  $d$  and all edges of  $D'$  incident to  $z$  (there are at most  $m$  of them) arise from edges incident to  $X$ . Hence

$$\begin{aligned} \rho^*(X) &\geq \rho(A) - m(k+1) - (k+1)(k+d)(|A| - |X|) + (k+1+d)(\|A\| - \|X\|) \\ &\geq \rho(A) - m(k+1) - (k+1)(k+d)(|A| - |X|) + (k+1+d)(k+1)(|A| - |X|) \\ &= \rho(A) - m(k+1) + (k+1)(|A| - |X|) \geq \rho(A) - \ell = k^2, \end{aligned}$$

using  $|A| \geq |X|$  and the definition of  $m$  in the last step.

Hence  $(G[A], f^*)$  is a smaller instance, and  $G[A]$  is  $(k, f^*)$ -decomposable. By Lemma 2.5,  $G$  is  $(k, f)$ -decomposable.  $\square$



Lemma 2.10 forbids edges with multiplicity  $k + 1$ . Within the set  $V_0$ , we can reduce the multiplicity further. In particular, when  $k = 1$  a minimal counterexample must be a simple graph in which  $V_0$  is an independent set. One can also prove that no vertex of  $V_0$  has an incident edge of multiplicity  $k$ , but we will not need that.

**Lemma 2.11.** *No two vertices of  $V_0$  are joined by  $k$  edges.*

*Proof.* If  $x$  and  $y$  are vertices of capacity 0 joined by  $k$  edges, then  $\rho(\{x, y\}) = 2k(k + 1) - k(k + 1) = k(k + 1)$ , contradicting Lemma 2.9.  $\square$

**Lemma 2.12.** *For  $x \in V(G)$  with  $f(x) > 0$ , every proper induced subgraph of  $G$  containing  $x$  has a  $(k, f)$ -decomposition  $(F, D)$  such that  $d_D(x) < f(x)$ .*

*Proof.* Consider  $A$  with  $x \in A \subset V(G)$ . If  $f(x) > 0$ , then define  $f'$  by  $f'(x) = f(x) - 1$  and  $f'(v) = f(v)$  for  $v \in A - \{x\}$ . For  $x \in A' \subseteq A$  with  $|A'| \geq 2$ , we have  $\rho'(A') \geq \rho(A') - (k + 1) \geq k^2$ , by Lemma 2.9 (if  $f(x) = 1$ , then the inequality may be strict). Also  $A$  has no overfull subset. Since  $(G[A], f')$  is smaller than  $(G, f)$ , it has a  $(k, f')$ -decomposition, which is a  $(k, f)$ -decomposition such that  $d_D(x) < f(x)$ .  $\square$

**Lemma 2.13.** *If  $f(u) > 0$  and  $d_G(u) = k + 1$ , then  $N_G(u) \subseteq V_0$ .*

*Proof.* If there exists  $x \in N_G(u)$  with  $f(x) > 0$ , then by Lemma 2.12  $G - u$  has a  $(k, f)$ -decomposition  $(F, D)$  such that  $d_D(x) < f(x)$ . Add one edge with endpoints  $\{x, y\}$  to  $D$  and the other edges at  $x$  to distinct forests in  $F$  to complete a  $(k, f)$ -decomposition of  $G$ .  $\square$

Recall that  $d_G(x) \geq k + 1 + f(x)$  when  $0 < f(x) < d$  (Lemma 2.4). When  $f(x) = d$ , our lower bound on degree is weaker.

**Lemma 2.14.** *If  $x \in V(G)$  and  $|N(x) \cap V_0| \geq 2$ , then  $d_G(x) > k + 1$ .*

*Proof.* By Lemma 2.3,  $d_G(x) \geq k + 1$ ; consider equality. Since  $G$  has no nontrivial full set, adding an edge joining vertices  $y, z \in N_G(x) \cap V_0$  to form  $G'$  from  $G - x$  creates no overfull set. The added edge decreases the potential of any set  $A$  containing  $\{y, z\}$  by  $k + 1$ . By Lemma 2.9,  $\rho(A) > k(k + 1)$  if  $\{y, z\} \subseteq A \subseteq V(G) - \{x\}$ , and therefore  $\rho'(A) \geq k^2$ .

Hence  $(G', f)$  is an instance smaller than  $(G, f)$ , and  $G'$  has a  $(k, f)$ -decomposition  $(F', D')$ . Since  $y, z \in V_0$ , the added edge  $yz$  lies in  $F'$ . Replace it in its forest with the  $y, z$ -path of length 2 through  $x$ . The remaining  $k - 1$  edges at  $x$  can be added to the  $k - 1$  other forests in  $F'$ .  $\square$

**Lemma 2.15.** *If  $x \in V(G)$ , then  $d_G(x) \geq k + 2$  unless  $f(x) = 0$  and  $|N_G(x) \cap V_0| \leq 1$ .*

*Proof.* If  $|N_G(x) \cap V_0| \geq 2$ , then Lemma 2.14 applies. When  $f(x) = 0$  there is nothing further to show. If  $f(x) > 0$ , then Lemma 2.3 yields  $d_G(x) \geq k + 1$ , Lemma 2.13 yields  $N_G(x) \subseteq V_0$  when  $d_G(x) = k + 1$ , and Lemma 2.10 prevents all  $k + 1$  incident edges from going to a single neighbor; hence  $|N_G(x) \cap V_0| \geq 2$ .  $\square$

### 3 Final reductions

Our final reductions restrict the edges leaving  $A$  when  $A$  has small potential. For  $A \subseteq V(G)$ , let the *boundary*  $\partial A$  denote the set of vertices in  $A$  having a neighbor outside  $A$ .

**Lemma 3.1.** *If  $A$  is a nontrivial subset of  $V(G)$  such that  $\rho(A) \leq k(k+1) + d$ , then no edge joining  $\partial A$  to  $V(G) - A$  has positive capacity at both endpoints.*

*Proof.* Suppose otherwise, and choose  $A$  among the counterexamples with smallest potential. Let  $xy$  be an edge with  $x \in A$ ,  $y \notin A$ , and  $f(x), f(y) > 0$ . Form  $G'$  from  $G$  by deleting this edge and then contracting  $A$  to a single vertex  $z$ . Define  $f'$  on  $G'$  by  $f'(z) = 0$  and  $f'(y) = f(y) - 1$  and  $f'(v) = f(v)$  for  $v \in V(G) - A - \{y\}$ .

We first prove that  $f'$  is feasible. Consider  $A' \subseteq V(G')$ . If  $y, z \notin A'$ , then  $\rho'(A') = \rho(A')$ . If  $A'$  contains  $y$  and not  $z$ , then  $\rho'(A') = \rho(A') - (k+1) \geq k^2$ , by Lemma 2.9. When  $z \in A'$ , let  $A^* = (A' - z) \cup A$ . If  $y, z \in A'$ , then since  $f'(y) = f(y) - 1$  and one copy of  $xy$  in  $G$  is missing from  $G'$ , we obtain  $\rho'(A') \geq \rho(A^*) \geq k^2$  from

$$\begin{aligned} \rho'(A') - \rho(A^*) &\geq -\rho(A) + \rho'(z) - (k+1) - \rho(xy) \\ &\geq -[k(k+1) + d] + k(k+1) - (k+1) + (k+1+d) = 0. \end{aligned}$$

If  $A'$  contains  $z$  and not  $y$ , then  $\rho(A^*) \leq \rho(A) + \rho'(A') - \rho(z)$ . If  $\rho'(A') < k^2$ , then  $A' \neq \{z\}$  and  $\rho(A^*) < \rho(A) - k$ . Since the edge  $xy$  joining  $\partial A^*$  to  $V(G) - A^*$  has positive capacity at both endpoints, and  $A^*$  is a nontrivial set, this contradicts our choice of  $A$  as a counterexample with smallest potential. We conclude  $\rho'(A') \geq k^2$ .

Hence  $f'$  is feasible. Since  $G$  has no overfull set, an overfull set must contain  $z$ . However, since  $f'$  is feasible on  $G'$  and  $f'(z) = 0$ , Lemma 2.6 implies that no overfull set in  $G'$  contains  $z$ . Hence  $G'$  contains no overfull set.

Since  $(G', f')$  is smaller than  $(G, f)$ , we now have a  $(k, f')$ -decomposition of  $G'$ . By Lemma 2.12,  $G[A]$  has a  $(k, f)$ -decomposition  $(F^*, D^*)$  such that  $d_{D^*}(x) < f(x)$ . By Lemma 2.5, the two decompositions combine to form a  $(k, f)$ -decomposition  $(F, D)$  of  $G - xy$  that becomes a  $(k, f)$ -decomposition of  $G$  by adding a copy of the edge  $xy$  to  $D$ .  $\square$

**Lemma 3.2.** *If  $A$  is a nontrivial subset of  $V(G)$  such that  $\rho(A) \leq k(k+1) + d$ , then  $\partial A \subseteq V_0$ .*

*Proof.* If not, then let  $A$  be a largest nontrivial subset with  $\rho(A) \leq k(k+1) + d$  and  $\partial A \not\subseteq V_0$ . Choose  $x \in \partial A$  with  $f(x) > 0$ , and choose  $y \in N_G(x) - A$ . By Lemma 3.1,  $f(y) = 0$ .

Let  $(G', f')$  be the  $A$ -contraction of  $(G, f)$ . If  $f'$  is feasible, then  $G$  is  $(k, f)$ -decomposable, by Lemma 2.8. Consider  $A' \subseteq V(G')$  with  $z \in A'$ , and let  $A^* = (A' - \{z\}) \cup A$ .

If  $\rho'(A') < k^2$ , then

$$\rho(A^*) \leq \rho'(A') - \rho'(z) + \rho(A) \leq \rho(A) - k < \rho(A).$$

This contradicts the choice of  $A$  if  $A^*$  is nontrivial and  $\partial A^* \not\subseteq V_0$ . Hence we may assume  $A^* = V(G)$  or  $\partial A^* \subseteq V_0$ . In either case, since  $x \in A$  and  $f(x) > 0$ , we must have  $y \in A'$ . Now  $\rho(xy) = -(k+1+d)$ , but  $\rho'(zy) = -(k+1)$ , so  $\rho'(zy) - \rho(xy) = d$ . Hence

$$\rho'(A') \geq \rho(A^*) - \rho(A) + \rho'(z) + d \geq \rho(A^*),$$

by the hypothesis  $\rho(A) \leq k(k+1) + d$ . We conclude that  $f'$  is feasible, as desired.  $\square$

We now give a discharging argument to show what remains to be excluded when  $d \geq k$ .

**Theorem 3.3.** *Let  $(G, f)$  be a smallest counterexample, and for  $v \in V(G)$  let  $h(v)$  be the number of edges joining  $v$  to vertices of  $V_0$ . If  $d \geq k$ , then some  $v \in V(G)$  satisfies*

(1)  $f(v) = 0$  with  $h(v) > 2(d_G(v) - k - 1)\frac{k}{k-1}$  and  $k > 1$ , or

(2)  $f(v) = d$  with  $h(v) < \frac{(2k+2-d_G(v))(k+1+d)-2(k+1)}{d+1-k}$ .

In particular, (1) requires  $d_G(v) < 2k$ , and (2) requires  $d_G(v) < 2k + 2$ .

*Proof.* A smallest counterexample has all the properties derived in the prior lemmas, which impose no restriction on  $(k, d)$ . For  $d \geq k$ , we use discharging to show that the total potential, which must be at least  $k^2$ , is nonpositive when vertices as specified above are also forbidden, thereby prohibiting counterexamples.

Give each vertex and edge initial charge equal to its potential, yielding total charge at least  $k^2$ . The edges now take charge from their endpoints by the following rules:

Rule 1: Every edge  $xy$  with  $f(x) = 0$  and  $f(y) > 0$  takes  $k$  from  $x$  and  $d+1$  from  $y$ .

Rule 2: Every edge joining vertices not in  $V_0$  takes  $(k+1+d)/2$  from each endpoint.

Rule 3: Every edge joining vertices in  $V_0$  takes  $(k+1)/2$  from each endpoint.

By construction, edges end with charge 0. It suffices to show that all vertices also end with nonpositive charge. Consider  $v \in V(G)$ . By Lemma 2.3,  $d_G(v) \geq k+1$ .

If  $f(v) = 0$ , then  $v$  loses charge  $h(v)\frac{k+1}{2} + (d_G(v) - h(v))k$ . This is at least  $k(k+1)$  if and only if  $h(v)\frac{k-1}{2} \leq (d_G(v) - k - 1)k$ , which always holds when  $k = 1$ . Hence retaining positive charge at a vertex  $v$  of capacity 0 requires (1).

If  $1 \leq f(v) < d$ , then each vertex with positive capacity loses at least  $\frac{k+1+d}{2}$  to each incident edge, by Rule 1 or Rule 2, since  $d+1 \geq \frac{k+1+d}{2}$ . Also  $d_G(v) \geq k+1+f(v)$ , by Lemma 2.4. Hence  $v$  loses charge at least  $\frac{k+1+d}{2}(k+1+f(v))$ . If  $d \geq k+1$ , then  $\frac{k+1+d}{2} \geq k+1$ , and the lower bound  $(k+1)(k+1+f(v))$  on the lost charge exceeds the initial charge  $(k+1)(k+f(v))$ . The remaining case is  $k = d > f(v)$ , where

$$(k+1)(k+f(v)) - \frac{k+1+d}{2}(k+1+f(v)) = \frac{1}{2}(k+f(v)) - k - \frac{1}{2} < 0.$$

Finally, consider  $f(v) = d$ . The charge lost by  $v$  is  $h(v)(d+1) + (d_G(v) - h(v))\frac{k+1+d}{2}$ , which must be at least  $(k+1)(k+d)$  for  $v$  to reach nonpositive charge. The inequality simplifies to  $h(v) \geq \frac{(2k+2-d_G(v))(k+1+d)-2(k+1)}{d+1-k}$ . Hence retaining positive charge at a vertex  $v$  of capacity  $d$  requires (2).  $\square$

For  $k = 2$ , the case not covered is  $(k, d) = (2, 1)$ , where there is no vertex  $v$  with  $1 \leq f(v) < d$ . The computation for failing to reach nonpositive charge when  $f(v) = d = 1$  then reduces to  $d_G(v) \leq 9/2$ . In this situation with  $d_G(v) \leq \{3, 4\}$ , our discharging rules cannot produce nonpositive charge for any choice of  $h(v)$ .

For  $d \geq k$ , Theorem 3.3 implies that vertices with large degree cause no trouble.

**Corollary 3.4.** *Let  $v$  be a vertex remaining with positive capacity in the setting of Theorem 3.3. If  $f(v) = 0$ , then  $d_G(v) < 2k$ . If  $f(v) = d$ , then  $d_G(v) < 2k + 2$ . If  $f(v) = d$  and  $d_G(v) = 2k + 1$ , then  $h(v) = 0$  and  $d \leq k + 1$ .*

*Proof.* In all other cases, the conditions (1) and (2) in Theorem 3.3 cannot be satisfied.  $\square$

## 4 The Case $k \leq 2$

Our remaining task for  $d \geq k$  with  $k \leq 2$  is to prohibit the exceptions in Corollary 3.3 when  $(G, f)$  is a smallest counterexample. The first lemma holds whenever  $k \leq 2$  and enables us to complete the proof for  $k = 1$ .

**Lemma 4.1.** *For  $k \leq 2$ , let  $(G, f)$  be a smallest counterexample. If  $d_G(x) = k + 2$  with  $f(x) \geq 2$ , then  $N_G(x) \subseteq V_0$ .*

*Proof.* If the conclusion fails, then  $x$  has a neighbor  $y$  with  $f(y) > 0$ . Since  $G$  has no overfull set,  $x$  has at least two neighbors; choose  $u \in N_G(x) - \{y\}$ . Let  $u'$  be a third vertex of  $N_G(x)$ , if possible; otherwise, let  $u' = y$ . Form  $G'$  from  $G - x$  by adding one copy of the edge  $uu'$ . Define  $f'$  by  $f'(y) = f(y) - 1$  and  $f'(v) = f(v)$  for  $v \in V(G) - \{x, y\}$ . By Lemma 2.10,  $G$  has no full set, and hence  $G'$  has no overfull set.

To show that  $f'$  is feasible, we need  $\rho'(A') \geq k^2$  for  $A' \subseteq V(G')$ . By Lemma 2.9,  $\rho(A') > k(k + 1)$  if  $|A'| \geq 2$ . Even with  $\rho'(y) = \rho(y) - (k + 1)$ , we thus have  $\rho'(A') \geq k^2$  unless  $u, u' \in A'$ . If  $y \notin A'$ , then  $\rho'(A') = \rho(A') + \rho'(uu')$ . Again we have  $\rho'(A') \geq k^2$  unless  $\rho'(uu') = -(k + 1 + d)$  and  $\rho(A') \leq k(k + 1) + d$ . Since  $u, u' \in N_G(x)$ , we have  $u, u' \in \partial A'$ . Lemma 3.2 then requires  $u, u' \in V_0$ , so  $\rho'(uu') = -(k + 1)$  and again  $\rho'(A) \geq k^2$ .

Hence  $\rho'(A') < k^2$  requires  $y, u, u' \in A'$ . Let  $r$  be the number of edges joining  $x$  to  $A'$ ; each has potential  $-(k + 1 + d)$ , since  $f(x) > 0$ . Since  $\rho(y) - \rho'(y) = k + 1$ , we have

$$\begin{aligned} \rho'(A') &\geq \rho(A' \cup \{x\}) - \rho(x) - (k + 1) + \rho'(uu') + r(k + 1 + d) \\ &\geq \rho(A' \cup \{x\}) - (k + 1)(k + d + 1) + (r - 1)(k + 1 + d). \end{aligned}$$

It thus suffices to show  $r \geq k + 1$  when  $\rho(A' \cup \{x\}) > k(k + 1) + d$  and  $r = k + 2$  otherwise. If  $\rho(A' \cup \{x\}) \leq k(k + 1) + d$  and  $A' \neq V(G)$ , then  $\partial(A' \cup \{x\}) \subseteq V_0$ , by Lemma 3.2. Since  $f(x) > 0$ , this requires  $N_G(x) \subseteq A'$ , which also holds if  $A' \cup \{x\} = V(G)$ . Hence  $r = k + 2$ .

When  $\rho(A' \cup \{x\}) > k(k+1) + d$ , we only need  $r \geq k+1$ . Here we use  $k \leq 2$ . If  $y \neq u'$ , then  $x$  has three neighbors in  $A'$ . If  $y = u'$ , then  $N_G(x) = \{u, y\} \subseteq A'$ , and  $r = k+2$ .

Hence  $(G', f')$  is a smaller instance, and  $G'$  has a  $(k, f')$ -decomposition  $(F', D')$ . Since  $f(x) \geq 2$ , the added edge  $uu'$  can be replaced in its forest by a  $u, u'$ -path  $P$  of length 2 through  $x$ . Adding the remaining  $k$  edges at  $x$  to the other forests will yield a  $(k, f)$ -decomposition of  $G$ . When  $P$  is added to a forest other than  $D'$ , we must add one of the remaining edges to  $D'$ . This edge can be  $xy$  if  $y \neq u'$ , since  $f'(y) < f(y)$  and  $f(x) > 0$ . If  $y = u'$ , then  $N_G(x) = \{y, u\}$ ; since there is no full set of size 2 (Lemma 2.10),  $G$  has two copies of the edge  $xy$ . Hence also in this case a copy of  $xy$  is not absorbed by  $P$  and can be added to  $D'$ .  $\square$

**Corollary 4.2.** *The NDT Conjecture is true when  $k = 1$ .*

*Proof.* When  $k = 1$ , Lemma 2.3 implies that every vertex has degree at least 2. Hence by Corollary 3.4, exceptions with capacity 0 cannot occur. For an exception  $v$  with capacity  $d$ , by Corollary 3.4 we have  $d_G(v) \leq 3$ , and by Lemma 2.15 we have  $d_G(v) \geq 3$ . Hence  $d_G(v) = 3$ , and then by Corollary 3.4 it suffices to have one neighbor with capacity 0. In fact, Lemma 4.1 yields  $h(x) = d_G(x) = 3$  when  $d \geq 2$ . In the case  $(k, d) = (1, 1)$  with  $d_G(v) = 3$ , condition (2) in Theorem 3.3 actually requires  $h(v) < -1$ .  $\square$

When  $k = 2$ , again not many possibilities remain for a vertex  $v$  whose charge in Theorem 3.3 does not become nonpositive. First suppose  $f(v) = 0$ . By Lemma 2.3,  $d(v) \geq 3$ . By Corollary 3.4,  $d(v) \leq 3$ . By Theorem 3.3,  $v$  being problematic requires  $h(v) > 0$ , so it suffices to prohibit edges induced by  $V_0$ .

Next suppose  $f(v) = d$ . By Lemma 2.15,  $d_G(v) \geq 4$ . By Corollary 3.4,  $d_G(v) \leq 5$ . By Theorem 3.3, when  $d_G(v) = 4$  and  $v$  is problematic we need  $h(v) \leq 2$ , and when  $d_G(v) = 5$  we need  $h(v) = 0$ . Lemma 4.1 takes care of  $d_G(x) = 4$ , and when  $d_G(x) = 5$  we only need one neighbor in  $V_0$ .

We consider these remaining cases in two lemmas.

**Lemma 4.3.** *For  $k = 2$ , a 3-vertex in  $V_0$  has no neighbor in  $V_0$ .*

*Proof.* Suppose  $d_G(x) = 3$  with  $f(x) = 0$  and  $N_G(x) \cap V_0 \neq \emptyset$ . Since  $G$  has no full set (Lemma 2.10),  $|N_G(x)| \geq 2$ . Let  $u$  be a neighbor of  $x$  in  $V_0$ , and let  $u'$  be another neighbor of  $x$ . Form  $G'$  by adding to  $G - x$  an edge joining  $u$  and  $u'$ . Since  $G$  has no full subgraph,  $G'$  has no overfull subgraph.

Let  $f'$  be the restriction of  $f$  to  $V(G')$ . Always  $\rho'(A') = \rho(A')$  unless  $u, u' \in A'$ . If  $\rho'(A') < k^2$ , then

$$\begin{aligned} \rho(A' \cup \{x\}) &\leq \rho'(A') + \rho(x) + \rho(xu) + \rho(xu') - \rho(uu') \\ &\leq \rho'(A') + k(k+1) - (k+1) \leq 2(k-1)(k+1) \end{aligned}$$

When  $k = 2$ , we have  $2(k - 1)(k + 1) = k(k + 1)$ . This contradicts Lemma 2.9 unless  $A' \cup \{x\} = V(G)$ . In that case we add the potential of the third edge at  $x$ , obtaining  $\rho(A' \cup \{x\}) \leq (2k - 3)(k + 1)$ , again a contradiction when  $k = 2$ . Thus in all cases  $\rho'(A') \geq k^2$ .

Hence  $(G', f')$  is a smaller instance, and  $G'$  has a  $(2, f')$ -decomposition  $(F', D')$ . Since  $f(u) = 0$ , the added edge  $uu'$  lies in a forest in  $F'$ . Replace it with a path of length 2 through  $x$ , and add the third edge at  $x$  to the other forest to complete a  $(2, f)$ -decomposition of  $G$ .  $\square$

Hence it remains only to consider 5-vertices with capacity  $d$ . Although it is possible to prove that all neighbors of such a vertex  $v$  lie in  $V_0$ , by Theorem 3.3 we only need the weaker conclusion that some neighbor is in  $V_0$ . The proof in the next lemma is valid only for  $d \geq 3$ .

**Lemma 4.4.** *For  $k = 2$  and  $d \geq 3$ , if  $f(v) = d$  and  $d_G(v) = 5$ , then  $N_G(v) \cap V_0 \neq \emptyset$ .*

*Proof.* Let  $x$  be a 5-vertex with capacity  $d$ , and let  $U = N_G(x)$  and  $U' = U \cup \{x\}$ . Suppose  $N_G(x) \cap V_0 = \emptyset$ . Since  $G$  has no edge with multiplicity at least 3, we have  $|U| \geq 3$ . If equality holds, then some  $u \in U$  is the endpoint of at least two edges incident to  $x$ .

Form  $G'$  from  $G - x$  by adding a matching on  $U$  if  $|U| \geq 4$ , and adding an edge from  $u$  to each other vertex of  $U$  if  $|U| = 3$ . For each endpoint of each added edge, reserve an edge joining it to  $x$ , thereby reserving four of the five edges incident to  $x$  (if  $|U| = 3$ , then two copies of  $ux$  are reserved). Define  $f'$  on  $V(G')$  by  $f'(y) = f(y) - 1$  and  $f'(v) = f(v)$  for  $v \in V(G') - \{y\}$ , where  $y$  is the endpoint in  $U$  of the unreserved edge at  $x$ .

Since  $G$  has no full set, an overfull set  $A'$  in  $G'$  must contain the endpoints of both added edges. The set  $A' \cup \{x\}$ , which has one more vertex and induces at least  $\|A'\|_{G'} + 4$  edges, is then full in  $G$ , a contradiction. (This argument uses  $k \leq 5$ .) Hence  $G'$  has no overfull set.

Now suppose  $\rho'(A') < k^2 = 4$ . By Lemma 2.9,  $G'[A']$  must contain at least one added edge. If it contains one added edge  $e$  but not the vertex  $y$ , then  $\rho(A') \leq k(k + 1) + d$ ; the edges joining  $e$  to  $x$  then contradict Lemma 3.1.

If  $G'[A']$  contains exactly one added edge and also  $y$ , then  $A' \cup \{x\}$  induces at least three edges incident to  $x$ . Now

$$\begin{aligned} \rho(A' \cup \{x\}) &\leq \rho'(A') + \rho(x) - 3(k + 1 + d) + (k + 1 + d) + (k + 1) \\ &= \rho'(A') + (k + 1)(k - 1) + d(k - 1) \leq 6 + d = k(k + 1) + d, \end{aligned}$$

using  $k = 2$ . Again we contradict Lemma 3.1. If  $G'[A']$  contains both added edges, then moving to  $G[A' \cup \{x\}]$  loses two edges instead of one in  $A'$  but also gains four edges instead of three at  $x$ . If  $A'$  also contains  $y$ , then the bound increases by  $k + 1$  for  $f(y)$  but decreases by  $k + 1 + d$  for inducing the fifth edge at  $x$ . Hence in each case we obtain  $\rho'(A') \geq k^2$  by essentially the same contradiction.

Hence  $(G', f')$  is a smaller instance, and  $G'$  has a  $(k, f')$ -decomposition  $(F', D')$ . If the two added edges lie in distinct forests in the decomposition, then replace them by paths of

length 2 through  $x$  with the same endpoints, and add the edge  $xy$  to the third forest. This causes no problem when the third forest is  $D'$ , since  $f'(y) = f(y) - 1$ .

If the two added edges lie in the same forest, then deleting them yields at least three components in that forest, with three endpoints of the added edges in distinct components. Extend that forest by edges from  $x$  to those three (distinct) specified vertices. Add the remaining two edges at  $x$  to the other two forests. Again, if the forest containing the two specified edges is not  $D'$ , then the unreserved edge  $xy$  can be added to  $D'$ .  $\square$

The last step in this proof is not valid for  $d = 2$ , because we may be giving  $x$  three incident edges in the  $d$ -bounded forest. Fortunately, when  $d = 2$  we do not need the conclusion of Lemma 4.4; the discharging is always strong enough.

**Corollary 4.5.** *The NDT Conjecture is true when  $k = 2$  and  $d \geq 2$ .*

*Proof.* As we have remarked, Lemma 4.4 completes the proof for  $k = 2$  and  $d \geq 3$ . When  $d = k = 2$ , a vertex with capacity 2 has potential 12, and it loses charge at least  $5/2$  along every edge by the rules in Theorem 3.3. Hence a 5-vertex loses at least 12.5 and ends with negative charge. The rest of the proof remains the same as for  $d \geq 3$ .  $\square$

## Acknowledgment

We thank Hehui Wu for useful comments and the referees for careful reading and suggested improvements in exposition.

## References

- [1] J. Balogh, M. Kochol, A. Pluhár, and X. Yu, Covering planar graphs with forests. *J. Combin. Theory Ser. B* 94 (2005), 147–158.
- [2] A. Bassa, J. Burns, J. Campbell, A. Deshpande, J. Farley, M. Halsey, S.-Y. Ho, D. Kleitman, S. Michalakis, P.-O. Persson, P. Pylyavskyy, L. Rademacher, A. Riehl, M. Rios, J. Samuel, B.E. Tenner, A. Vijayasarathy, and L. Zhao, Partitioning a planar graph of girth ten into a forest and a matching, *Stud. Appl. Math.* 124 (2010), 213–228.
- [3] O.V. Borodin, A.O. Ivanova, B.S. Stechkin, Decomposing a planar graph into a forest and a subgraph of restricted maximum degree, *Sib. Elektron. Mat. Izv.*, 4 (2007), 296–299.
- [4] O.V. Borodin, A.O. Ivanova, A.V. Kostochka, and N.N. Sheikh, Planar graphs decomposable into a forest and a matching, *Discrete Math.* 309 (2009), 277–279.
- [5] O.V. Borodin, A.O. Ivanova, A.V. Kostochka and N.N. Sheikh, Decompositions of quadrangle-free planar graphs, *Discussiones Math. Graph Theory* 29 (2009), 87–99.
- [6] O.V. Borodin, A.V. Kostochka, N.N. Sheikh, and G. Yu. Decomposing a planar graph with girth 9 into a forest and a matching. *European J. Combinatorics* 29(5) (2008), 1235–1241.

- [7] D. Gonçalves. Covering planar graphs with forests, one having bounded maximum degree. *J. Combin. Theory (B)* 99 (2009), 314–322.
- [8] W. He, X. Hou, K.-W. Lih, J. Shao, W. Wang, and X. Zhu. Edge-decompositions of planar graphs and their game coloring numbers. *Journal of Graph Theory* 41 (2002), 307–317.
- [9] H. Jiang, D. Yang, and D. Zhang, Decomposing a graph into forests: the Nine Dragon Tree Conjecture is true. *Combinatorica* (accepted)
- [10] S.-J. Kim, A.V. Kostochka, H. Wu, D.B. West, and X. Zhu. Decomposition of sparse graphs into forests and a graph with bounded degree. *Journal of Graph Theory* 74 (2013), 369–391.
- [11] D. J. Kleitman, Partitioning the edges of a girth 6 planar graph into those of a forest and those of a set of disjoint paths and cycles, Manuscript, 2006.
- [12] M. Montassier, A. Pêcher, A. Raspaud, D. B. West, and X. Zhu. Decomposition of sparse graphs, with application to game coloring number. *Discrete Math.* 310 (2010), 1520–1523.
- [13] M. Montassier, P. Ossona de Mendez, A. Raspaud, and X. Zhu. Decomposing a graph into forests. *J. Combin. Theory (B)* 2012, 38–52.
- [14] C. St. J. A. Nash-Williams. Decompositions of finite graphs into forests. *J. London Math. Soc.* 39 (1964), 12.
- [15] C. Payan, Graphes equilibre et arboricité rationnelle. *Europ. J. Combin.*, 7 (1986), 263–270.
- [16] Y. Wang and Q. Zhang, Decomposing a planar graph with girth 8 into a forest and a matching, *Discrete Math.* 311 (2011), 844–849.
- [17] D. Yang, Decompose a graph into forests and a matching, submitted.