Trifold Arrangements and Cevian Dissections

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Abstract

We determine precisely how many triple points can be formed in the plane by an arrangement of n lines lying in three parallel families of p, q, r lines, respectively. Using this result we solve the Euclidean realization problem for such arrangements. We apply our results to solve an analogous problem in which a triangle is dissected by three families of cevians. We conclude by mentioning some related unsolved problems.

1 Introduction

In 1826, Steiner [12] showed by recursion that the maximum number of regions formed by n lines in the plane that fall into m parallel families $\pi_1, \pi_2, ..., \pi_m$ of $k_1, k_2, ..., k_m$ lines, respectively, is given by

$$R_{\max} = 1 + \sigma_1 + \sigma_2, \tag{1}$$

where $\sigma_1 = \sum_{1 \le i \le m} k_i$ and $\sigma_2 = \sum_{1 \le i < j \le m} k_i k_j$ are the first two elementary symmetric functions on the so-called *Steiner data* $(k_1, k_2, ..., k_m)$. Steiner's formula (1), which is correct even when some of the lines have no parallel partners, is a generalization of the familiar formula $R_{\max} = 1 + n + \binom{n}{2}$ for generic arrangements of n lines.

The more difficult problem of determining how *few* regions can be formed by such an arrangement remains unsolved. In this article we find this minimum in the simplest case of three parallel families, and we show that every integer between the maximum and the minimum can be realized as the region count of some such arrangement. Our results depend on a determination of the number of triple points that can be formed by such arrangements. We give a complete solution of the Euclidean realization problem in this case.

Finally, we solve a closely related triangle dissection problem by transforming the dissected triangle into a combinatorially equivalent arrangement of lines in three parallel families. We conclude with a few brief remarks about some related unsolved problems.

2 Arrangements of Lines

We consider the general situation briefly before specializing to the case of three parallel families. Let Λ be an arrangement of n lines in the plane with Steiner data $(k_1, k_2, ..., k_m)$. The multiplicity $\lambda(P)$ of a point P formed by Λ is the number of lines of Λ on which it lies. For each $i \geq 2$ let t_i be the number of points P formed by Λ that have multiplicity $\lambda(P) = i$. The number R of regions that are formed by the lines of Λ is completely determined by the Steiner data and the counters t_i; and we have the following simple formula, which reflects the fact that $\binom{\lambda-1}{2}$ regions are lost at each point of multiplicity λ. We give a sweep-line argument (cf. Wetzel [14]).

Lemma 1. $R = 1 + n + \sigma_2 - \sum_{i\geq 3} \binom{i-1}{2} t_i$.

Proof. Let ℓ be an auxiliary line not parallel to any of the lines of Λ and placed so that all of the points formed by Λ lie on the same side of ℓ. In this initial position ℓ meets the lines of Λ in n points, which form $1 + n$ segments and rays on ℓ each of which lies in, and so identifies, a region of Λ. When ℓ is translated across the points formed by Λ, new regions appear precisely at the points where the lines of Λ intersect, with exactly $\lambda - 1$ new regions at each point of multiplicity λ. Consequently

$$R = 1 + n + \sum_P (\lambda(P) - 1).$$

But

$$\sigma_2 = \sum_P \binom{\lambda(P)}{2}, \tag{2}$$

because each side of the equation counts the intersecting pairs of lines of Λ. Hence

$$\begin{aligned} R &= 1 + n + \sigma_2 - \sum_P \left[\binom{\lambda(P)}{2} - \lambda(P) + 1 \right] \\ &= 1 + n + \sigma_2 - \sum_P \binom{\lambda(P) - 1}{2} \\ &= 1 + n + \sigma_2 - \sum_{i\geq 3} \binom{i-1}{2} t_i. \ \square \end{aligned}$$

To minimize the region count one must find arrangements with given Steiner data for which the *total multiplicity* $\mu = \sum_{i\geq 3} \binom{i-1}{2} t_i$ is as large as possible. Very little is known about questions of this sort, mainly because they involve the realizability of arrangements of lines with prescribed Steiner and multiplicity data, about which little is known: *for given* $m \geq 3$, *positive integers* $k_1, k_2,, ..., k_m$, *and nonnegative integers* $t_2, ..., t_m$, *when does there exist an arrangement of* $n = \sum_{1 \leq i \leq m} k_i$ *lines with Steiner data* $(k_1, k_2, ..., k_m)$ *that forms exactly* t_j *points of each multiplicity* j, $2 \leq j \leq m$?[1]

3 Triple Points in a Trifold Arrangement

We turn now to the case $m = 3$, and we simplify the notation a little. Positive integers p, q, r are given. We call an arrangement Λ of n lines a ***trifold arrangement*** with *Steiner data* (p, q, r) if $n = p + q + r$ and the lines of Λ form three parallel families having p, q, r lines respectively. It will be convenient to assume throughout that $p \leq q \leq r$.

According to Lemma 1, the number of regions formed by such an arrangement is determined by the number t_3 of points of multiplicity three and is given by the formula:

$$R = 1 + n + \sigma_2 - t_3. \tag{3}$$

How many triple points can be formed by an arrangement Λ with Steiner data (p, q, r)? We answer this question first for arrangements in which the lines of the first two parallel families are perpendicular to each other and equally spaced. Then we use an argument having its roots in Katona [10] to settle the general case.

To facilitate the description we work in the coordinate plane $\mathbb{R}^2$. Let positive integers p, q be given with $p \leq q$, and let Γ be the $p \times q$ rectangular lattice array $\{(x, y) \in \mathbb{N} \times \mathbb{N} : 1 \leq x \leq p, 1 \leq y \leq q\}$. A *diagonal* ℓ, that is, a line of slope $+1$, has *weight* $|\ell|$ if it meets the array Γ in exactly $|\ell|$ points. The array Γ has $p + q - 1$ diagonals of positive weights, and their weights, in order of increasing x-intercept, are:

$$1, 2, 3, ..., p-1, \underbrace{p, p, ..., p}_{q-p+1 \text{ terms}}, p-1, ..., 3, 2, 1. \tag{4}$$

We shall need an auxiliary function $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$, defined in the following way. Set $\varphi(k) = 0$ for $k \leq 0$, and for $k > 0$ set

$$\varphi(k) \quad = \quad \underbrace{1 + 1 + 2 + 2 + 3 + 3 + ...}_{k \text{ terms}} = \left\lfloor \frac{1}{4}(k+1)^2 \right\rfloor.$$

Note that for $k > 0$, $\varphi(k)$ counts, by diagonals, the points in a $\lfloor (k+1)/2 \rfloor \times \lceil (k+1)/2 \rceil$ array.

[1] A more felicitous setting for the realizability problem and related questions is the real projective plane. See, for example, Grünbaum [8], Erdös and Purdy [6].

Theorem 2. For any positive integers p, q, r with $p \leq q \leq r$, the number t_3 of triple points formed by a trifold arrangement Λ with Steiner data (p, q, r) lies in the range $0 \leq t_3 \leq \omega(p, q, r)$, where $\omega(p, q, r) = pq - \varphi(p + q - 1 - r)$. For each integer t with $0 \leq t \leq \omega(p, q, r)$ there is such a trifold arrangement whose lines form exactly t triple points.

Proof. We begin by considering trifold arrangements formed by equally spaced horizontal and vertical lines with lines of slope $+1$. Let π_1 be the family of p parallel lines with equations $x = i$ for $1 \leq i \leq p$, and let π_2 be the family of parallel lines $y = j$ for $1 \leq j \leq q$. Let Γ be the $p \times q$ rectangular lattice array in which π_1 and π_2 intersect (Figure 1). We show first how to add r diagonals to $\pi_1 \cup \pi_2$ so as to create as many triple points as possible. Let $m = p + q - 1$, the number of diagonals of Γ that have positive weight. When $r \geq m$, we take all of those m diagonals and such additional diagonals as are needed. This creates $pq = \omega(p, q, r)$ triple points. When $r < m$, at least $m - r$ diagonals of Γ must be omitted. To generate as many triple points as possible, we select the r diagonals of greatest weight and omit the $m - r$ diagonals of least weight, whose weights are the $m - r$ smallest entries in the list (4). So at most $pq - \varphi(m - r) = \omega(p, q, r)$ triple points can be formed in this case as well.dtbpF2.4431in2.7285in0ptFigure

Next we show that for each k in the range $0 \leq k \leq \omega(p, q, r)$ it is possible to place r diagonals so as to form precisely k triple points. Since $\omega(p, q, r)$ triple points can be achieved with r diagonals, it suffices to show for $0 < k \leq \omega(p, q, r)$ that $k - 1$ triple points can be so achieved whenever k can. Let π be a parallel family of r diagonals that form k triple points, let w be the smallest positive weight among the diagonals in π, and let ℓ be a diagonal in π whose weight is w. To achieve $k - 1$ triple points we replace ℓ in π by any diagonal that is disjoint from Γ if $w = 1$ and by any diagonal of weight $w - 1$ if $w > 1$. In either case, the new family has r diagonals, and it forms $k - 1$ triple points.dtbpF4.2791in3.2396in0ptFigure

To complete the argument we show that one can do no better if the lines of the parallel families π_1 and π_2 are not evenly spaced or perpendicular. Suppose families π_1, π_2 of p, q parallel lines, respectively, are given (Figure 2), and suppose a third direction is given (pictured as horizontal in Figure 2). The pq points in which the families π_1 and π_2 meet lie in p disjoint ell-shaped layers L_j, indicated in Figures 1 and 2, and each line in the given direction meets each such layer in at most one point of intersection. When r lines in the given direction are put in place, the number of points selected from each layer L_j is at most $\min\{r, |L_j|\}$, where $|L_j|$ is the number of points in the layer L_j. Consequently the total number t of points that are transformed into triple points by the r new lines is bounded by the layer-wise sum

$$t \leq \sum_{j=1}^{p} \min\{r, |L_j|\}.$$

The conclusion now follows from the observation that this inequality becomes an equality in the construction described in the first paragraph of the proof, where the lines of π_1 and π_2 are perpendicular and evenly spaced and the r lines of the third family are optimally placed.
This completes the proof. □

The solution to the realizability problem for trifold arrangements is an immediate consequence of this theorem.

Corollary 3. Given positive integers p, q, r with $p \leq q \leq r$ and nonnegative integers t_2 and t_3, there is a trifold arrangement with Steiner data (p, q, r) that forms t_2 points of multiplicity two and t_3 points of multiplicity three if and only if (a) $t_2 + 3t_3 = qr + rp + pq$ and (b) $t_3 \leq \omega(p, q, r)$.

Proof. The existence of a trifold arrangement Λ with Steiner data (p, q, r) having t_3 triple points follows immediately from Theorem 2, and Λ has $qr + rp + pq - 3t_3$ points of multiplicity two, according to formula (2). On the other hand, if Λ is such a trifold arrangement, then (a) follows from (2) and (b) from Theorem 2. □

4 Regions of a Trifold Arrangement

The precise range of the region counter for a trifold arrangement with Steiner data (p, q, r) is determined by formula (3) and Theorem 2. Here is the result.

Theorem 4. Let p, q, r be positive integers with $p \leq q \leq r$, let $n = p + q + r$, and let $m = p + q - 1$. The number R of regions that are formed by a trifold arrangement of n lines with Steiner data (p, q, r) falls in the range

$$1 + n + \sigma_2 = R_{\max}(p, q, r) \geq R \geq R_{\min}(p, q, r) = 1 + n + \begin{cases} \lfloor n^2/4 \rfloor & \text{if } r \leq m \\ r(n - r) & \text{if } r \geq m. \end{cases}$$

Furthermore, for each integer R between $R_{\min}(p, q, r)$ and $R_{\max}(p, q, r)$ there is a trifold arrangement with Steiner data (p, q, r) that forms exactly R regions.

Proof. According to formula (3) and Theorem 2, $R = 1 + n + \sigma_2 - t_3$, where t can take any value between 0 and $\omega(p, q, r)$. It follows that $R_{\min} = 1 +$

$n + \sigma_2 - \omega(p,q,r)$, and every R between $R_{\min}$ and $R_{\max}$ is achieved. Since $\omega(p,q,r) = pq - \varphi(m-r)$ we see that $R_{\min} = 1 + n + r(n-r) + \varphi(m-r)$. If $r \le m$, then $\varphi(m-r) = \lfloor (m-r+1)^2/4 \rfloor = \lfloor (n-2r)^2/4 \rfloor = \lfloor n^2/4 \rfloor - r(n-r)$. If $r \ge m$, then $\varphi(m-r) = 0$. The assertion follows. □

Similar results can be given for the number S of segments and the number V of vertices formed by trifold arrangements. We summarize the results in a theorem but leave the detailed proofs to the reader with the observation that the analogues of formula (3) for segments and vertices are $S = n + 2\sigma_2 - 3t_3$ and $V = \sigma_2 - 2t_3$.

Theorem 5. Let p,q,r be positive integers with $p \le q \le r$, let $n = p+q+r$, and let $m = p+q-1$. The number S of segments (including rays) and the number V of points that are formed by a trifold arrangement of n lines with Steiner data (p,q,r) fall in the ranges

$$n + 2\sigma_2 = S_{\max}(p,q,r) \ge S \ge S_{\min}(p,q,r) = n + \begin{cases} 3\lfloor n^2/4 \rfloor - \sigma_2 & \text{if } r \le m \\ 3r(n-r) - \sigma_2 & \text{if } r \ge m, \end{cases}$$

$$\sigma_2 = V_{\max}(p,q,r) \ge V \ge V_{\min}(p,q,r) = \begin{cases} 2\lfloor n^2/4 \rfloor - \sigma_2 & \text{if } r \le m \\ 2r(n-r) - \sigma_2 & \text{if } r \ge m. \end{cases}$$

Every third integer from $S_{\min}$ to $S_{\max}$ is the segment count of some trifold arrangement with Steiner data (p,q,r), and every second integer from $V_{\min}$ to $V_{\max}$ is the point count of some such trifold arrangement.

We can now answer an interesting related question. For given n, how many and how few regions can be formed by a trifold arrangement of n lines?

Corollary 6. The number R of regions formed by a trifold arrangement Λ of $n \ge 3$ lines lies in the range

$$R_{\max}(n) = 1 + n + \lfloor n^2/3 \rfloor \ge R \ge R_{\min}(n) = 3n - 3.$$

Proof. Since $p,q,r \ge 1$, the fewest possible regions are formed when Λ has Steiner data $(1,1,n-2)$, as the formula of Theorem 4 shows. It follows from the elementary algebraic identity

$$\sigma_2 = qr + rp + pq = \frac{1}{3}n^2 - \frac{1}{6}\left((r-q)^2 + (p-r)^2 + (q-p)^2\right)$$

that the most regions are formed when p,q,r are as nearly equal as they can be and satisfy the constraint $p+q+r = n$. This gives the desired formula. □

As we have seen, for each fixed (p, q, r), every integer R between $R_{\min}(p, q, r)$ and $R_{\max}(p, q, r)$ is the region count of some trifold arrangement with Steiner data (p, q, r). It is not true that every R between $R_{\min}(n)$ and $R_{\max}(n)$ is the region count of some trifold arrangement of n lines. For $n = 9$, for example, $R_{\min}(9) = 24$ and $R_{\max}(9) = 37$, but it is not possible to arrange nine lines in three parallel families so as to form either 26 or 27 regions: $R_{\min}(1, 1, 7) = 24$, $R_{\max}(1, 1, 7) = 25$, and $R_{\min}(p, q, r) \geq 28$ for every other triple (p, q, r) with $p + q + r = 9$.

5 Triangles Dissected by Cevians

The following problem appears in Yaglom and Yaglom [13, p. 13]:

> Each of the three vertices of a triangle is joined by straight lines to n points on the opposite side of the triangle. Into how many parts do these $3n$ lines divide the interior of the triangle if no three of them pass through the same point?

More generally, when a triangle ABC is dissected by three families of cevians, p to side BC, q to side CA, and r to side AB, then at most

$$R_{\max} = 1 + \sigma_1 + \sigma_2 \tag{5}$$

regions are formed inside ABC, where $\sigma_1 = p + q + r$ and $\sigma_2 = qr + rp + pq$ (cf. Alexanderson and Wetzel [3]). A proof of (5) can easily be given by recursion, by a sweep-line argument, by Euler's formula, or indeed by direct counting.

Again, natural companion problems are to determine how *few* regions can be formed by a dissected triangle for given p, q, r and which potential region counts between the minimum and maximum can actually be realized.

The analogy between a triangle dissected by three families of cevians and the plane partitioned by three families of parallels is striking, and one is soon convinced that the two situations probably are abstractly identical. We substantiate that belief by constructing an explicit homeomorphism that maps the interior of the given triangle onto the plane and carries cevians from a vertex of that triangle to lines parallel to the side opposite that vertex. It follows that for each dissected triangle there is a combinatorially equivalent[2] trifold arrangement, and conversely for each trifold arrangement there is a combinatorially equivalent dissected triangle.

To describe the mapping, we employ *cevian coordinates* in the dissected triangle and the variant of barycentric coordinates called *areal coordinates* in the range

[2] A triangle dissected by cevians and a trifold arrangement of lines are *combinatorially equivalent* if there is an incidence-preserving bijection between their respective vertices, segments, and regions.

plane. We begin with a brief description of these two coordinate systems. Each requires a fixed reference triangle. It will be convenient to use the same reference triangle ABC, whose vertices we assume are named in the counterclockwise sense, for both coordinate systems.

Cevian coordinates. Write T for the interior of ABC, and let P be a point of T. If the feet X, Y, Z of the three cevians AX, BY, CZ through P divide the sides BC, CA, AB in the ratios[3] u, v, w, respectively, we call the triple (u, v, w) the *cevian coordinates* of P (with respect to ABC). If (u, v, w) are cevian coordinates of a point P, Ceva's Theorem (see Eves [7, p. 14]) requires that $uvw = 1$. Conversely, any three positive real numbers u, v, w with $uvw = 1$ are the cevian coordinates of a unique point inside ABC.

If a point X divides side BC in the ratio k, then a point P of T with cevian coordinates (u, v, w) lies on the cevian AX if and only if $u = k$. The closed half-triangles[4] ABX and AXC into which the cevian $u = k$ divides T are described in cevian coordinates by the inequalities $u \leq k$ and $u \geq k$, respectively.

Areal coordinates. For any point P in the plane let e, f, g be the directed distances from P to the three sidelines BC, CA, AB of ABC, the signs chosen in such a way that all three distances are positive when P lies in the interior of ABC. Let h_a, h_b, h_c be the altitudes of ABC, and write x, y, z for the ratios $e/h_a, f/h_b, g/h_c$, respectively. We call the triple (x, y, z) the *areal coordinates* of P (with respect to ABC) (see Coxeter [5, p. 218-221]). Writing a for the length BC and (XYZ) for the signed area[5] of XYZ, we see that

$$x = \frac{e}{h_a} = \frac{ae/2}{ah_a/2} = \frac{(PBC)}{(ABC)}.$$

So x is the signed fraction of the area of ABC that lies in triangle PBC. But $(PBC)+(PCA)+(PAB) = (ABC)$, and it follows that $x+y+z = 1$. Conversely, any three real numbers x, y, z with $x + y + z = 1$ are the areal coordinates of a unique point in the plane.

A point with areal coordinates (x, y, z) lies on the line ℓ parallel to and at directed distance kh_a from the sideline BC of ABC precisely when $x = k$. The two closed half-planes into which ℓ divides the plane are described in areal coordinates by the inequalities $x \leq k$ and $x \geq k$.

[3] The ratio $u = u(X)$ in which a point X of a segment BC divides BC is defined to be $u(X) = BX/XC$. The ratio u gives an increasing homeomorphism of the open segment BC onto the positive real numbers.

[4] By *closed* we mean that the half-triangle contains the points of the cevian that is its edge.

[5] If triangle XYZ has area K, the signed area (XYZ) is K if the vertices of XYZ labeled counterclockwise and $-K$ if they are labeled clockwise.

The homeomorphism. Suppose that a reference triangle ABC with interior T is given. Let $\mathbb{P} = \{(u,v,w) : u,v,w > 0\}$ be the positive octant in $\mathbb{R}^3$. The coordinate mapping

$$\gamma : T \longrightarrow \{(u,v,w) \in \mathbb{P} : uvw = 1\}$$

that carries a point P of T to its triple of cevian coordinates (with respect to the reference triangle ABC) is clearly a homeomorphism. The mapping $\theta : \mathbb{P} \longrightarrow \mathbb{R}^3$ defined by

$$\theta(u,v,w) = (\frac{1}{3} + \ln u, \frac{1}{3} + \ln v, \frac{1}{3} + \ln w)$$

is a homeomorphism that carries the surface $uvw = 1$ in $\mathbb{P}$ one-to-one onto the plane $x + y + z = 1$ in $\mathbb{R}^3$. The coordinate map

$$\tau : \mathbb{R}^2 \longrightarrow \{(x,y,z) \in \mathbb{R}^3 : x + y + z = 1\}$$

that carries a point Q of $\mathbb{R}^2$ to its triple of areal coordinates (with respect to the reference triangle ABC) is also clearly a homeomorphism.

Consequently the composite $\Theta = \tau^{-1} \circ \theta \circ \gamma$ is a homeomorphism that maps the interior T of triangle ABC onto the plane $\mathbb{R}^2$.

Now let positive integers p,q,r be given, and suppose the triangle ABC is dissected by families of p,q,r cevians, respectively, to form a dissected triangle Δ. We call the triple (p,q,r) the *Steiner data* of Δ. What is the image of Δ under this homeomorphism?

A cevian from the vertex A with cevian equation $u = k$ is transformed by Θ to the line with areal equation $x = \frac{1}{3} + \ln k$, parallel to the sideline BC of ABC. So Θ carries cevians of ABC to lines parallel to the sidelines of ABC. Let Λ be the trifold arrangement formed by the images under Θ of the dissecting cevians of Δ. Evidently Λ has Steiner data (p,q,r).

Since $\ln t$ is an increasing function of t, the closed half-triangles in T given by the cevian inequalities $u \geq k$, $u \leq k$ are mapped to the closed half-planes given by the areal inequality $x \geq \frac{1}{3} + \ln k$, $x \leq \frac{1}{3} + \ln k$; and it follows that Θ carries every closed half-triangle in T whose edge is a cevian to a closed half-plane in $\mathbb{R}^2$ whose edge is parallel to a sideline of ABC. Now, every point, segment, and region of Δ is the intersection of the collection of the closed half-triangles in which it lies, and the intersection of the corresponding closed half-planes is the corresponding point, segment (or ray), or region of Λ. Since the correspondence so described plainly preserves incidence, the trifold arrangement Λ is combinatorially equivalent to Δ.

A similar argument gives the converse assertion. We collect our results in a theorem.

Theorem 7. For each dissected triangle Δ there is a combinatorially equivalent trifold arrangement Λ, and conversely for each trifold arrangement Λ there is a combinatorially equivalent dissected triangle Δ.

It follows that formulas analogous to those of Theorems 4 and 5 and Corollary 6 hold for dissected triangles.

Corollary 8. Suppose $R_{\max}, R_{\min}, S_{\max}, S_{\min}, V_{\max}, V_{\min}$ are defined for dissected triangles as in Theorem 4 and Theorem 5. Let positive integers p, q, r be given with $p \leq q \leq r$, let $n = p + q + r$ and $m = p + q - 1$, and write R, S, V for the number of regions, segments, and points formed by a triangle dissected by n cevians with Steiner data (p, q, r). Then $R_{\max} \geq R \geq R_{\min}$, and every value between $R_{\max}$ and $R_{\min}$ is achieved; $S_{\max} \geq S \geq S_{\min}$, and every third value from $S_{\min}$ to $S_{\max}$ is achieved; and $V_{\max} \geq V \geq V_{\min}$, and every second value from $V_{\min}$ to $V_{\max}$ is achieved. Further, if $R_{\max}(n), R(n), R_{\min}(n)$ are defined for dissected triangles as in Corollary 6, then $R_{\max}(n) \geq R(n) \geq R_{\min}(n)$.

Figure 3 shows a dissected triangle with Steiner data $(5, 8, 9)$ that forms as few regions as possible for such a dissected triangle.dtbpF2.7146in2.7882in0ptFigure

6 Related Questions

In the projective setting, Martinov [11] has completely characterized the pairs (n, R) for which some projective arrangement of n lines forms R regions. Analogous results for arrangements of n lines in the Euclidean plane that lie in $m \geq 3$ parallel families would generalize Theorems 4 and 5 (and extend Corollary 6), and they might give some insights into the general realizability problem for arrangements in both the Euclidean and the real projective planes.

The many analogous problems in higher dimensions are mostly unexamined. For example, for each k with $0 \leq k \leq d - 2$, how should n hyperplanes in m parallel families of prescribed sizes be arranged in $\mathbb{R}^d$ so as to create as few k-flats of intersection as possible? For context, see Alexanderson and Wetzel [1], [4]. Inequalities like those of Corollary 6 in higher dimensions would also be of interest.

In 1971, Alexanderson and Wetzel [3] considered the analog in $\mathbb{R}^3$ of a dissected triangle in the plane and determined the maximum number of cells that can be formed when a tetrahedron is dissected by families of triangular plates hinged on its six edges. Two years later they announced (in [2]) the corresponding results for a dissected simplex in $\mathbb{R}^d$. A graph-theoretic proof of their formulas was given in 1976 by Zaslavsky [15]. These results were recently revisited by Hudelson [9]. The question of the minimum number of cells that can be formed in a dissected tetrahedron in $\mathbb{R}^3$ or, more generally, in a dissected simplex in $\mathbb{R}^d$, remains open. It is likely that a dissected simplex in $\mathbb{R}^d$ is combinatorially equivalent to a suitable arrangement of hyperplanes, but to our knowledge that equivalence has not been established for $d \geq 3$, although it was regarded as obvious in $\mathbb{R}^3$ by Alexanderson and Wetzel [3].

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