The Museum Theorem:

Thick Face-Paths and Hamiltonian-Connectedness in Plane Graphs

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Cycles in Graphs, Vanderbilt

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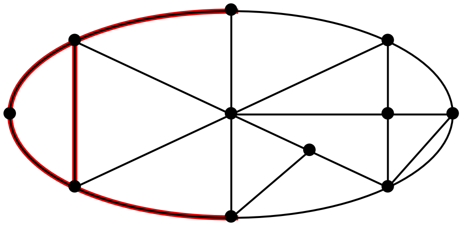
Thm. (Whitney [1931]) 4-connected planar triangulations are Hamiltonian.

Thm. (Tutte [1956]) 4-connected plane graphs are Hamiltonian.

Thm. (Thomassen [1983]) 4-connected plane graphs are Hamiltonian-connected.
($\forall x, y \in V(G)$, there is a spanning x, y-path).

Needed Concepts

Def. A thin subgraph of G is a subgraph H such that in each block of H at most three vertices have neighbors in G outside the block. (**Ex:** any tree.)



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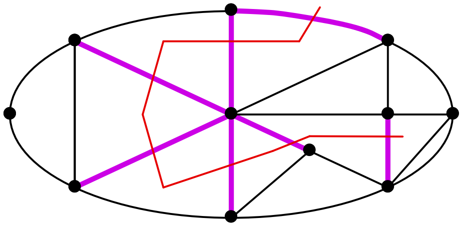

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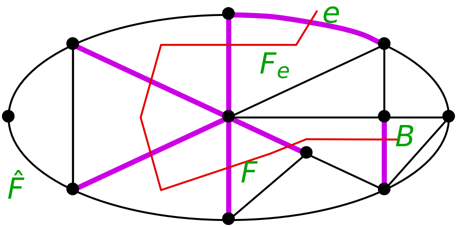

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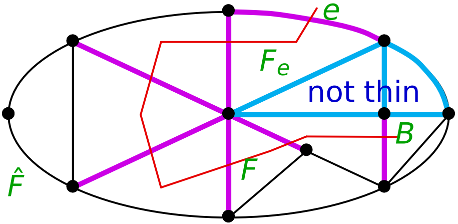
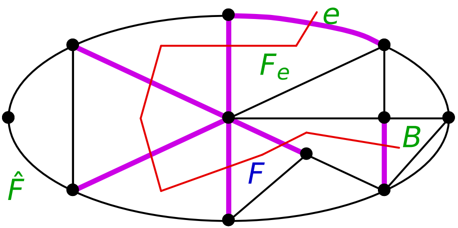
A face-path P is thick if $G - E(P)$ is a thin subgraph of G.



Statement of the Result

Def. Let $\hat{C}$ be the external cycle in G. A face F touches $\hat{C}$ if it shares a vertex (or more) with $\hat{C}$.



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Thm. Let G be a 2-connected plane graph; $e \in E(\hat{C})$. Let F be a face touching $\hat{C}$, and let B be a bounded face. If G has an $[e, F, B]$-face-path, then G has a thick $[e, F, B]$-face-path.

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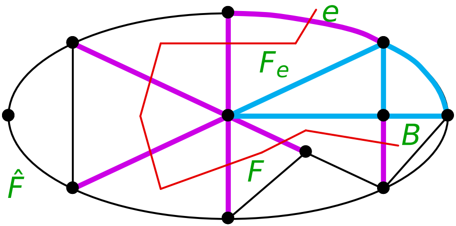

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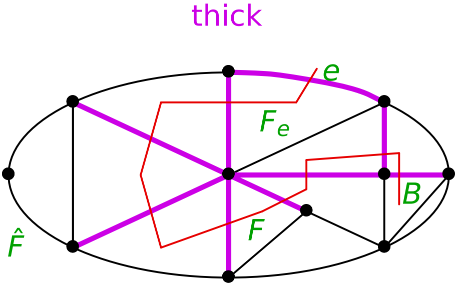
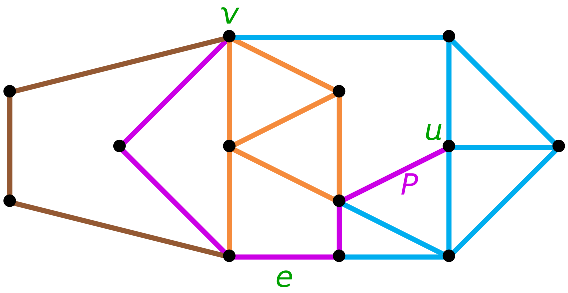


Thomassen's Result

Thm. (Thomassen [1983]) Let G be a 2-connected plane graph with outer cycle C. If $v \in V(C)$ and $e \in E(C)$ and $u \in V(G - v)$, then G has a u, v-path P through e such that each component of $G - V(P)$ has at most three neighbors in P (just two for components containing an edge of C).



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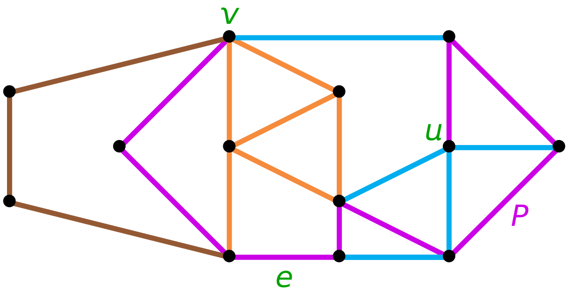

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Thomassen	**Us**
paths in H	face-paths in $G = H^*$
start at external v	start at $\hat{F}$
via external edge e	visit F touching $\hat{C}$
end at any given u	end at any given face B
≤ 3 nbrs of each fragment	$G - E(P)$ is thin
4-connected H $\Rightarrow$ no leftover fragment	4-connected H $\Rightarrow$ no leftover bounded face

We believe that neither directly implies the other.

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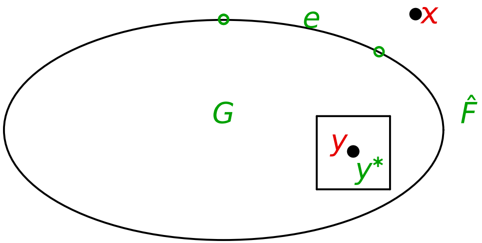
Pf. Given $x, y \in V(H)$, we seek a spanning x, y-path in H. Embed H with x on the outer face. Draw its dual G with the face $\hat{F}$ for x as the outer face. Choose e on $\hat{F}$.



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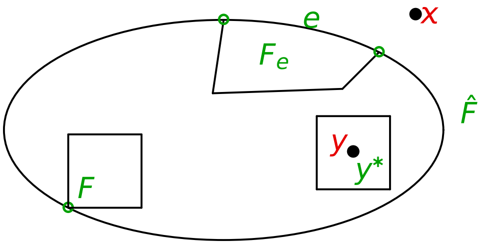

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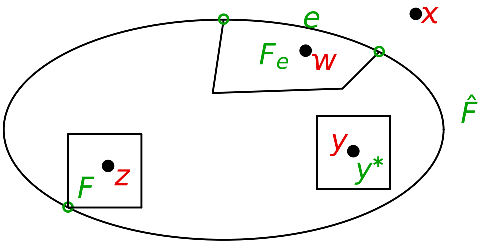
Let w and z in $V(H)$ be dual to F and F_e.



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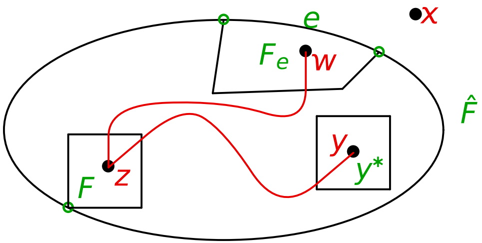
Let w and z in $V(H)$ be dual to F and F_e.

$\kappa(H) \geq 3$ $\Rightarrow$ $\exists\, w, y$-path in H via z and not x.



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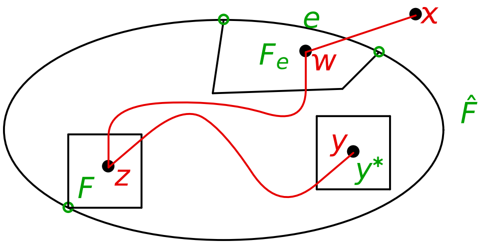
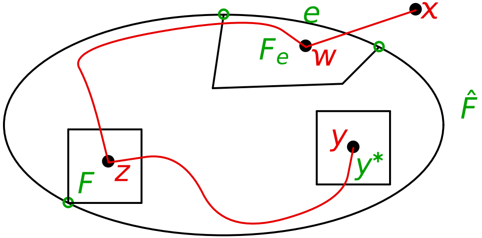
This yields an $[e, F, y^*]$-face-path in G.



Spanning x, y-path

Now Theorem $\Rightarrow$ G has a thick $[e, F, y^*]$-face-path P.
(Thick $[e, y^*]$-path is enough; no need to require F.)



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Let R be a maximal bounded region not entered by P;
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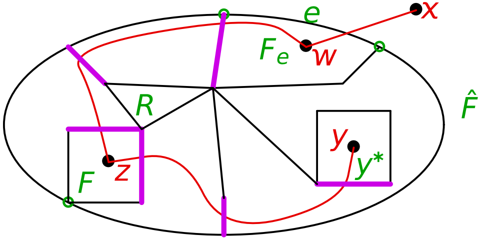

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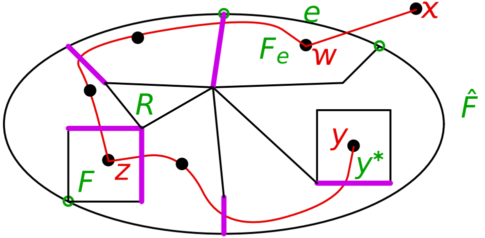
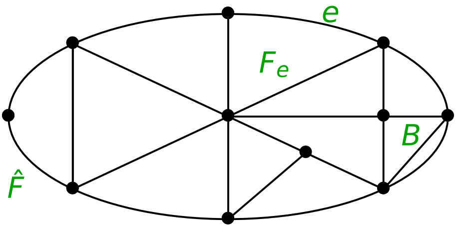
Since P is a thick face-path, at most three vertices of Q have nbrs outside Q. $\therefore$ at most three faces of P share edges with R, separating R from other faces, including one of $\{\hat{F}, F_e, F, y^*\}$. This contradicts $\kappa(H) \geq 4$. ∎



The "Desired" Result

Ignore requirement of visiting F.

Thm. Given 2-connected plane G, edge $e \in E(\hat{C})$, and bounded face B, there is a thick $[e, B]$-face-path.



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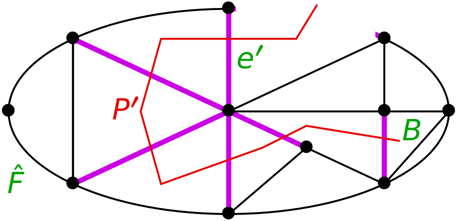
Naive approach: Delete e to get smaller graph G'. Use external edge e' in G' & induction hypothesis.



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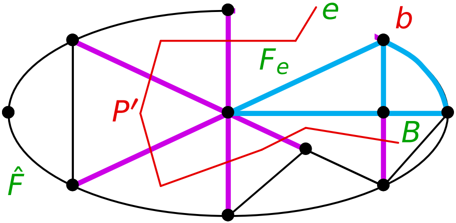

Problem! The boundary vertex b is dangerous. It may belong to a thin block in $G' - E(P')$ but become a fourth boundary vertex of the block when e is restored.

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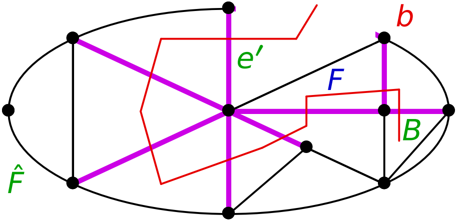

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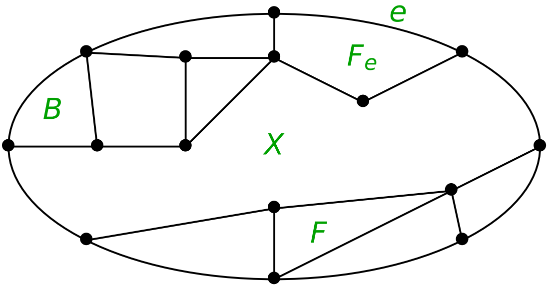
Remedy! Require visiting a face F containing b.

Another Difficulty - Separating Faces

When we require visiting F, there may be no $[e, F, B]$-face-path. In the weak dual a face X (cut-vertex of the weak dual) may separate F from F_e and B.



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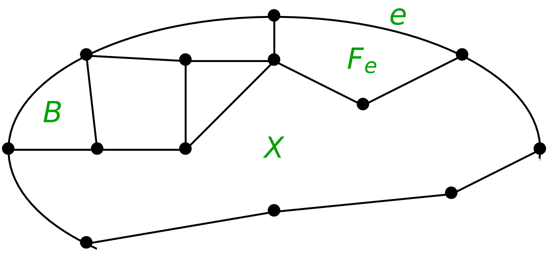

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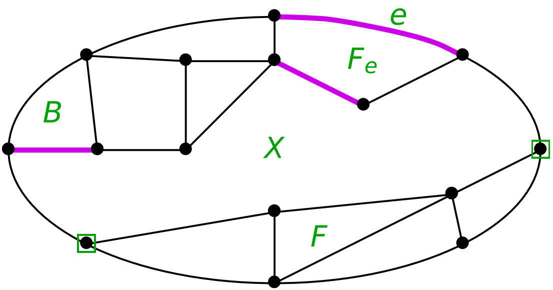

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Since P' visits X, the part separated from F_e and B by X is a block of $G - E(P')$ with only two boundary vertices.

The Real Theorem

Thm. With G, e, F, B as specified, G has a thick $[e, F, B]$-face-path or has a thick $[e, B]$-face-path P with F inside a block of $G - E(P)$ having only two boundary vertices.

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The proof of this final statement is simple. The desired face-path is assembled inductively, using face-paths from various subgraphs that satisfy the hypotheses. The idea is indicated in the following figure:

The Final Induction Step