# A new proof that 4-connected planar graphs are Hamiltonian-connected

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**Abstract**

We prove a theorem guaranteeing special paths of faces in 2-connected plane graphs. As a corollary, we obtain a new proof of Thomassen's theorem that every 4-connected planar graph is Hamiltonian-connected.

## 1   Introduction

Imagine planning a visit to a museum. The entrance and the last room (the museum shop) are fixed, and there is one important room you must visit. Other than that, you don't insist on visiting every room, but you want to skip only "small" pieces, in some sense.

Our theorem models this situation. The rooms are the faces of a plane graph. Following a list of rooms is equivalent to following a path in the dual graph, so we call this a "face-path" to emphasize the original graph. Edges correspond to walls separating rooms, and we can cross any internal edge via a door in the wall. Vertices are not very important in this scenario, but the notion of "small pieces" in what is skipped involves them.

Well-known prior results for 4-connected plane graphs follow from our main result. Whitney [3] proved that 4-connected triangulations are *Hamiltonian*, meaning that they have spanning cycles. Tutte [2] extended the conclusion to all 4-connected planar graphs. Thomassen [1] showed further that 4-connected planar graphs are *Hamiltonian-connected*, meaning that for any two vertices $u$ and $v$, there is a spanning path with endpoints $u$ and $v$.

Tutte and Thomassen proceeded by proving technically stronger statements that facilitate inductive proof, showing the existence of special paths satisfying additional conditions. Our approach also has this flavor, but our technical result is somewhat simpler. It is almost implied by Thomassen's technical result. Our graphs allow multiple edges but no loops; since we study 2-connected plane graphs, the dual graphs also have no loops.

Stating our theorem requires additional terminology. A subgraph $G'$ of a graph $G$ is a *thin subgraph* of $G$ if every block of $G'$ has at most three vertices that have neighbors in $G$ outside the block. Such vertices are *boundary vertices* of the block.

Given a plane graph $G$, let $\hat{F}$ denote the unbounded face of $G$, and let $\hat{C}$ denote the boundary of $\hat{F}$; in a 2-connected plane graph, $\hat{C}$ is a cycle. The *weak dual* of $G$ is the subgraph $\hat{G}$ obtained from the dual graph $G^*$ by deleting the vertex corresponding to $\hat{F}$.

Faces are *adjacent* if they share an edge. A *face-path* in $G$ is a list of faces whose corresponding dual vertices form a path in $G^*$ in that order. When $P$ is a face-path in $G$, let $P^*$ denote the corresponding path in $G^*$. With $E(P^*)$ being the edge set of that dual path, let $\hat{E}(P)$ denote the corresponding edges of $G$; these are the edges of $G$ *crossed* while following $P$ in $G$ (Fig. 1 shows $\hat{E}(P)$ in bold). A face-path $P$ is a *thick face-path* if $G - \hat{E}(P)$ is a thin subgraph of $G$. We seek a thick face-path that begins with the unbounded face $\hat{F}$, crosses a specified edge $e$ of $\hat{C}$ to the bounded face $F_e$, and ends at a specified bounded face $B$. Such a path is a *thick $[e, B]$-face-path*. Fig. 1 shows a thick $[e, B]$-face-path, denoted $P$.
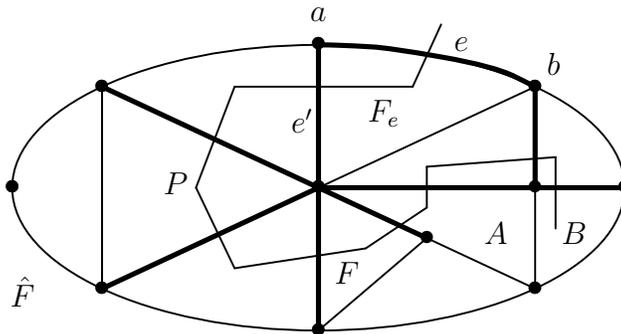


Figure 1: A thick $[e, B]$-face-path through $F$

A stronger conclusion is needed for an inductive proof. We could seek a thick $[e, B]$-face-path inductively as follows. Let $G' = G - e$, with $e = ab$. In the case that $G'$ is 2-connected, let $e'$ be an external edge of $G'$ that is not external in $G$, incident to $a$. Let $P'$ be a thick $[e', B]$-face-path in $G'$, and prepend the step across $e$ to form $P$ in $G$. In Fig. 1, $P'$ could differ from $P$ by going directly from $A$ to $B$ instead of the last three steps. In that case $b$ lies in a block of $G' - \hat{E}(P')$ having three boundary vertices as a subgraph of $G'$, but $b$ becomes a fourth boundary vertex in a block of $G - \hat{E}(P)$ as a subgraph of $G$ when $e$ is restored.

To overcome the difficulty and obtain an inductive proof, we specify another face to visit along the way (the "important" room in the museum). An *$[e, F, B]$-face-path* is an $[e, B]$-face-path in $G$ that visits a specified face $F$. We will seek a thick $[e, F, B]$-face-path for a face $F$ that touches $\hat{C}$, where a face of $G$ *touches* a subgraph $G'$ if the face and $G'$ share at least one vertex (see Fig. 1; note that *neighboring* faces still must have a common edge). Our main result is the following.

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**Theorem 1.1.** *Given a 2-connected plane graph $G$ with external cycle $\hat{C}$, let $e$ be an edge of $\hat{C}$, let $B$ be a bounded face of $G$, and let $F$ be a bounded face touching $\hat{C}$. If $F$ lies on some $[e, B]$-face-path, then $G$ has a thick $[e, F, B]$-face-path.*


**Corollary 1.2.** (Thomassen [1]) *Every 4-connected planar graph is Hamiltonian-connected.*

*Proof.* Given vertices $x$ and $y$ in a 4-connected (simple) planar graph $H$; we seek a spanning $x, y$-path. Embed $H$ with $x$ on the outer face. Let $G$ be the dual of $H$, drawn so that the face $\hat{F}$ corresponding to $x$ is the unbounded face. Let $e$ be an edge of the boundary $\hat{C}$ of $\hat{F}$ in $G$, bounding $\hat{F}$ and $F_e$. Let $B$ be the face of $G$ that is dual to $y$. Since $d_H(x) \geq 4$, in $G$ some face $F$ not in $\{F_e, B\}$ has a boundary edge on $\hat{C}$.

Since it is the dual of a loopless planar graph, $G$ is 2-connected. Since $H-x$ is 3-connected, $F$ lies on some $[e, B]$-face-path. Hence by Theorem 1.1, $G$ has a thick $[e, F, B]$-face-path $P$; Fig. 2 illustrates this with $\hat{E}(P)$ in bold. Vertices of $H$ are solid; those of $G$ are hollow. If $P$ visits all faces of $G$, then $P^*$ is a Hamiltonian $x, y$-path in $H$.
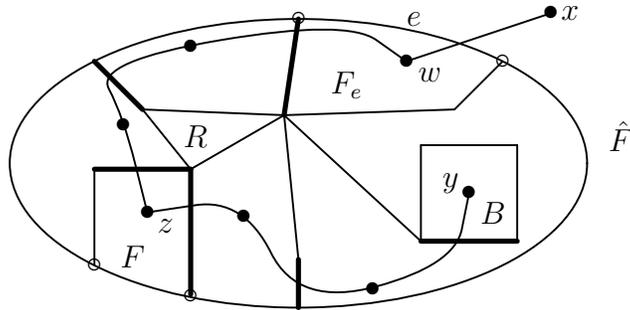


Figure 2: Thick $[e, F, B]$-face-path missing a region $R$

Otherwise, $P$ misses some face of $G$. Hence $G - \hat{E}(P)$ has a bounded face. Let $R$ be a maximal bounded region not entered by $P$, that is, a union of faces in $G$ corresponding to a connected subgraph of the dual, such that each face neighboring this union is a face in $P$. The vertices and edges of $G$ in $R$ form a block $Q$ in $G - \hat{E}(P)$, since enlarging the 2-connected subgraph $Q$ would require enlarging $R$. Since $P$ is a thick face-path, at most three vertices of $Q$ have neighbors outside $Q$ in $G$. By the definition of $R$, the faces of $G$ neighboring $R$ are in $P$.

Since $Q$ has at most three boundary vertices, at most three faces of $G$ neighbor $R$. Since $P$ has at least four faces ($\hat{F}$, $F_e$, $F$, and $B$) and all faces on $P$ are outside $R$, deleting the faces neighboring $R$ separates the faces inside $R$ from at least one face on $P$. This contradicts the hypothesis that $H$ is 4-connected, so in fact $P^*$ is Hamiltonian. $\qquad\square$

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The face $F$ lies on no $[e, B]$-face-path if in $\hat{G}$ the vertices for $F_e$ and $B$ lie outside the block containing the vertex for $F$. The face in $G$ corresponding to the cut-vertex of $\hat{G}$ between them then *separates* $F$ from $\{F_e, B\}$. To incorporate this possibility in a single inductive statement that applies in all situations, we prove Theorem 1.1 in a more detailed form.

**Theorem 1.3.** *Given a 2-connected plane graph $G$ with external cycle $\hat{C}$, let $e$ be an edge of $\hat{C}$, let $F$ be a face touching $\hat{C}$, and let $B$ be a bounded face of $G$. Either $G$ has a thick $[e, F, B]$-face-path, or $G$ has a thick $[e, B]$-face-path $P$ such that $F$ is inside a block of $G - \hat{E}(P)$ having only two boundary vertices.*

For convenience, let a face-path as specified in Theorem 1.3 be a *suitable* path. Thomassen obtained his theorem on 4-connected planar graphs by proving the following stronger result.

**Theorem 1.4.** (Thomassen [1]) *Let $H$ be a 2-connected plane graph with external cycle $C$. If $v \in V(C)$ and $e \in E(C)$ and $u \in V(H - v)$, then $H$ has a $u, v$-path $P$ containing $e$ such that each component of $H - V(P)$ has at most three neighbors on $P$, and components of $H - V(P)$ containing an edge of $C$ have at most two neighbors on $P$.*

When $H$ is 4-connected, Thomassen's conclusion immediately implies that $P$ is a spanning $u, v$-path, since otherwise $H$ would have a separating set of size at most 3 in $V(P)$. With $G$ being the dual of $H$, Theorem 1.4 almost implies our Theorem 1.3. To explain the relationship, we describe what Theorem 1.4 says about the dual graph.

Draw $H$ with $v$ on the external cycle; in $G$ the face corresponding to $v$ is $\hat{F}$. The face corresponding to $u$ is $B$. Thomassen's edge $e$ becomes adjacent faces $F$ and $F'$, both touching $\hat{C}$. Thus he specifies more than our single face $F$ touching $\hat{C}$, but he does not specify the initial edge of $P$ the way we specific the initial edge $e$ of $\hat{E}(P)$. A component of $H - V(P)$ corresponds to a maximal region of $G - \hat{E}(P)$ not entered by the face path, which forms a block in $G - \hat{E}(P)$. Having at most three neighbors on $P$ corresponds to being surrounded by at most three faces of the face-path in $G$ and hence having at most three boundary vertices. The final clause is analogous to our case with $F$ separated from $\{F_e, B\}$.

The aspect of our result that is not implied by Thomassen's result is the specification of the initial edge. The freedom to specify this edge in combining subpaths is a key reason why our proof is simpler.

# 2   Proof of Theorem 1.3

We are given a 2-connected plane graph $G$ with external cycle $\hat{C}$, an edge $e$ of $\hat{C}$, a face $F$ touching $\hat{C}$, and a bounded face $B$ of $G$. We seek a suitable path $P$, defined as in Theorem 1.3. We emphasize that boundary vertices for blocks in $G - \hat{E}(P)$ are those having outside neighbors in $G$, not just outside neighbors in $G - \hat{E}(P)$.

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Our approach is inductive. When $G$ has only one bounded face, it serves as all of $\{F_e, F, B\}$, the $[e, F, B]$-face-path $P$ has length 1, and $G - \hat{E}(P)$ is a path (in which each block has only two vertices). Otherwise, we consider a minimal counterexample $G$ (fewest vertices). In various cases, we construct a suitable path in $G$ from suitable paths in subgraphs.

A *separating face* in a 2-connected plane graph $G$ is a face $X$ whose corresponding dual vertex is a cut-vertex of the weak dual $\hat{G}$. In terms of $G$, deleting the vertices of $X \cap \hat{C}$ disconnects $\hat{C}$. An $X$-*slice* of $G$ is a maximal subgraph of $G$ containing $X$ (that is, its boundary) in which $X$ is not a separating face.

**Lemma 2.1.** *A minimal counterexample $G$ has no separating face.*

*Proof.* If $G$ has a separating face $X$, then let $G'$ and $G''$ be subgraphs of $G$ such that $G' \cup G'' = G$, each is a union of $X$-slices of $G$, and each $X$-slice appears in only one of $\{G', G''\}$. Each of $G'$ and $G''$ is 2-connected and smaller than $G$ (see Figure 3).

**Case 1:** $F_e$ *and* $B$ *lie in one of* $\{G', G''\}$. By symmetry, let $G'$ be the subgraph containing $F_e$ and $B$. By the minimality of $G$, in $G'$ there is a suitable path $P$; it is an $[e, F, B]$-face-path if $F$ is contained in $G'$, and otherwise it may be any thick $[e, B]$-face-path. If $P$ does not visit $X$, then the full boundary of $X$ lies in one block of $G' - \hat{E}(P)$, and including the rest of $G''$ just enlarges that block without changing the boundary vertices. If $P$ visits $X$, then edges on the boundary of $X$ that are shared with other faces in $G''$ are single-edge blocks in $G' - \hat{E}(P)$ (and none of them can be $e$). Adding the rest of $G''$ absorbs them into a single block of $G - \hat{E}(P)$ having two boundary vertices as a subgraph of $G$, without changing other blocks. Regardless of where $F$ is, $P$ is suitable for $G$.

**Case 2:** $F_e$ *and* $B$ *do not both lie in one of* $\{G', G''\}$. In particular, $X \notin \{F_e, B\}$. We assemble a suitable path by combining suitable paths for $G'$ and $G''$. By symmetry, we may assume $e \in E(G')$. Let $P'$ be a thick $[e, F, X]$-face-path in $G'$ if $F$ is a face in $G'$; otherwise $P'$ is any thick $[e, X]$-face-path in $G'$. Now let $e'$ be the edge across which $P'$ enters $X$. The second path $P''$ is a thick $[e', F, B]$-face-path in $G''$ if $F$ is a face in $G''$; otherwise it is any thick $[e', B]$-face-path in $G''$. Combining $P'$ and $P''$ yields a suitable path $P$ in $G$. □
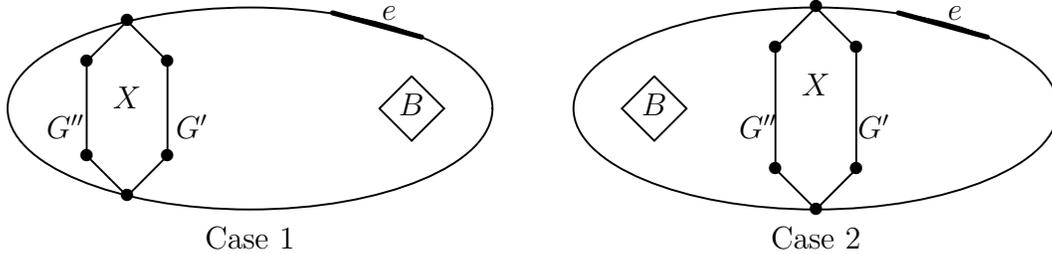


Figure 3: Excluding a separating face

With separating faces forbidden, we will decompose $G$ in a different way, using subgraphs $G_1$ and $G_2$. After defining them, we will restrict the form of $G_1$.

**Definition 2.2.** *In a minimal counterexample $G$, the outer boundary $\hat{C}$ contains two paths that join $e$ and $F$. Since $G$ has no separating face (by Lemma 2.1), at least one of these paths is not touched by $B$. Let $\hat{Q}$ be a path from $e$ to $F$ along $\hat{C}$ that is not touched by $B$. Let $\mathcal{F}$ be the set of bounded faces of $G$ from $F_e$ to $F$ that touch $\hat{Q}$. Let $G_1$ be the subgraph of $G$ consisting of the union of the faces in $\mathcal{F}$ and all faces enclosed by $\mathcal{F}$. Let $G_2$ be the union of all bounded faces of $G$ that are not in $G_1$ (this includes $F$).*

**Lemma 2.3.** *In a minimal counterexample $G$, the weak dual of $G_1$ is a path.*

*Proof.* The plane graph $G_1$ is bounded by $\hat{Q}$ and portions of the boundaries of the faces in $\mathcal{F}$. We consider ways in which the weak dual $\hat{G}_1$ of $G_1$ may fail to be a path.

If some face of $G_1$ is not in $\mathcal{F}$, then let $\hat{H}$ be a component of $\hat{G} - \mathcal{F}$. By the construction of $G_1$, there are two faces $X$ and $Y$ in $G_1$ whose corresponding vertices $x$ and $y$ in $\hat{G}_1$ form a separating set in $\hat{G}_1$. Let the plane graph $H$ be the union of the faces in $G$ corresponding to the vertices of $\hat{H}$. The outer boundary of $H$ consists of part of the boundary of $X$, part of the boundary of $Y$, and possibly part of $\hat{C}$ joining $X$ and $Y$. Figure 4 shows two ways this may occur, labeled $H_1$ and $H_2$. Note that $H$ has no cut-vertex, since $\hat{H}$ is connected.

Such a subgraph $H$ has at most three boundary vertices as a subgraph of $G$. At most two boundary vertices lie along $\hat{C}$. All other boundary vertices of $H$ are incident to both $X$ and $Y$, since every edge bounding $H$ lies along $X$ or $Y$ or $\hat{C}$. Also, if $H$ has two boundary vertices that are incident to both $X$ and $Y$, then it has no other boundary vertices. In the case labeled $H_2$ in Figure 4, the dashed edges indicate the possibility of other similar blocks or single edges caught between $X$ and $Y$.

The remaining case is when faces $X$ and $Y$ in $\mathcal{F}$ have two common boundary edges sharing a vertex $z$. This is shown in Figure 4 as $H_3$.

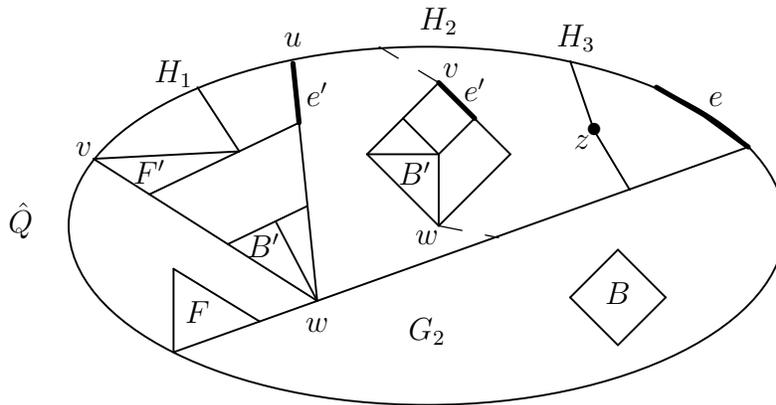In each case, we use the minimality of $G$ to obtain suitable paths in smaller graphs.



Figure 4: Restricting $G_1$

**Case 1:** *$H$ exists and has two boundary vertices on $\hat{C}$. Let $u$ and $v$ be the boundary*

vertices of $H$ on $\hat{C}$, with $u$ closer to $e$ than $v$ along $\hat{Q}$ (see $H_1$ in Figure 4.) Since $H$ contains a vertex $z$ interior to $G_1$, in $H$ there is more than one face. Also, since $H$ corresponds to a component of $G_1 - \{x, y\}$, there is a common vertex $w$ of $X$ and $Y$ that is a boundary vertex of $H$. Hence $H$ has exactly three boundary vertices as a subgraph of $G$. (Note that $w$ need not be on the boundary of $G_2$ as in Figure 4. There may be additional components of $\hat{G} - \{x, y\}$ or single edges as shown under $H_2$.)

Let $G'$ be the graph obtained from $G$ by replacing $H$ with a single face $K$ having vertex set $\{u, v, w\}$, occupying the same region as $H$. The graph $G'$ is 2-connected; let $P'$ be a face-path of $G'$ as guaranteed by the minimality of $G$. If $P'$ does not visit $K$, then $P'$ has the desired properties in $G$, since replacing $K$ with $H$ does not change the boundary vertices of any block obtained by deleting $\hat{E}(P')$.

If $P'$ visits $K$, then it must cross both $uw$ and $vw$, crossing $uw$ first. We obtain a suitable path $P$ in $G$ by replacing $K$ along $P'$ with a suitable path in $H$. Let $e'$ be the edge of the boundary of $H$ that is incident to $u$ and not on $\hat{C}$, let $F'$ be a face of $H$ touching $\hat{C}$ at $v$, and let $B'$ be the bounded face of $H$ that is incident to $w$ and contains the first edge of the path from $w$ to $v$ along the boundary that does not pass through $u$ (see Figure 4). Since $H$ is 2-connected and smaller than $G$, we obtain a suitable path $P''$. Delete the initial outer face of $H$ from $P''$ and insert the rest into $P'$ in place of $K$ to form $P$.

In any block of $G' - \hat{E}(P')$ containing $w$, already $w$ is a boundary vertex, since the edges $wu$ and $wv$ lie in $G'$ but not in the block. Hence replacing $H$ does not cause trouble for these blocks. If $P''$ does not enter $F'$ (note that $H$ may have separating faces), then $F'$ lies in a block of $H - \hat{E}(P'')$ having at most two boundary vertices as a subgraph of $H$; adding the boundary vertex $v$ when viewing the block as a subgraph of $G$ causes no trouble. Hence for the discussion of $v$ we may assume that $P''$ enters $F'$.

For the role of $u$, $v$, or $w$ as a boundary vertex, if $F_{e'}$, $F'$, or $B'$ (respectively) is the only face of $H$ incident to that vertex, then the block(s) of $H - \hat{E}(P'')$ containing that vertex are single edges and hence thin. If there are other such faces, then the edges of the visited face already ensure that the specified vertex will be a boundary vertex of its block in $H - \hat{E}(P'')$ as a subgraph of $H$. Hence expanding the graph to $G$ does not add boundary vertices.

**Case 2:** *H exists and has two boundary vertices, each a common vertex of $X$ and $Y$.* Let $v$ and $w$ be the boundary vertices of $H$, with $v$ closer to $\hat{C}$ along $X$ and $Y$ (see $H_2$ in Figure 4.) Form $G'$ by replacing $H$ with the single edge $vw$. Again $G'$ is 2-connected; let $P'$ be a resulting suitable path. If $P'$ does not cross $vw$, then $P'$ is suitable in $G$.

If $P'$ crosses $vw$, then we obtain the suitable path $P$ in $G$ by inserting into $P'$ (between $X$ and $Y$) a suitable path $P''$ for $H$ (after deleting the outer face at which it starts). Let $e'$ be the edge of the boundary of $H$ incident to $v$ that is nearer to $F_e$ along $\hat{Q}$. Let $B'$ be the bounded face of $H$ containing the edge of the boundary incident to $w$ that is farther from $F_e$ along $\hat{Q}$. Since $H$ is 2-connected and smaller than $G$, we obtain a thick $[e', B']$-face-path

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in $H$. As in Case 1, the visiting of $F_{e'}$ and $B'$ ensures that $v$ and $w$ are boundary vertices (as subgraphs of $H$) for the blocks containing them in $H - \hat{E}(P'')$. Hence inserting $P''$ into $P'$ to obtain $P$ causes no trouble, and $P$ is a suitable path in $G$.

**Case 3:** *$z$ is a common vertex of two edges in the boundary of both $X$ and $Y$* (see $H_3$ in Figure 4). In this case, form $G'$ by contracting one of the two edges incident to $z$. Whether the resulting suitable path $P'$ crosses the remaining edge at $z$ or not, re-expanding $z$ allows $P'$ to serve as a suitable path in $G$. □

The next lemma completes the proof of the theorem.

**Lemma 2.4.** *There is no minimal counterexample to Theorem 1.3.*

*Proof.* Suppose that $G$ with inputs $[e, F, B]$ is a minimal counterexample, and define $G_1$ and $G_2$ as in Definition 2.2. By Lemma 2.3, the weak dual of $G_1$ is a path, corresponding to a face-path $P_1$ in $G_1$ that enters $G_1$ across $e$ and ends in $F$.

Since $G$ has no separating face, by Lemma 2.1, $G_2$ is 2-connected. Let $Q$ be the path common to the boundaries of $G_1$ and $G_2$. Let $e_2$ be the end edge of $Q$ that lies in the boundary of $F$. Let $F_2$ be the bounded face in $G_2$ that is bounded by the edge at the other end of $Q$, which is shared also by $F_e$. Applying the induction hypothesis to $G_2$ with inputs $[e_2, F_2, B]$, we obtain in $G_2$ a suitable path $P_2$ (see Figure 5).

We aim to combine $P_1$ and $P_2$ (after deleting the unbounded face that starts $P_2$) to obtain a thick $[e, F, B]$-face-path in $G$. However, blocks in $G_2 - \hat{E}(P_2)$ may gain additional boundary vertices as subgraphs of $G$. For example, in Figure 5, the block $K$ in $G_2 - \hat{E}(P_2)$ has boundary vertices $\{u, v, w\}$ as a subgraph of $G_2$, but it is also a block in $G - \hat{E}(P_1 \cup P_2)$ and then it gains $x$ as a boundary vertex when viewed as a subgraph of $G$.
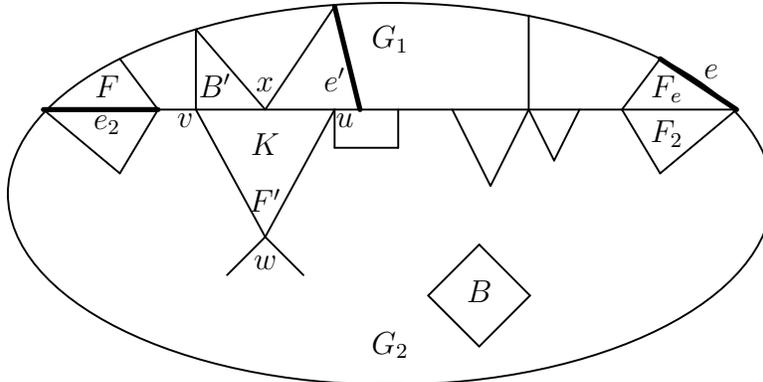


Figure 5: Combining $G_1$ and $G_2$

Any added boundary vertices for a block in $G_2 - \hat{E}(P_2)$ lie along $Q$. Because $P_2$ visits $F_2$ (or $F_2$ lies in a block of $G_2 - \hat{E}(P_2)$ having two boundary vertices), each block $K$ of $G_2 - \hat{E}(P_2)$

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that touches $Q$ contains a subpath $Q'$ of $Q$, and the endpoints of $Q'$ are boundary vertices of $K$ as a subgraph of $G_2$. If $K$ gains a boundary vertex due to an edge in $G_1$, then the endpoints $u$ and $v$ of $Q'$ are distinct. Since $K$ is thin in $G_2 - \hat{E}(P_2)$ as a subgraph of $G_2$, it has at most one more boundary vertex in $G_2$; call it $w$ if it exists.

We remedy the difficulty by making a detour from $P_1$. Let $u$ be the endpoint of $Q'$ closer to $F_e$. Let $e'$ be the internal edge of $G_1$ that is the last edge crossed by $P_1$ before reaching a face $F_{e'}$ sharing an edge of $Q'$ with $K$. That edge is the edge of $Q'$ incident to $u$; let $B'$ be the face of $G_1$ bounded by the edge of $Q'$ incident to $v$. (If $F_{e'}$ and $B'$ are the same face, then no internal edges of $G_1$ are incident to $Q'$, and $K$ remains thin as a subgraph of $G$, so no detour is needed.)

Since the weak dual of $G_1$ is a path, $P_1$ contains a face-path from $F_{e'}$ to $B'$. Furthermore, adding these faces to $K$ yields a 2-connected graph $G'$. Let $F'$ be a face of $G'$ touching $w$, if $w$ exists (otherwise $F'$ is arbitrary). Apply the induction hypothesis to $G'$ with inputs $[e', F', B']$ to obtain a suitable path $P'$ in $G'$. Replace the portion of $P_1$ from $F_{e'}$ to $B'$ with $P'$ (deleting the unbounded face at the start of $P'$).

After making such detours for each problematic block $K$, we have the final face-path $P$, which we claim is suitable for $G$ with inputs $[e, F, B]$. Besides $u$, $v$, and possibly $w$, no vertices of $K$ have neighbors in $V(G_2)$ outside $V(K)$, due to the properties of $P_2$. Since $P$ enters $G'$ across $e'$ and leaves it from $B'$, vertices $u$ and $v$ are already boundary vertices for their blocks in $G - \hat{E}(P)$. Since $F'$ touches $w$, the same property holds for $w$ if $P'$ visits it. However, if $P'$ does not visit $w$, then $w$ lies in a block of $G' - \hat{E}(P')$ having two boundary vertices in $G'$, and adding $w$ as a boundary vertex for this block causes no problem.

We have shown that $G - \hat{E}(P)$ is a thin subgraph of $G$. $\qquad\qquad\square$

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