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REGRESSIONS AND MONOTONE CHAINS: A RAMSEY-TYPE EXTREMAL PROBLEM FOR PARTIAL ORDERS

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Dedicated to Paul Erdős on his seventieth birthday

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A *regression* is a function g from a partially ordered set to itself such that $g(x)\leqq x$ for all z. A *monotone k-chain* is a chain of k elements $x_1<x_2<\ldots<x_k$ such that $g(x_1)\leqq g(x_2)\leqq\ldots\leqq g(x_k)$. If a partial order has sufficiently many elements compared to the size of its largest antichain, every regression on it will have a monotone $(k+1)$-chain. Fixing w, let $f(w, k)$ be the smallest number such that every regression on every partial order with size least $f(w, k)$ but no antichain larger than w has a monotone $(k+1)$-chain. We show that $f(w, k)=(w+1)^k$.

1. Introduction

At the Symposium on Ordered Sets in Banff, Klaus Leeb posed to us the following question. Consider a finite partially ordered set (poset). Define a *regression* on a poset to be a function g mapping the poset to itself such that $g(x)\leqq x$ for all x. Define a *monotone k-chain* to be a chain of k elements $x_1<x_2<\ldots<x_k$ such that $g(x_1)\leqq g(x_2)\leqq\ldots\leqq g(x_k)$. Problem: find bounds such that any regression on any poset must have a monotone $(k+1)$-chain if the size of the poset exceeds those bounds.

In this note we solve this question for posets of bounded width. (The *width* of a poset is the size of its largest antichain. Any poset terminology not explicitly defined here can be found in [4].) If a poset has width at most w and has a regression with no monotone $(k+1)$-chain, then we call it a (w, k)-poset. Let $f(w, k)$ be the smallest integer such that there is no (w, k)-poset with that many elements. In other words, every regression on every poset that has at least that many elements but width at most w has a monotone $(k+1)$-chain. This is analogous to the Ramsey problem, where every coloring on a large enough structure must have a substructure of a certain type. However, for this problem we obtain the optimal solution:

Theorem 1. $f(w, k)=(w+1)^k$.

In [2] Rado shows that such a bound also exists in term of another parameter. Specifically, he shows that if a poset has a regression with no monotone $(k+1)$-chain, then it cannot be an arbitrory large Boolean algebra. The forbidden size is not known. Harzheim [1] generalizes Rado's result and several other related result. Both [1] and [2] use Ramsey's Theorem [3] as a lemma; since we obtain the complete solution, it is not surprising that we do not use Ramsey's Theorem.

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2. Proof of the upper bound

Assume that P is a (w, k)-poset and that g is a regression on it with no monotone $(k+1)$-chain. We proceed by induction on k. If $k=1$, every element of P must be a minimal element. Otherwise, a non-minimal x and a minimal point below $g(x)$ will form a monotone 2-chain. Minimal elements form an antichain, so $|P| \leqq w$, the bound placed on the width.

Now assume $k > 1$. Let $G(x)$ be the longest monotone chain in P that has x as its top element. We say that x has *effective height* $|G(x)|-1$. No point in P can have effective height k or more; let M be the set of points with effective height $k-1$. If M is deleted from P, then g restricted to $P-M$ is a well-defined regression on $P-M$ having no monotone k-chain. It would fail to be a regression if $g(x) \in M$ for some $x \notin M$, but then x could be placed atop $G(g(x))$ to obtain a monotone $(k+1)$-chain in P. The points in $P-M$ have effective height less than $k-1$ in P, so there are no more monotone k-chains. We conclude that $P-M$ is a $(w, k-1)$-poset.

We claim that M contains at most w fixed-points of g, at most w points mapped by g to any specified point of $P-M$, and no points mapped by g to other points of M. Suppose g has two related fixed-points $x > y$ in M, or two related points $x > y$ in M with $g(x)=g(y)=z$, or two points $x, y \in M$ with $g(x)=y$. Then x can be placed at the top of $G(y)$ to obtain a monotone $(k+1)$-chain in P, since $g(x) \geqq g(y)$. The bound on antichain size completes the claim.

Since $P-M$ is a $(w, k-1)$-poset, its size is at most $f(w, k-1)-1$. Including the points of M yields a total of at most $w+(w+1)[(w+1)^{k-1}-1]=(w+1)^k-1$ elements in P.

3. The construction for the lower bound

We construct a poset P_k having width w and size $(w+1)^k-1$, and we define on it a regression g_k with no monotone $(k+1)$-chain. The explicit construction contains many interesting patterns, but it is simpler and faster to describe the poset and regression inductively. To achieve the upper bound, P_k must add w fixed-points and w points mapping to each point of P_{k-1}.

Let P_1 be a single antichain of size w, at rank 0. For $k > 1$, let $P_k - P_{k-1}$ consist of w disjoint chains, each having $(w+1)^{k-1}$ elements. Add these chains to the top of P_{k-1}, so that every new element lies above every old element. The new elements form $(w+1)^{k-1}$ ranks of w elements each. Thus it follows by induction that P_k has $\frac{1}{w}((w+1)^k-1)$ full ranks of w elements each.

Let g_k restricted to P_{k-1} be g_{k-1}. Let the minimal points of $P_k - P_{k-1}$ be fixed-points. Let g map each remaining rank of $P_k - P_{k-1}$ to a single distinct element of P_{k-1}, as follows. The uppermost w ranks of P_k map to the w minimal elements of P_{k-1}. The next highest w ranks of P_k map to the w elements of P_{k-1} at rank 1, and so on. Since the number of non-minimal ranks in $P_k - P_{k-1}$ equals $|P_{k-1}|$, this is all well-defined.

We must show that g_k on P_k has no monotone $(k+1)$-chain. Since g_{k-1} on P_{k-1} has no monotone k-chain, it suffices to show that no two elements of $P_k - P_{k-1}$

can appear in a single monotone chain. This is immediate from the construction of g_k. Whenever $x > y$ in $P_k - P_{k-1}$, either $g(x)$ and $g(y)$ are unrelated, or $g(x) < g(y)$.

It is also easy to show that each element of $P_k - P_{k-1}$ has effective height $k-1$ under g_k, appearing at the top of a monotone k-chain that has one element from each $P_j - P_{j-1}$.

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