Extensions of matroid covering and packing

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Abstract

Let M be a loopless matroid on E with rank function r_M. Let $\beta(M) = \max_{\emptyset \neq X \subseteq E} \frac{|X|}{r_M(X)}$ and $\varphi(M) = \min_{r_M(X)<r(M)} \frac{|E|-|X|}{r_M(E)-r_M(X)}$. The Matroid Covering and Packing Theorems state that the minimum number of independent sets whose union is E is $\lceil \beta(M) \rceil$, and the maximum number of disjoint bases is $\lfloor \varphi(M) \rfloor$.

We prove refinements of these theorems. If $\beta(M) = k + \varepsilon$, where $k \geq 0$ is an integer and $0 \leq \varepsilon < 1$, then E can be partitioned into $k + 1$ independent sets with one of size at most $\varepsilon \cdot r_M(E)$. If $\varphi(M) = k + \varepsilon$, then M has $k + 1$ disjoint independent sets such that k are bases and the other has size at least $\varepsilon \cdot r_M(E)$.

We apply these results to truncations of cycle matroids to refine graph-theoretic results due to Chen, Koh, and Peng in 1993 and to Chen and Lai in 1996.

Keywords: Matroid, Matroid Covering Theorem, Matroid Packing Theorem, Nash-Williams' Theorem, Tutte's Theorem.

1 Introduction

Given a finite set E, let 2^E be the family of subsets of E. A *hereditary family* on E is a family $\mathcal{I} \subseteq 2^E$ such that every subset of a member of $\mathcal{I}$ belongs to $\mathcal{I}$. A nonempty hereditary family can be specified by its members, its maximal members, its minimal

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nonmembers, etc.; together these aspects form a *hereditary system*. Matroids are special hereditary systems that can be defined in many equivalent ways; we use the following:

> A nonempty hereditary family $\mathcal{I} \subseteq 2^E$ is the family of *independent sets* of a *matroid with ground set* E if whenever $I, I' \in \mathcal{I}$ with $|I'| > |I|$, there is an element $e \in I' - I$ such that $I \cup \{e\} \in \mathcal{I}$. This added property is the *augmentation property*.

For a matroid M on E, we use $\mathcal{I}_M$ to denote its family of independent sets. The maximal independent sets, which by augmentation all have the same size, are called *bases*; $\mathcal{B}_M$ denotes the family of bases. The *rank function* r_M assigns to each $X \subseteq E$ its *rank*, which is the maximum size of a member of $\mathcal{I}_M$ contained in X (equivalently, the maximum size of the intersection of X with a base). The *rank* of the matroid is $r_M(E)$, also written as $r(M)$. For additional background and terminology on matroids, see [16, 21].

An element e of E is a *loop* if it appears in no independent set. Since we are interested in packing and covering E with independent sets, all matroids in this paper have no loops, and we do not further mention this restriction. For a matroid M on E, the following two parameters are our main interest:

$$\beta(M) = \max_{\varnothing \neq X \subseteq E} \frac{|X|}{r_M(X)}, \quad \varphi(M) = \min_{r_M(X) < r(M)} \frac{|E| - |X|}{r_M(E) - r_M(X)}.$$

Letting $X = E$ and $X = \varnothing$ respectively yield $\beta(M) \geq \frac{|E|}{r(M)} \geq \varphi(M)$.

The Matroid Covering Theorem (Theorem 1.1) gives the minimum number of independent sets in a matroid M needed to cover (equivalently, partition) the ground set E. The Matroid Packing Theorem (Theorem 1.2) gives the maximum number of pairwise disjoint bases contained in E.

Theorem 1.1 (Edmonds [5]). *For matroid M on E, the minimum number of independent sets whose union is E is $\lceil \beta(M) \rceil$.*

Theorem 1.2 (Edmonds [6]). *For matroid M on E, the maximum number of disjoint bases is $\lfloor \varphi(M) \rfloor$.*

The next two theorems refine Theorems 1.1 and 1.2.

Theorem 1.3. *Let M be a matroid on E. If $\beta(M) = k + \varepsilon$, where $k \in \mathbb{N}$ and $0 \leq \varepsilon < 1$, then E can be partitioned into $k+1$ independent sets with one of size at most $\varepsilon \cdot r(M)$.*

Theorem 1.4. *Let M be a matroid on E. If $\varphi(M) = k + \varepsilon$, where $k \in \mathbb{N}$ and $0 \leq \varepsilon < 1$, then M has k disjoint bases plus an independent set of size at least $\varepsilon \cdot r(M)$.*

In Section 2, we prove Theorems 1.3 and 1.4 and extend them further to seek coverings and packings by sets of restricted size. In Section 3, we apply these results to graphs, where sharpness examples also prove sharpness of the matroid results.

2 Extensions of Matroid Covering and Packing

Edmonds–Fulkerson [7] and Nash-Williams [15] independently proved the Matroid Union Theorem (Theorem 2.1 below). The *union* of matroids $M_1, \ldots, M_k$ on a set E, denoted $M_1 \vee \cdots \vee M_k$, is the hereditary system whose independent sets are $\{I_1 \cup \cdots \cup I_k \colon\ I_i \in \mathcal{I}_{M_i}\}$.

Theorem 2.1 (Matroid Union Theorem; Edmonds–Fulkerson [7], Nash-Williams [15]). *The union of matroids $M_1, \ldots, M_k$ on E with rank functions $r_1, \ldots, r_k$ is a matroid, with rank function r given by $r(X) = \min_{Y \subseteq X}(|X - Y| + \sum_{i=1}^{k} r_i(Y))$.*

Edmonds ([5, 6]) observed that the Matroid Union Theorem implies Theorem 1.1 and Theorem 1.2. Here we similarly use the Matroid Union Theorem to prove the refined versions, Theorems 1.3 and 1.4. We need another matroid concept. The *j-truncation* of a matroid M is the hereditary system whose independent sets are all the sets of size at most j that are independent in M. Since the j-truncation inherits the augmentation property from the augmentation property for M, the j-truncation is also a matroid.

Theorem 1.3. *Let M be a matroid on E. If $\beta(M) = k + \varepsilon$, where $k \in \mathbb{N}$ and $0 \le \varepsilon < 1$, then E can be partitioned into $k + 1$ independent sets with one of size at most $\varepsilon \cdot r(M)$.*

Proof. Let $M_1, \ldots, M_k$ be copies of M on E, and let M_{k+1} be the $\varepsilon r(M)$-truncation of M. Let $M' = M_1 \vee \cdots \vee M_k \vee M_{k+1}$. The set E is the union of $k + 1$ independent sets in M with one of size at most $\varepsilon r(M)$ if and only if E is independent in M'.

By the Matroid Union Theorem, requiring $r_{M'}(E) \ge |E|$ is equivalent to requiring $|E| - |Y| + \sum_{i=1}^{k+1} r_i(Y) \ge |E|$ for all $Y \subseteq E$. Since $r_i(Y) = r_M(Y)$ for $1 \le i \le k$, the set E decomposes as desired if and only if $kr_M(Y) + r_{k+1}(Y) \ge |Y|$ for all $Y \subseteq E$.

The definition of $\beta(M) = k + \varepsilon$, plus $r_{k+1}(Y) = \min\{\varepsilon r(M), r_M(Y)\}$, yields $|Y| \le (k+\varepsilon) r_M(Y) \le kr_M(Y) + r_{k+1}(Y)$ for all $Y \subseteq E$. Hence the needed condition holds. □

Theorem 1.4. *Let M be a matroid on E. If $\varphi(M) = k+\varepsilon$, where $k \in \mathbb{N}$ and $0 \le \varepsilon < 1$, then M has k disjoint bases plus an independent set of size at least $\varepsilon \cdot r(M)$.*

Proof. Let $M_1, \ldots, M_k$ be copies of M on E, and let M_{k+1} be the $\varepsilon r(M)$-truncation of M. Let $M' = M_1 \vee \cdots \vee M_k \vee M_{k+1}$. The set E contains k disjoint bases plus an independent set of size at least $\varepsilon r(M)$ that is disjoint from them if and only if $r_{M'}(E) \ge (k+\varepsilon) r_M(E)$.

By the Matroid Union Theorem, this requires $|E| - |Y| + \sum_{i=1}^{k+1} r_i(Y) \ge (k + \varepsilon) r_M(E)$ for all $Y \subseteq E$. The inequality holds when $r_M(Y) = r_M(E)$, so suppose $r_M(Y) < r_M(E)$.

By definition, $\varphi(M) = k+\varepsilon$ yields $|E| - |Y| \ge (k+\varepsilon)(r_M(E) - r_M(Y)) \ge (k+\varepsilon) r_M(E) - \sum_{i=1}^{k+1} r_i(Y)$, where the last inequality follows from $r_{k+1}(Y) = \min\{\varepsilon r(M), r_M(Y)\}$. Since the inqualities hold for all $Y \subseteq E$, the needed condition holds. □

To further extend the theorems by restricting the sizes of the independent sets being used, we introduce additional concepts.

Definition 2.2. Let t denote a fixed positive integer. A *t-independent set* in a matroid M is an independent set of size at most t. A *t-base* is an independent set of size exactly t. Define parameters $\beta_t(M)$ and $\varphi_t(M)$ by

$$\beta_t(M) = \max_{\varnothing \neq X \subseteq E} \frac{|X|}{\min\{t, r_M(X)\}}, \quad \varphi_t(M) = \min_{r_M(X)<t} \frac{|E| - |X|}{t - r_M(X)}.$$

We next apply Theorems 1.3 and 1.4 to the matroid truncation M_t.

Corollary 2.3. *Let M be a matroid on E with rank at least t. If $\beta_t(M) = k + \varepsilon$, where $k \in \mathbb{N}$ and $0 \leq \varepsilon < 1$, then E is covered by $k+1$ t-independent sets, one with size at most εt.*

Proof. Since $r_{M_t}(X) = \min\{t, r_M(X)\}$, we have $\beta_t(M) = \beta(M_t)$. Hence applying Theorem 1.3 to M_t yields a covering of E by $k+1$ independent sets in M_t, with one of size at most $\varepsilon \cdot r(M_t)$. □

Corollary 2.4. *Let M be a matroid on E with rank at least t. If $\varphi_t(M) = k + \varepsilon$, where $k \in \mathbb{N}$ and $0 \leq \varepsilon < 1$, then M has k disjoint t-bases plus an independent set of size at least εt.*

Proof. Since $r(M_t) = t$, and $r_{M_t}(X) = r_M(X)$ when $r_{M_t} < t$, we have $\varphi_t(M) = \varphi(M_t)$. Hence applying Theorem 1.4 to M_t yields k disjoint bases in M_t (t-bases in M) plus an independent set of size at least $\varepsilon \cdot r(M_t)$. □

3 Applications to Graphs

In this section, we consider packing and covering of finite loopless multigraphs by forests, using the term "graph" to allow multiedges. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G, respectively.

A *decomposition* of a graph G is a set of edge-disjoint subgraphs with union G. The *arboricity* of G is the minimum size of a decomposition of G into forests. The *fractional arboricity* of G, introduced by Payan [18] (also [1]) and here denoted $\Upsilon_1(G)$, is defined by

$$\Upsilon_1(G) = \max_{\varnothing \neq H \subseteq G} \frac{|E(H)|}{|V(H)| - 1}.$$

The Arboricity Theorem of Nash-Williams [14] characterizes when a graph has arboricity at most k. It has been proved many ways (see [3, 8, 10, 11, 17, 19, 22, 23]).

Theorem 3.1 ([14])**.** *A graph G decomposes into k forests if and only if $\Upsilon_1(G) \le k$.*

Let G be a graph and let X be a subset of $V(G)$. To *shrink* X is to delete all edges joining vertices of X and then merge the vertices of X into a single vertex. We denote the resulting graph by G/X. More generally, given a partition $\mathcal{P}$ of $V(G)$ into nonempty parts $V_1, \ldots, V_t$, we *shrink* $\mathcal{P}$ by shrinking each set V_j in $\mathcal{P}$. We denote the resulting graph by $G/\mathcal{P}$; note that $G/\mathcal{P}$ may have multiedges but not loops. The *size* $|\mathcal{P}|$ of a partition $\mathcal{P}$ is the number of parts.

We introduce the *fractional packing number* $\nu_1(G)$, defined by

$$\nu_1(G) = \min_{|\mathcal{P}|>1} \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|-1}.$$

Nash-Williams [13] and Tutte [20] independently characterized when a graph has k edge-disjoint spanning trees.

Theorem 3.2 ([13, 20])**.** *A graph G has k edge-disjoint spanning trees if and only if $\nu_1(G) \ge k$.*

A graph with maximum degree at most d is *d-bounded*. Our research was motivated by [9] and by a conjecture of Montassier, Ossona de Mendez, Raspaud, and Zhu [12] known as the Nine Dragon Tree Conjecture (honoring a tree in Kaohsiung, Taiwan that is far from acyclic). The conjecture was eventually proved by Jiang and Yang [11].

Theorem 3.3 (Nine Dragon Tree (NDT) Theorem [11])**.** *Let G be a graph, and let k and d be non-negative integers. If $\Upsilon_1(G) \le k + \frac{d}{k+d+1}$, then G decomposes into $k+1$ forests with one being d-bounded.*

From the general viewpoint of refining theorems that involve conditions given by integer bounds, an appealing restatement of the theorem may be the following.

Theorem 3.3′. *If $\Upsilon_1(G) \le k + \varepsilon$, then G decomposes into $k+1$ forests with one being $\frac{\varepsilon}{1-\varepsilon}(k+1)$-bounded.*

Chen, Koh, and Peng [2] generalized Theorems 3.1 and 3.2 by restricting the number of edges in the forests used in the decomposition or packing. The proofs in [2] were graph-theoretic. Matroidal proofs of these results, using matroid truncation, were given by Chen and Lai [4]. In order to state the results, we introduce additional terminology.

Definition 3.4. Let i denote a fixed positive integer. An *i-forest* in a graph G is a spanning forest of G that has at least i components (that is, at most $|V(G)| - i$ edges). An *i-tree* in G is an i-forest having exactly i components (that is, $|V(G)| - i$ edges). For a graph G with more than i vertices, define the *fractional i-arboricity* $\Upsilon_i(G)$ and the *fractional i-packing number* $\nu_i(G)$ by

$$\Upsilon_i(G) = \max_{\varnothing \neq H \subseteq G} \frac{|E(H)|}{\min\{|V(G)| - i, |V(H)| - 1\}}, \quad \nu_i(G) = \min_{|\mathcal{P}| > i} \frac{|E(G/\mathcal{P})|}{|\mathcal{P}| - i},$$

where the minimum is taken over all nontrivial partitions of $V(G)$.

The names of these parameters reflect that Theorems 3.1 and 3.2 are the special cases given by setting $i = 1$ in the following.

Theorem 3.5 ([2, 4]). *For $k \in \mathbb{N}$, a graph G decomposes into k i-forests if and only if $\Upsilon_i(G) \leq k$.*

Theorem 3.6 ([2, 4]). *For $k \in \mathbb{N}$, a graph G has k edge-disjoint i-trees if and only if $\nu_i(G) \geq k$.*

The *cycle matroid* $M(G)$ of a graph G has ground set $E(G)$, and its independent sets are the edge sets of forests. Since this hereditary family satisfies the augmentation property, $M(G)$ is a matroid. A spanning forest with i components in an n-vertex graph has $n - i$ edges. Hence the rank of a set $X \subseteq E(G)$ in $M(G)$ is $n - i$, where i is the number of components of G_X. Henceforth let $n = |V(G)|$.

Using only i-forests in the covering and packing of G is equivalent to using only independent sets in the $(n-i)$-truncation of $M(G)$. The content of [4] is that Theorems 3.5 and 3.6 are the special cases of Theorems 1.1 and 1.2 for truncations of cycle matroids.

We extend Theorems 3.5 and 3.6 by considering non-integer bounds on the parameters. The results (except for the sharpness) are special cases of Corollaries 2.3 and 2.4.

Corollary 3.7. *Let G be an n-vertex graph. If $\Upsilon_i(G) = k + \varepsilon$ with $0 \leq \varepsilon < 1$, then G decomposes into $k+1$ i-forests with one having at most $\varepsilon(n-i)$ edges, and this is sharp.*

Proof. The i-forests in G are precisely the $(n-i)$-independent sets in $M(G)$. Hence the first claim follows immediately from Corollary 2.3 if $\Upsilon_i(G) = \beta_{n-i}(M(G))$.

In computing $\Upsilon_i(G)$, we may restrict the maximization to connected induced subgraphs. For such subgraphs H, the rank of $E(H)$ in $M(G)$ is $|V(H)| - 1$. Hence for such H, the ratio being maximized is the ratio being maximized in $\beta_{n-i}(M(G))$.

In computing $\beta_t(M)$ for a matroid M, we may restrict the maximization to *closed* sets X, meaning that no element can be added without increasing the rank. When M is a cycle matroid, such sets are the edge sets of induced subgraphs. Considering components of induced subgraphs further restricts the computation to be the same as for $\Upsilon_i(G)$.

For sharpness, let G be a graph formed by picking k edge-disjoint i-trees on the same n vertices plus an i-forest F sharing no edges with them. Let $\varepsilon = \frac{|E(F)|}{n-i}$. Existence of the given decomposition requires $\Upsilon_i(G) \le k + \varepsilon$, and $\Upsilon_i(G) \le \frac{|E(H)|}{|V(H)|-i} = k + \varepsilon$ when $H = G$. Hence the condition holds and there is no decomposition of size $k+1$ in which any i-forest has fewer than $\varepsilon(n-1)$ edges, since every i-forest has at most the number of edges in an i-tree. □

Corollary 3.8. *Let G be an n-vertex graph. If $\nu_i(G) = k + \varepsilon$ with $0 \le \varepsilon < 1$, then G has $k+1$ edge-disjoint i-forests such that k are i-trees and the other has at least $\varepsilon(n-i)$ edges, and this is sharp.*

Proof. For i-trees in G, we seek independent sets of size $n-i$ in $M(G)$, so we set $t = n-i$ in Corollary 2.4. For $X \subseteq E(G)$, let G_X be the spanning subgraph of G with edge set X, having s_X components. A set $X \in E(G)$ has rank less than $n-i$ in $M(G)$ if and only if $s_X > i$. Note that $r_{M(G)}(X) = n - s_X$. The components of G_X form a partition $\mathcal{P}$ of $V(G)$ with s_X parts. To minimize $\frac{|E|-|X|}{t-r(X)}$ without changing the denominator, we may assume that all edges whose endpoints lie in the same part in $\mathcal{P}$ belong to X. In that case, $|E| - |X| = |E(G/\mathcal{P})|$. Also $|\mathcal{P}| - i = n - i - (n - |\mathcal{P}|) = t - r(X)$. Hence the condition of Corollary 2.4 reduces to $\nu_i(G) \ge k$.

For sharpness, let G be a graph formed by picking k edge-disjoint i-trees with the same vertex set plus an i-forest F sharing no edges with them. Let $\varepsilon = \frac{|E(F)|}{|V(G)|-i}$. Existence of the given subgraphs requires $\nu_i(G) \ge k + \varepsilon$, and $\nu_i(G) \le \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|-i} = k + \varepsilon$ when $\mathcal{P}$ is the partition consisting of $V(G)$ singleton sets. Hence the condition holds and there are no edges beyond the packing guaranteed above. □

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