Matching and Edge-connectivity in Regular Graphs

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Abstract

Henning and Yeo proved a lower bound for the minimum size of a maximum matching in a connected k-regular graphs with n vertices; it is sharp infinitely often. In an earlier paper, we characterized when equality holds. In this paper, we prove a lower bound for the minimum size of a maximum matching in an l-edge-connected k-regular graph with n vertices, for $l \geq 2$ and $k \geq 4$. Again it is sharp for infinitely many n, and we characterize when equality holds in the bound.

1 Introduction

Petersen [11] proved that every cubic graph with no cut-edges has a perfect matching. It is natural to ask what happens when there are cut-edges. The *matching number* of a graph G, written $\alpha'(G)$, is the maximum size of a matching in G. Biedl et al. [2] determined the smallest matching number among connected cubic graphs with n vertices. Henning and Yeo [7] extended this to connected k-regular n-vertex graphs for appropriate n. In [10], we gave a short proof of their bound for odd k, characterized the extremal graphs, and studied the relationship between the matching number and the number of cut-edges.

Chartrand et al. [4] determined the minimum number of vertices in a k-regular $(k-2)$-edge-connected graph with no perfect matching. Niessen and Randerath [9] extended this to k-regular l-edge-connected graphs. In another direction, Broere et al. [3] gave a formula for the minimum size of a matching among k-regular $(k-2)$-edge-connected graphs with a fixed number of vertices. Katerinis [8] considered the analogue for vertex connectivity. Our

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lower bound for the minimum size of a matching in a k-regular l-edge-connected graph with n vertices implies these various results when the parameters are set to appropriate values. Although this bound is sharp infinitely often when $l > 0$, for $l = 0$ the bound in [7, 10] is stronger. In Section 3, we characterize the graphs achieving equality in the bound for $l > 0$, and in Section 4 we show that there are infinitely many of them.

We note that a forthcoming paper by Cioabă and O [5] explores the relationship among matching, edge-connectivity, and eigenvalues.

2 Lower Bound

We use the Berge–Tutte Formula for the matching number. The deficiency $\mathrm{def}(S)$ of a vertex set S in G is defined by $\mathrm{def}(S) = o(G - S) - |S|$, where $o(H)$ is the number of odd components in a graph H. Tutte [13] proved that a graph G has a 1-factor if and only if $\mathrm{def}(S) \le 0$ for all $S \in V(G)$. The equivalent Berge–Tutte Formula (see Berge [1]) states that $\alpha'(G) = \min_{S \subseteq V(G)} \frac{1}{2}(n - \mathrm{def}(S))$.

In our counting arguments based on the Berge–Tutte Formula, we consider edge cuts that separate an odd number of vertices from the rest of the graph. Since the degree sum of any graph is even, it follows that for such a cut in a k-regular graph, the size of the cut has the same parity as k. Thus the bound when the edge-connectivity has opposite parity from the degree is the same as the bound for the next larger value of edge-connectivity. That is, it suffices to study $(2t + 1)$-edge-connected $(2r + 1)$-regular graphs and $2t$-edge-connected $2r$-regular graphs.

Since $2r^2 + r = 2(r + \frac{1}{4})^2 - \frac{1}{8}$, the formula in Theorem 2.2 has a very similar flavor to that in Theorem 2.1. In the special case $t = r - 1$, the formulas in Theorem 2.1 and Theorem 2.2 reduce to essentially the formula in Broere et al. [3]. Also when n is even and less than $2(k\lceil k/2\rceil + k - 1)$, those formulas imply that a $(k - 2)$-edge-connected k-regular graph with n vertices has a perfect matching; this is the result of Chartrand et al. [4]. More generally, for l-edge-connected graphs, the threshold on the number of vertices for graphs without perfect matchings in Niessen and Randerath [9] also follows.

Theorem 2.1. *If G is a $(2t+1)$-edge-connected $(2r+1)$-regular graph with n vertices, where $0 \le t \le r$, then $\alpha'(G) \ge \frac{n}{2} - (\frac{r-t}{2(r+1)^2+t})\frac{n}{2}$.*

Proof. Let S be a set with maximum deficiency. Thus, $\alpha'(G) = \frac{1}{2}(n - \mathrm{def}(S))$, where $\mathrm{def}(S) = o(G - S) - |S|$. Let c_i count the odd components of $G - S$ having exactly i edges to S; note that c_i is nonzero only when i is odd. Let $c = c_{(2t+1)} + \cdots + c_{(2r-1)}$, and let $c' = o(G - S) - c$. Each odd component counted by c' has at least $2r + 1$ edges to S. Note that for $2t + 1 \le i \le 2r - 1$, each odd component of $G - S$ having exactly i edges to S has at least $2r+3$ vertices. (For any vertex set Q with q vertices, $[Q,\overline{Q}] \geq q(2r+2-q)$, and this lower bound is at least $2r+1$ when $q \leq 2r+1$.)

Since the edges incident to S include the edges joining S to odd components of $G-S$, we have $(2r+1)|S| \geq (2r+1)c' + (2t+1)c$, and hence $|S| \geq c' + (\frac{2t+1}{2r+1})c \geq (\frac{2t+1}{2r+1})c$. Therefore, $n \geq |S| + c(2r+3) \geq (\frac{2t+1}{2r+1} + 2r+3)c$, which yields $c \leq (\frac{2r+1}{4r^2+8r+4+2t})n$. We compute

$$\begin{aligned} \mathrm{def}(S) &= (c+c') - |S| \leq c - \frac{2t+1}{2r+1}c = \frac{2(r-t)}{2r+1}c \\ &\leq \frac{2(r-t)}{2r+1}\left(\frac{2r+1}{4r^2+4r+4+2t}\right)n = \frac{(r-t)n}{2(r+1)^2+t}. \end{aligned}$$

□

As noted earlier, the same bound holds for $2t$-edge-connected $(2r+1)$-regular graphs. Similarly, the bound in the next theorem also holds for $(2t-1)$-edge-connected $2r$-regular graphs.

Theorem 2.2. *If G is a $2t$-edge-connected $2r$-regular graph with n vertices, where $1 \leq t \leq r$ and $r \geq 2$, then $\alpha'(G) \geq \frac{n}{2} - (\frac{r-t}{2r^2+r+t})\frac{n}{2}$.*

Proof. The proof is similar to that of Theorem 2.1. Defining S and c_i as in that proof, here the contributions are nonzero only when i is even and at least $2t$. Also, for $2t \leq i \leq 2r-2$, the odd components of $G-S$ having i edges to S have at least $2r+1$ vertices. The same steps as before then lead to $\mathrm{def}(S) \leq \frac{(r-t)n}{2r^2+r+t}$. □

3 Characterization

We begin by developing properties that graphs achieving equality in the bounds of Theorem 2.1 and Theorem 2.2 must satisfy. We will show that all graphs with these properties meet the bound, thereby characterizing equality. In the next section, we explicitly construct infinitely many graphs achieving equality, for each fixed r and t with $r > t > 0$. The needed properties lead us to define special families. Note that $\delta(H)$ and $\Delta(H)$ denote the minimum and maximum vertex degrees in a graph H.

Definition 3.1. A *nontrivial cut* in a graph G is an edge cut with at least two vertices on each side. A $(2r+1, 2t+1)$-*bullet* is a graph H satisfying the following conditions :
(1) $|V(H)| = 2r+3$,
(2) $\delta(H) \geq \max\{2r-2t, r+1\}$,
(3) $\Delta(H) = 2r+1$,
(4) $|E(\overline{H})| = r+t+2$ and
(5) every nontrivial cut has at least $2r+1$ edges.

Definition 3.2. Let B be a graph with $\Delta(B) = a$ such that $\sum_{v\in V(B)}(a - d_B(v)) = b$. If u is a vertex of degree b in a graph H, then *splicing B into u* means deleting u and replacing each edge of the form uw in it with an edge from w to a vertex of B, in such a way that each vertex of B now has degree a.

Definition 3.3. An (a, b)-*biregular* graph is a bipartite graph with partite sets X and Y such that vertices in X have degree a, and those in Y have degree b. For $r > t \geq 1$, let $\mathcal{H}_{r,t}$ be the family of $(2t+1)$-edge-connected $(2r+1, 2t+1)$-biregular graphs, let $\mathcal{B}_{r,t}$ be the family of $(2r + 1, 2t + 1)$-bullets, and let $\mathcal{F}_{r,t}$ be the family of graphs obtained from graphs H in $\mathcal{H}_{r,t}$ by splicing a $(2r + 1, 2t + 1)$-bullet into each vertex having degree $2t + 1$ in H.

Lemma 3.4. *Every graph in $\mathcal{F}_{r,t}$ is $(2t + 1)$-edge-connected and $(2r + 1)$-regular.*

Proof. Let H be a graph in $\mathcal{H}_{r,t}$ with partite sets R and T such that vertices in R have degree $2r+1$ and those in T have $2t+1$. If G is derived from H by splicing bullets into all vertices of T, then by the construction, G is $(2r+1)$-regular. For the edge-connectivity, it suffices to show that splicing a bullet B into one vertex u of degree $2t+1$ in a $(2t+1)$-edge-connected graph J yields a $(2t + 1)$-edge-connected graph J'.

Let F be a set of edges in J' with $|F| \leq 2t$. Since J is $(2t+1)$-edge-connected, $G - F$ is connected, where edges joining $V(J' - u)$ to $V(B)$ correspond to edges joining $V(J - u)$ to u. Thus $J' - F$ has a path from each vertex of $V(J') - V(B)$ to $V(B)$. Hence it suffices to show that each vertex of B can reach every other vertex of B in $J' - F$.

If $V(B)$ does not induce a connected subgraph in $J' - F$, then F contains an edge cut of B. Since B is a bullet, every nontrivial edge cut has size at least $2r + 1$. Hence F contains all edges of B incident to some vertex x. Let $S = V(B) - \{x\}$. In $J' - F$, the subgraph induced by S is connected, since otherwise F contains a nontrivial edge cut of B. Hence it suffices to show that $J' - F$ has a path from x to S through vertices outside B.

Since $d_{J'}(x) = 2r + 1 > 2t$ and F contains all edges of B incident to x, some edge from x to $V(J') - V(B)$ remains in $J' - F$; let y be a neighbor of x via such an edge. Also, since $d_B(x) \geq r + 1 > t + 1$, there are fewer edges from x to $V(J') - V(B)$ than to S. Hence $|[S, \overline{S}]| \geq 2t + 1$, and an edge e remains in $J' - F$ from S to $V(J') - V(B)$; let w be the endpoint outside S. Since F has at least one edge in B, it follows that $J - F$ is 2-edge-connected. Hence it has a cycle C through uy and uw. Now $C - u$ completes a path with xy and e from x to S in $J' - F$. □

Theorem 3.5. *For $t, r \in \mathbb{N}$ with $t < r$, a $(2t + 1)$-edge-connected $(2r + 1)$-regular graph G achieves equality in the bound of Theorem 2.1 if and only if it is in $\mathcal{F}_{r,t}$.*

Proof. First, suppose that G arises from $H \in \mathcal{H}_{r,t}$ by splicing in bullets. By Lemma 3.4, G is $(2t + 1)$-edge-connected and $(2r + 1)$-regular. Let R and T be the sets of vertices with degree $2r+1$ and degree $2t+1$ in H, respectively. Note that $|T| = \frac{2r+1}{2t+1}|R|$ and $|V(G)| = |R| + (2r+3)|T|$. Hence $|R| = \frac{(2t+1)n}{4r^2+8r+4+2t}$. Also, there are $|T|$ odd components in $G-R$, which implies that

$$\operatorname{def}(G) = \operatorname{def}(R) = |T| - |R| = \left(\frac{2r+1}{2t+1} - 1\right)|R| = \frac{2(r-t)}{2t+1}\frac{(2t+1)n}{4r^2+8r+4+2t} = \frac{(r-t)n}{2(r+1)^2+t}.$$

Theorem 2.1 yields $\operatorname{def}(G) \le \frac{(r-t)n}{2(r+1)^2+t}$; hence equality holds.

Conversely, we want to show that every graph G achieving equality in Theorem 2.1 is in $\mathcal{F}_{r,t}$. By definition, G is $(2t+1)$-edge-connected and $(2r+1)$-regular. Let S be a maximal vertex subset with maximum deficiency in G. By this maximality, $G-S$ has no even components. Achieving equality in the computation of Theorem 2.1 requires the following conditions:
(i) for $i \ge 2t+3$, no odd component in $G-S$ has i edges to S,
(ii) every odd component of $G-S$ has exactly $2r+3$ vertices, and
(iii) S is an independent set.

Let H be the graph obtained from G by shrinking each odd component in $G-S$ to a single vertex. If H is not $(2t+1)$-edge-connected, then G would not be. Thus the resulting graph is in $\mathcal{H}_{r,t}$, and G is obtained by splicing each odd component of $G-S$ into a vertex of H with degree $2t+1$.

Now, we consider an odd component C of $G-S$. It remains only to show that C is a $(2r+1, 2t+1)$-bullet. Let A be a nonempty proper subset of $V(C)$, and let $l = |[A, A']|$, where $A' = V(C) - A$. We may assume that $|A| \le |A'|$. Letting $a = |A|$, we then have $a \le r+1$. We show that $l \ge 2r+1$, except possibly when $a = 1$. We have $(2r+1)a = \sum_{v \in A} d_G(v) \le a(a-1) + l + 2t + 1$, which implies that

$$a(2r+2-a) \le l + 2t + 1. \tag{1}$$

If $2 \le a \le r+1$, then $a(2r+2-a) \ge 4r$, and $l \ge 4r-2t-1 \ge 4r-2(r-1)-1 = 2r+1$. If $a = 1$, then (1) yields $l \ge 2r-2t$. Let $b = |[A, S]|$ and $c = |[A', S]|$. Note that $b + c = 2t+1$ and $l + b = 2r+1$, which yields $l - c = 2r - 2t$. Since G is $(2t+1)$-edge-connected, $l + c \ge 2t+1$. Adding $l - c = 2r - 2t$ yields $2l \ge 2r+1$, which yields $l \ge r+1$. Thus, $\delta(C) \ge \max\{2r-2t, r+1\}$.

Since G is $(2r+1)$-regular and the number of edges joining C to $G - V(C)$ is less than $|V(C)|$, it follows that $\Delta(C) = 2r+1$. Since C has $\frac{(2r+1)(2r+3)-(2t+1)}{2}$ edges, we have $|E(\overline{C})| = r+t+2$. Also, we have shown that nontrivial cuts in C have size at least $2r+1$. Hence $C \in \mathcal{B}_{r,t}$. □

Similarly, we can characterize when the matching number for even-regular graphs is minimized. When the parameters are even, we use a slightly different definition of bullet.

Definition 3.6. A $(2r, 2t)$-*bullet* is a graph H satisfying the following conditions :
(1) $|V(H)| = 2r + 1$,
(2) $\delta(H) \geq \max\{2r - 2t, r\}$,
(3) $\Delta(H) = 2r$,
(4) $|E(\overline{H})| = t$ and
(5) every nontrivial cut has at least $2r$ edges.

Let $\mathcal{H}'_{r,t}$ be the family of $2t$-edge-connected $(2r, 2t)$-biregular bipartite graphs, let $\mathcal{B}'_{r,t}$ be the family of $(2r, 2t)$-bullets, and let $\mathcal{F}'_{r,t}$ be the family of graphs obtained from a graph H in $\mathcal{H}'_{r,t}$ by splicing a $(2r, 2t)$-bullet in $\mathcal{B}'_{r,t}$ into each vertex having degree $2t$ in H'.

Arguments similar to the proofs of Lemma 3.4 and Theorem 3.5 yield the following results.

Lemma 3.7. *Every graph in $\mathcal{F}'_{r,t}$ is $2t$-edge-connected and $2r$-regular graph.*

Theorem 3.8. *For $t, r \in \mathbb{N}$ with $t < r$, a $2t$-edge-connected $2r$-regular graph G achieves equality in the bound of Theorem 2.2 if and only if it is in $\mathcal{F}'_{r,t}$.*

4 Construction of an Infinite Family

Finally, we construct infinitely many graphs in the families $\mathcal{F}_{r,t}$ and $\mathcal{F}'_{r,t}$. It suffices to have at least one $(2r+1, 2t+1)$-bullet and $(2r, 2t)$-bullet and infinitely many graphs in $\mathcal{H}_{r,t}$ and $\mathcal{H}'_{r,t}$. Let $B_r = \overline{P_3 + rK_2}$. For $0 \leq t \leq r$, let $B_{r,t}$ be a graph obtained from B_r by deleting a matching of size t whose edges are not incident to the unique vertex of degree $2r$ in B_r. Similarly, let $B'_{r,t}$ be the graph obtained from K_{2r+1} by deleting a matching of size t.

We show first that $B_{r,t} \in \mathcal{B}_{r,t}$ and $B'_{r,t} \in \mathcal{B}'_{r,t}$. By construction, the number of vertices, minimum degree, maximum degree, and number of missing edges are as required. It remains only to show that the nontrivial cuts are big enough. Every nontrivial cut in K_{2r+3} has size at least $2(2r+1)$, and we delete exactly $r + t + 2$ edges to form $B_{r,t}$, so nontrivial cuts in $B_{r,t}$ have size at least $2r + 1$ (since $t < r$). Similarly, every nontrivial cut in K_{2r+1} has size at least $2(2r - 1)$, and we delete only t edges to form $B'_{r,t}$.

When H is a (a, b)-biregular graph with edge-connectivity b, where $a = 2r + 1$ and $b = 2t + 1$, splicing $B_{r,t}$ into each vertex having degree $2t + 1$ in H preserves $(2t + 1)$-edge-connectedness, by the argument for Lemma 3.4. Similarly, for $a = 2r$ and $b = 2t$, splicing $B'_{r,t}$ into each vertex having degree $2t$ in H preserves $2t$-edge-connectedness, by the corresponding argument for Lemma 3.7.

Hence it remains only to show that there are infinitely many graphs in the families $\mathcal{F}_{r,t}$ and $\mathcal{F}'_{r,t}$. More generally, we show that there are infinitely many (a, b)-biregular graphs with edge-connectivity b when $a \geq b$.

Example 4.1. *Construction of G_k.* For $1 \le i \le k$, let H_i be a copy of $K_{a,b}$ with partite sets X_i and Y_i of sizes a and b, respectively. Choose $S_i \subseteq X_i$ of size b. Delete a matching of size b joining S_i and Y_i. Restore the original vertex degrees by adding a matching of size b joining Y_i and S_{i+1} for each i, with subscript taken modulo k. The resulting graph is G_k.

Elementary lemmas lead us to $\kappa'(G_k) = b$.

Lemma 4.2 ([12])**.** *If G is bipartite with diameter at most 3, then $\kappa'(G) = \delta(G)$.*

Corollary 4.3. *For $a \ge b$, if a graph H is a graph obtained from $K_{a,b}$ by deleting a matching of size b, then $\kappa'(H) = b - 1$.*

Proof. Since H has diameter 3, Lemma 4.2 applies. □

Theorem 4.4. *The edge-connectivity of G_k is b.*

Proof. Note that $\kappa'(G_k) \le \delta(G_k) = b$. To prove equality, consider $F \subseteq E(G_k)$ with $|F| < b$. Let $G' = G_k - F$; we show that G' is connected. Let H_i' be the subgraph of G_k induced by $X_i \cup Y_i$. By Corollary 4.3, each H_i' is $(b-1)$-edge-connected. Therefore, the subgraph of G' induced by $X_i \cup Y_i$ is connected unless $F \subseteq E(H_i')$. Also, since $|F| < b$, there exists an edge in G' joining Y_i and S_{i+1}. Either what remains of each H_i' is connected and has a remaining edge to H_{i+1}', or one H_i' is cut but all edges of G_k outside it remain.

In the latter case, the subgraph $G' - (X_i \cup Y_i)$ is connected. Each vertex in Y_i has a neighbor in X_{i+1}. Each vertex in $X_i - S_i$ has a neighbor in Y_i, since it has degree b in G_k. Each vertex in S_i has a neighbor in Y_{i-1}. Hence G' is connected. □

Corollary 4.5. *There are infinitely many graphs in $\mathcal{F}_{r,t}$ and $\mathcal{F}'_{r,t}$.*

Proof. Use Theorem 4.4 with $a = 2r + 1$ and $b = 2t + 1$ and arbitrary k. □

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