Decomposition of Sparse Graphs

Douglas B. West

Department of Mathematics
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

preprint & slides at
http://www.math.uiuc.edu/~west/pubs/publink.html

Joint work with
Mickael Montassier, Arnaud Pêcher,
André Raspaud, Xuding Zhu

Motivation

Aim: Decompose G into a forest and a subgraph with small max degree.

Why: Upper bound on the game chromatic number.

Motivation

Aim: Decompose G into a forest and a subgraph with small max degree.

Why: Upper bound on the game chromatic number.

Def. ordering game: Alice and Bob alternately choose vertices to make a linear order L.

Motivation

Aim: Decompose G into a forest and a subgraph with small max degree.

Why: Upper bound on the game chromatic number.

Def. ordering game: Alice and Bob alternately choose vertices to make a linear order L.

back degree $b_L(x)$ = $\#\{y \in N_G(x) \colon\ y < x \text{ in } L\}$.

game coloring bound (or "game coloring number")

$$\mathrm{col}_g(G) = \min\{k \colon \text{ Alice guarantees } b_L(x) < k \ \ \forall\, x\}.$$

Motivation

Aim: Decompose G into a forest and a subgraph with small max degree.

Why: Upper bound on the game chromatic number.

Def. ordering game: Alice and Bob alternately choose vertices to make a linear order L.

back degree $b_L(x) = \#\{y \in N_G(x)\colon\ y < x \text{ in } L\}$.

game coloring bound (or "game coloring number")

$\mathrm{col}_g(G) = \min\{k\colon \text{ Alice guarantees } b_L(x) < k \ \ \forall\, x\}$.

coloring bound (or "coloring number")

$\mathrm{col}(G) = \min\{k\colon\ \exists\, L \text{ with } b_L(x) < k \ \ \forall\, x\}$.

Motivation

Aim: Decompose G into a forest and a subgraph with small max degree.

Why: Upper bound on the game chromatic number.

Def. ordering game: Alice and Bob alternately choose vertices to make a linear order L.

back degree $b_L(x) = \#\{y \in N_G(x)\colon\ y < x \text{ in } L\}$.

game coloring bound (or "game coloring number")

$$\mathrm{col}_g(G) = \min\{k\colon \text{ Alice guarantees } b_L(x) < k \ \ \forall\, x\}.$$

coloring bound (or "coloring number")

$$\mathrm{col}(G) = \min\{k\colon\ \exists\, L \text{ with } b_L(x) < k \ \ \forall\, x\}.$$

Obs. $\chi(G) \le \mathrm{col}(G)$.

Motivation

Aim: Decompose G into a forest and a subgraph with small max degree.

Why: Upper bound on the game chromatic number.

Def. ordering game: Alice and Bob alternately choose vertices to make a linear order L.

back degree $b_L(x) = \#\{y \in N_G(x)\colon\ y < x \text{ in } L\}$.

game coloring bound (or "game coloring number")

$$\mathrm{col}_g(G) = \min\{k\colon \text{ Alice guarantees } b_L(x) < k\ \ \forall\, x\}.$$

coloring bound (or "coloring number")

$$\mathrm{col}(G) = \min\{k\colon\ \exists\, L \text{ with } b_L(x) < k\ \ \forall\, x\}.$$

Obs. $\chi(G) \le \mathrm{col}(G)$. $\chi_g(G) \le \mathrm{col}_g(G)$.

Motivation

Aim: Decompose G into a forest and a subgraph with small max degree.

Why: Upper bound on the game chromatic number.

Def. ordering game: Alice and Bob alternately choose vertices to make a linear order L.

back degree $b_L(x) = \#\{y \in N_G(x)\colon\ y < x \text{ in } L\}$.

game coloring bound (or "game coloring number")

$\mathrm{col}_g(G) = \min\{k\colon \text{ Alice guarantees } b_L(x) < k\ \ \forall\, x\}$.

coloring bound (or "coloring number")

$\mathrm{col}(G) = \min\{k\colon\ \exists\, L \text{ with } b_L(x) < k\ \ \forall\, x\}$.

Obs. $\chi(G) \le \mathrm{col}(G)$. $\chi_g(G) \le \mathrm{col}_g(G)$.

- "game" parameters model worst-case Nature

Decomposition and Planar Graphs

- Zhu [1999]: If G decomposes into G_1 and G_2, then $\mathrm{col}_g(G) \leq \mathrm{col}_g(G_1) + \Delta(G_2)$.

Decomposition and Planar Graphs

- Zhu [1999]: If G decomposes into G_1 and G_2, then $\mathrm{col}_g(G) \leq \mathrm{col}_g(G_1) + \Delta(G_2)$.

- Faigle-Kern-Kierstead-Trotter [1993]: If G is a forest, then $\mathrm{col}_g(G) \leq 4$.

Decomposition and Planar Graphs

- Zhu [1999]: If G decomposes into G_1 and G_2, then $\mathrm{col}_g(G) \le \mathrm{col}_g(G_1) + \Delta(G_2)$.

- Faigle-Kern-Kierstead-Trotter [1993]: If G is a forest, then $\mathrm{col}_g(G) \le 4$.

A planar graph with girth g decomposes into a forest and a subgraph with maxdegree k.

	g	k	$\mathrm{col}_g(G)$
Borodin-Kostochka-Sheikh-Yu [2008]	9	1	≤ 5
He-Hou-Lih-Shao-Wang-Zhu [2002]	7	2	≤ 6
"	5	4	≤ 8
Zhu [2008]			≤ 17

Decomposition and Planar Graphs

- Zhu [1999]: If G decomposes into G_1 and G_2, then $\mathrm{col}_g(G) \le \mathrm{col}_g(G_1) + \Delta(G_2)$.

- Faigle-Kern-Kierstead-Trotter [1993]: If G is a forest, then $\mathrm{col}_g(G) \le 4$.

A planar graph with girth g decomposes into a forest and a subgraph with maxdegree k.

	g	k	$\mathrm{col}_g(G)$
Borodin-Kostochka-Sheikh-Yu [2008]	9	1	≤ 5
He-Hou-Lih-Shao-Wang-Zhu [2002]	7	2	≤ 6
"	5	4	≤ 8
Zhu [2008]			≤ 17

Gonçalves [2009] Planar $G \vdash$ three forests, one with maxdegree ≤ 4 (conj. Balogh-Kochol-Pluhar-Yu [2005]).

Maximum Average Degree

- Planar graphs are sparse; a planar n-vertex graph has $\leq 3n - 6$ edges and hence average degree < 6 (same for every subgraph).

Maximum Average Degree

- Planar graphs are sparse; a planar n-vertex graph has $\leq 3n - 6$ edges and hence average degree < 6 (same for every subgraph).

Sparseness measured by Maximum average degree.

Def. $\mathrm{Mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$.

Maximum Average Degree

- Planar graphs are sparse; a planar n-vertex graph has $\leq 3n-6$ edges and hence average degree < 6 (same for every subgraph).

Sparseness measured by Maximum average degree.

Def. $\mathrm{Mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$.

- Euler's Formula $\Rightarrow$ $\mathrm{Mad}(G) < \frac{2g}{g-2}$ if G is planar with girth g.

Maximum Average Degree

- Planar graphs are sparse; a planar n-vertex graph has $\leq 3n-6$ edges and hence average degree < 6 (same for every subgraph).

Sparseness measured by Maximum average degree.

Def. $\mathrm{Mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$.

- Euler's Formula $\Rightarrow$ $\mathrm{Mad}(G) < \frac{2g}{g-2}$ if G is planar with girth g.

Some results on planar graphs (with girth at least g) only use the bound on $\mathrm{Mad}(G)$ and hold more generally.

Our result

Thm. If $\mathrm{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6}$, then G decomposes into a forest and a graph with maxdegree at most k.

Our result

Thm. If $\text{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6}$, then G decomposes into a forest and a graph with maxdegree at most k.

(When $k \leq 3$, both can be required to be forests.)

Our result

Thm. If $\mathrm{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6}$, then G decomposes into a forest and a graph with maxdegree at most k.

Cor. $\mathrm{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6} \Rightarrow \mathrm{col}_g(G) \leq 4 + k$.

Our result

Thm. If $\mathrm{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6}$, then G decomposes into a forest and a graph with maxdegree at most k.

Cor. $\mathrm{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6} \Rightarrow \mathrm{col}_g(G) \leq 4 + k$.

Def. Let $f(k) = \sup\{t : \mathrm{Mad}(G) < t \text{ suffices}\}$.

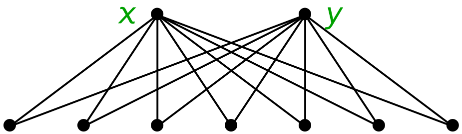
Our result

Thm. If $\text{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6}$, then G decomposes into a forest and a graph with maxdegree at most k.

Cor. $\text{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6} \Rightarrow \text{col}_g(G) \leq 4+k$.

Def. Let $f(k) = \sup\{t : \text{Mad}(G) < t \text{ suffices}\}$.

Ex. $K_{2,2k+2}$ has no such decomposition into F and D.



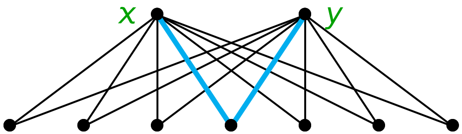
Our result

Thm. If $\mathrm{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6}$, then G decomposes into a forest and a graph with maxdegree at most k.

Cor. $\mathrm{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6} \Rightarrow \mathrm{col}_g(G) \leq 4+k$.

Def. Let $f(k) = \sup\{t : \mathrm{Mad}(G) < t$ suffices$\}$.

Ex. $K_{2,2k+2}$ has no such decomposition into F and D.



At most one 2-vertex has degree 2 in F.

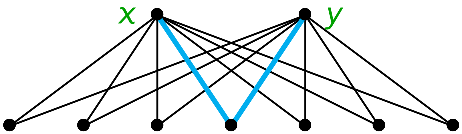
Our result

Thm. If $\mathrm{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6}$, then G decomposes into a forest and a graph with maxdegree at most k.

Cor. $\mathrm{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6} \Rightarrow \mathrm{col}_g(G) \leq 4+k$.

Def. Let $f(k) = \sup\{t : \mathrm{Mad}(G) < t \text{ suffices}\}$.

Ex. $K_{2,2k+2}$ has no such decomposition into F and D.



At most one 2-vertex has degree 2 in F.

$\therefore\ d_F(x) + d_F(y) \leq 2k+3$, and $\Delta(D) \geq k+1$.

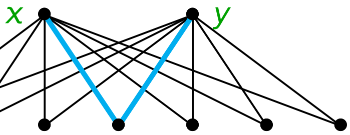
Our result

Thm. If $\mathrm{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6}$, then G decomposes into a forest and a graph with maxdegree at most k.

Cor. $\mathrm{Mad}(G) < 4\frac{k^2+4k+3}{k^2+6k+6} \Rightarrow \mathrm{col}_g(G) \leq 4+k$.

Def. Let $f(k) = \sup\{t : \mathrm{Mad}(G) < t$ suffices$\}$.

Ex. $K_{2,2k+2}$ has no such decomposition into F and D.



At most one 2-vertex has degree 2 in F.

$\therefore\ d_F(x) + d_F(y) \leq 2k+3$, and $\Delta(D) \geq k+1$.

Note $\mathrm{Mad}(K_{2,2k+2}) = \frac{4k+4}{k+2} = 4 - \frac{4}{k+2}$.

Not Quite Sharp

Def. Let $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6} = 4\frac{k^2+4k+3}{k^2+6k+6} = 4 - \frac{8k+12}{k^2+6k+6}$.

Not Quite Sharp

Def. Let $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6} = 4\frac{k^2+4k+3}{k^2+6k+6} = 4 - \frac{8k+12}{k^2+6k+6}$.

Cor. $4 - \frac{8k+12}{k^2+6k+6} = m_k \leq f(k) \leq 4 - \frac{4}{k+2}$.

Not Quite Sharp

Def. Let $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6} = 4\frac{k^2+4k+3}{k^2+6k+6} = 4 - \frac{8k+12}{k^2+6k+6}$.

Cor. $4 - \frac{8k+12}{k^2+6k+6} = m_k \leq f(k) \leq 4 - \frac{4}{k+2}$.

- Another example: subdivision of K_{2k+3}, same $\mathrm{M\alpha d}$.

Not Quite Sharp

Def. Let $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6} = 4\frac{k^2+4k+3}{k^2+6k+6} = 4 - \frac{8k+12}{k^2+6k+6}$.

Cor. $4 - \frac{8k+12}{k^2+6k+6} = m_k \leq f(k) \leq 4 - \frac{4}{k+2}$.

- Another example: subdivision of K_{2k+3}, same Mad.

- m_k is not big enough to imply the earlier results on planar graphs, but $4 - \frac{4}{k+2}$ would be.

Not Quite Sharp

Def. Let $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6} = 4\frac{k^2+4k+3}{k^2+6k+6} = 4 - \frac{8k+12}{k^2+6k+6}$.

Cor. $4 - \frac{8k+12}{k^2+6k+6} = m_k \leq f(k) \leq 4 - \frac{4}{k+2}$.

- Another example: subdivision of K_{2k+3}, same Mad.

- m_k is not big enough to imply the earlier results on planar graphs, but $4 - \frac{4}{k+2}$ would be.

Our method can't improve on m_k.

The Method

Induction on $|V(G)| + |E(G)|$. Since $\mathrm{Mad}(H) \leq \mathrm{Mad}(G)$ when $H \subseteq G$, every proper subgraph has an (F, D)-decomposition, where F is a forest and $\Delta(D) \leq k$.

The Method

Induction on $|V(G)| + |E(G)|$. Since $\mathrm{Mad}(H) \leq \mathrm{Mad}(G)$ when $H \subseteq G$, every proper subgraph has an (F, D)-decomposition, where F is a forest and $\Delta(D) \leq k$.

Step 1: Prove that certain configurations cannot occur in minimal counterexamples.

The Method

Induction on $|V(G)| + |E(G)|$. Since $\mathrm{Mad}(H) \leq \mathrm{Mad}(G)$ when $H \subseteq G$, every proper subgraph has an (F, D)-decomposition, where F is a forest and $\Delta(D) \leq k$.

Step 1: Prove that certain configurations cannot occur in minimal counterexamples.

Step 2: If G has no such configuration, $\mathrm{Mad}(G) \geq m_k$.

The Method

Induction on $|V(G)| + |E(G)|$. Since $\mathrm{Mad}(H) \leq \mathrm{Mad}(G)$ when $H \subseteq G$, every proper subgraph has an (F, D)-decomposition, where F is a forest and $\Delta(D) \leq k$.

Step 1: Prove that certain configurations cannot occur in minimal counterexamples.

Step 2: If G has no such configuration, $\mathrm{Mad}(G) \geq m_k$.

Step 2 uses discharging. Each vertex v has initial "charge" $d(v)$. Redistribute charge via "discharging rules" to achieve charge at least m_k at *every* vertex.
Conclusion: $\mathrm{Mad}(G) \geq m_k$.

The Method

Induction on $|V(G)| + |E(G)|$. Since $\text{Mad}(H) \le \text{Mad}(G)$ when $H \subseteq G$, every proper subgraph has an (F, D)-decomposition, where F is a forest and $\Delta(D) \le k$.

Step 1: Prove that certain configurations cannot occur in minimal counterexamples.

Step 2: If G has no such configuration, $\text{Mad}(G) \ge m_k$.

Step 2 uses discharging. Each vertex v has initial "charge" $d(v)$. Redistribute charge via "discharging rules" to achieve charge at least m_k at *every* vertex.
Conclusion: $\text{Mad}(G) \ge m_k$.

Extra wrinkle: long-distance discharging.
Special subgraphs start with charge 0, can gain and lose charge, end with nonnegative charge.

Reducible Configurations

Def. A j-vertex is a vertex of degree j.

An (r, s)-alternating cycle is an even cycle alternating between r-vertices and s-vertices.

A vertex v is small if $d(v) \leq k+1$; otherwise, v is big.

Reducible Configurations

Def. A j-vertex is a vertex of degree j.

An (r, s)-alternating cycle is an even cycle alternating between r-vertices and s-vertices.

A vertex v is small if $d(v) \leq k+1$; otherwise, v is big.

Lem. A minimal counterexample G has
(a) no 1-vertex, (b) no adjacent small vertices, and
(c) no $(k+2, 2)$-alternating cycle.

Reducible Configurations

Def. A j-vertex is a vertex of degree j.

An (r, s)-alternating cycle is an even cycle alternating between r-vertices and s-vertices.

A vertex v is small if $d(v) \leq k + 1$; otherwise, v is big.

Lem. A minimal counterexample G has
(a) no 1-vertex, (b) no adjacent small vertices, and
(c) no $(k + 2, 2)$-alternating cycle.

Pf. Every $H \subset G$ satisfies $\mathrm{Mad}(H) \leq \mathrm{Mad}(G)$, and hence H has an (F, D)-decomposition with $\Delta(D) \leq k$.

Reducible Configurations

Def. A j-vertex is a vertex of degree j.

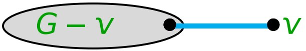
An (r, s)-alternating cycle is an even cycle alternating between r-vertices and s-vertices.

A vertex v is small if $d(v) \leq k+1$; otherwise, v is big.

Lem. A minimal counterexample G has
(a) no 1-vertex, (b) no adjacent small vertices, and
(c) no $(k+2, 2)$-alternating cycle.

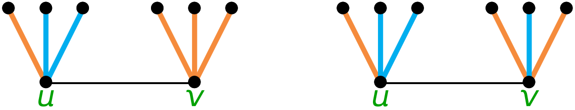
Pf. Every $H \subset G$ satisfies $\mathrm{Mad}(H) \leq \mathrm{Mad}(G)$, and hence H has an (F, D)-decomposition with $\Delta(D) \leq k$.

(a) Add the pendant edge to F in a decomp of $G - v$.



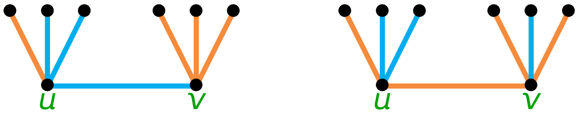
Reducible Configurations

(b) Let u and v be adjacent small vertices. In a decomp of $G-uv$, if u or v has k neighbors in D, then add uv to F; otherwise, add uv to D.



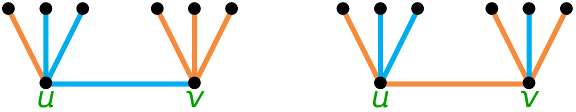
Reducible Configurations

(b) Let u and v be adjacent small vertices. In a decomp of $G - uv$, if u or v has k neighbors in D, then add uv to F; otherwise, add uv to D.

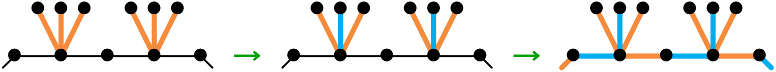


Reducible Configurations

(b) Let u and v be adjacent small vertices. In a decomp of $G - uv$, if u or v has k neighbors in D, then add uv to F; otherwise, add uv to D.



(c) If C is such a cycle, we may assume its big vertices have an incident edge in F in a decomp of $G - E(C)$. Now add $E(C)$ by alternating F and D.



Banks and Cores

Given such G, we extract special subgraphs.

Banks and Cores

Given such G, we extract special subgraphs.

Def. Let X = $\{x \in V(G) \colon d_G(x) = k + 2$ and all nbrs are 2-vertices (except possibly one when $k \leq 3$)$\}$.

Let Y = $\{y \in N_G(X) \colon d_G(y) = 2\}$.

Banks and Cores

Given such G, we extract special subgraphs.

Def. Let X = $\{x \in V(G)\colon\ d_G(x) = k + 2$ and all nbrs are 2-vertices (except possibly one when $k \le 3)\}$.

Let Y = $\{y \in N_G(X)\colon\ d_G(y) = 2\}$.

The bank B has $V(B) = X \cup Y$ and $E(B) = [X, Y]$.

A core of G = a component of the bank.

Banks and Cores

Given such G, we extract special subgraphs.

Def. Let X = $\{x \in V(G)\colon\ d_G(x) = k + 2$ and all nbrs are 2-vertices (except possibly one when $k \le 3)\}$.

Let Y = $\{y \in N_G(X)\colon\ d_G(y) = 2\}$.

The bank B has $V(B) = X \cup Y$ and $E(B) = [X, Y]$.

A core of G = a component of the bank.

- B is a forest (G has no $(k+2, 2)$-alternating cycle).

Banks and Cores

Given such G, we extract special subgraphs.

Def. Let X = $\{x \in V(G)\colon\ d_G(x) = k + 2$ and all nbrs are 2-vertices (except possibly one when $k \le 3)\}$.

Let Y = $\{y \in N_G(X)\colon\ d_G(y) = 2\}$.

The bank B has $V(B) = X \cup Y$ and $E(B) = [X, Y]$.

A core of G = a component of the bank.

- B is a forest (G has no $(k + 2, 2)$-alternating cycle).
- Each leaf is in Y (each x has $> k$ neighbors in B).

Banks and Cores

Given such G, we extract special subgraphs.

Def. Let X = $\{x \in V(G)\colon\ d_G(x) = k+2$ and all nbrs are 2-vertices (except possibly one when $k \le 3$)$\}$.

Let Y = $\{y \in N_G(X)\colon\ d_G(y) = 2\}$.

The bank B has $V(B) = X \cup Y$ and $E(B) = [X, Y]$.

A core of G = a component of the bank.

- B is a forest (G has no $(k+2, 2)$-alternating cycle).

- Each leaf is in Y (each x has $> k$ neighbors in B).

Initial Charge: Each v gets $d_G(v)$; each core gets 0.

Charge at cores permits charge to move long distances.

Discharging Rules

Thm. If $\delta(G) \geq 2$ and G has no adjacent small vertices or $(k+2, 2)$-alternating cycle, then $\mathrm{Mad}(G) \geq m_k$ (recall $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$ and small means $d_G(v) \leq k+1$).

Discharging Rules

Thm. If $\delta(G) \geq 2$ and G has no adjacent small vertices or $(k+2, 2)$-alternating cycle, then $\mathrm{Mad}(G) \geq m_k$ (recall $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$ and small means $d_G(v) \leq k+1$).

Pf. We want final charge $\omega(v) \geq m_k$ and $\omega(C) \geq 0$.

Discharging Rules

Thm. If $\delta(G) \geq 2$ and G has no adjacent small vertices or $(k+2,2)$-alternating cycle, then $\mathrm{Mad}(G) \geq m_k$ (recall $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$ and small means $d_G(v) \leq k+1$).

Pf. We want final charge $\omega(v) \geq m_k$ and $\omega(C) \geq 0$.

Discharging Rules

R1: Each big vertex gives r_1 to each 2-neighbor.

Discharging Rules

Thm. If $\delta(G) \geq 2$ and G has no adjacent small vertices or $(k+2,2)$-alternating cycle, then $\mathrm{Mad}(G) \geq m_k$ (recall $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$ and small means $d_G(v) \leq k+1$).

Pf. We want final charge $\omega(v) \geq m_k$ and $\omega(C) \geq 0$.

Discharging Rules

R1: Each big vertex gives r_1 to each 2-neighbor.

R2: Each big vertex v in no core gives lr_2 to each core C where v has l neighbors that are leaves of C.

Discharging Rules

Thm. If $\delta(G) \geq 2$ and G has no adjacent small vertices or $(k+2,2)$-alternating cycle, then $\mathrm{Mad}(G) \geq m_k$ (recall $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$ and small means $d_G(v) \leq k+1$).

Pf. We want final charge $\omega(v) \geq m_k$ and $\omega(C) \geq 0$.

Discharging Rules

R1: Each big vertex gives r_1 to each 2-neighbor.

R2: Each big vertex v in no core gives lr_2 to each core C where v has l neighbors that are leaves of C.

R3: Each core gives r_3 to each of its big vertices whose neighbors all have degree 2.

Discharging Rules

Thm. If $\delta(G) \geq 2$ and G has no adjacent small vertices or $(k+2,2)$-alternating cycle, then $\mathrm{Mad}(G) \geq m_k$ (recall $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$ and small means $d_G(v) \leq k+1$).

Pf. We want final charge $\omega(v) \geq m_k$ and $\omega(C) \geq 0$.

Discharging Rules

R1: Each big vertex gives r_1 to each 2-neighbor.

R2: Each big vertex v in no core gives lr_2 to each core C where v has l neighbors that are leaves of C.

R3: Each core gives r_3 to each of its big vertices whose neighbors all have degree 2.

R4: *($k \geq 4$ only)* Each big vertex gives r_4 to each 3-nbr.

($m_k > 3 \iff k \geq 4$, so 3-vertices need more charge then.)

Charge Amounts, Happy Vertices

$$r_1 = \frac{m_k - 2}{2}, \qquad r_2 = 1 - r_1 - \frac{m_k}{k+3},$$
$$r_4 = \frac{m_k - 3}{3}, \qquad r_3 = m_k - (k+2)(1 - r_1).$$

Charge Amounts, Happy Vertices

$$r_1 = \frac{m_k - 2}{2}, \qquad r_2 = 1 - r_1 - \frac{m_k}{k+3},$$
$$r_4 = \frac{m_k - 3}{3}, \qquad r_3 = m_k - (k+2)(1 - r_1).$$

(1) 2-vertices are happy. 3-vertices are happy ($k \geq 4$).

Charge Amounts, Happy Vertices

$$r_1 = \frac{m_k - 2}{2}, \qquad r_2 = 1 - r_1 - \frac{m_k}{k+3},$$

$$r_4 = \frac{m_k - 3}{3}, \qquad r_3 = m_k - (k+2)(1 - r_1).$$

(1) 2-vertices are happy. 3-vertices are happy ($k \geq 4$).

(2) If $d(v) = 3 = k$ or $4 \leq d(v) \leq k+1$, then $\omega(v) = d(v) > m_k$, so v is happy.

Charge Amounts, Happy Vertices

$$r_1 = \frac{m_k - 2}{2}, \qquad r_2 = 1 - r_1 - \frac{m_k}{k+3},$$
$$r_4 = \frac{m_k - 3}{3}, \qquad r_3 = m_k - (k+2)(1 - r_1).$$

(1) 2-vertices are happy. 3-vertices are happy ($k \geq 4$).

(2) If $d(v) = 3 = k$ or $4 \leq d(v) \leq k+1$, then $\omega(v) = d(v) > m_k$, so v is happy.

(3) If $d(v) \geq k+2$, then v gives at most r_1 to each nbr and at most r_2 to a core for each nbr (if $v \notin X$).

Charge Amounts, Happy Vertices

$$r_1 = \frac{m_k - 2}{2}, \qquad r_2 = 1 - r_1 - \frac{m_k}{k+3},$$
$$r_4 = \frac{m_k - 3}{3}, \qquad r_3 = m_k - (k+2)(1 - r_1).$$

(1) 2-vertices are happy. 3-vertices are happy ($k \geq 4$).

(2) If $d(v) = 3 = k$ or $4 \leq d(v) \leq k+1$, then $\omega(v) = d(v) > m_k$, so v is happy.

(3) If $d(v) \geq k+2$, then v gives at most r_1 to each nbr and at most r_2 to a core for each nbr (if $v \notin X$).
Hence $\omega(v) \geq d(v)(1 - r_1 - r_2) = \frac{d(v) m_k}{k+3}$.

Charge Amounts, Happy Vertices

$$r_1 = \tfrac{m_k - 2}{2}, \qquad r_2 = 1 - r_1 - \tfrac{m_k}{k+3},$$
$$r_4 = \tfrac{m_k - 3}{3}, \qquad r_3 = m_k - (k+2)(1 - r_1).$$

(1) 2-vertices are happy. 3-vertices are happy ($k \geq 4$).

(2) If $d(v) = 3 = k$ or $4 \leq d(v) \leq k+1$, then $\omega(v) = d(v) > m_k$, so v is happy.

(3) If $d(v) \geq k+2$, then v gives at most r_1 to each nbr and at most r_2 to a core for each nbr (if $v \notin X$).
Hence $\omega(v) \geq d(v)(1 - r_1 - r_2) = \frac{d(v)m_k}{k+3}$.
So, v is happy if $d(v) \geq k+3$.

Charge Amounts, Happy Vertices

$$r_1 = \frac{m_k - 2}{2}, \qquad r_2 = 1 - r_1 - \frac{m_k}{k+3},$$
$$r_4 = \frac{m_k - 3}{3}, \qquad r_3 = m_k - (k+2)(1 - r_1).$$

(1) 2-vertices are happy. 3-vertices are happy ($k \geq 4$).

(2) If $d(v) = 3 = k$ or $4 \leq d(v) \leq k+1$, then $\omega(v) = d(v) > m_k$, so v is happy.

(3) If $d(v) \geq k+2$, then v gives at most r_1 to each nbr and at most r_2 to a core for each nbr (if $v \notin X$).
Hence $\omega(v) \geq d(v)(1 - r_1 - r_2) = \frac{d(v) m_k}{k+3}$.
So, v is happy if $d(v) \geq k+3$.

Left to check: Happiness of $(k+2)$-vertices and cores.
(uses definition of m_k, etc.)

Vertices with degree $k+2$ are happy

$$r_2 = 1 - r_1 - \frac{m_k}{k+3} \qquad r_3 = m_k - (k+2)(1 - r_1)$$

$$r_1 = \frac{m_k}{2} - 1 \qquad r_4 = \frac{m_k}{3} - 1 \qquad m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$$

Vertices with degree $k+2$ are happy

$$r_2 = 1 - r_1 - \frac{m_k}{k+3} \qquad r_3 = m_k - (k+2)(1-r_1)$$

$$r_1 = \frac{m_k}{2} - 1 \qquad r_4 = \frac{m_k}{3} - 1 \qquad m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$$

- $v \notin X$

$k \leq 3$: at most k neighbors of degree 2

$$\omega(v) \geq 2 + k(1 - r_1 - r_2) = 2 + m_k - \frac{3m_k}{k+3} > m_k$$

(since $2 - \frac{3m_k}{k+3} = \frac{2k^2}{k^2+6k+6} > 0$)

Vertices with degree $k+2$ are happy

$$r_2 = 1 - r_1 - \frac{m_k}{k+3} \qquad r_3 = m_k - (k+2)(1-r_1)$$

$$r_1 = \frac{m_k}{2} - 1 \qquad r_4 = \frac{m_k}{3} - 1 \qquad m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$$

- $v \notin X$

$k \leq 3$: at most k neighbors of degree 2

$$\omega(v) \geq 2 + k(1 - r_1 - r_2) = 2 + m_k - \frac{3m_k}{k+3} > m_k$$

$k \geq 4$: at most $k+1$ nbrs of degree 2 and one 3-nbr.

$$\omega(v) \geq (k+1)(1-r_1-r_2) + 1 - r_4 = m_k - \frac{2m_k}{k+3} + 2 - \frac{m_k}{3} > m_k.$$

(since $2 - \frac{2m_k}{k+3} - \frac{m_k}{3} = \frac{2k(k-2)}{3(k^2+6k+6)} > 0$)

Vertices with degree $k+2$ are happy

$$r_2 = 1 - r_1 - \frac{m_k}{k+3} \qquad r_3 = m_k - (k+2)(1-r_1)$$

$$r_1 = \frac{m_k}{2} - 1 \qquad r_4 = \frac{m_k}{3} - 1 \qquad m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$$

- $v \notin X$

$k \leq 3$: at most k neighbors of degree 2

$$\omega(v) \geq 2 + k(1 - r_1 - r_2) = 2 + m_k - \frac{3m_k}{k+3} > m_k$$

$k \geq 4$: at most $k+1$ nbrs of degree 2 and one 3-nbr.

$$\omega(v) \geq (k+1)(1-r_1-r_2) + 1 - r_4 = m_k - \frac{2m_k}{k+3} + 2 - \frac{m_k}{3} > m_k.$$

- $v \in X$

If every neighbor has degree 2, then

$$\omega(v) = (k+2)(1-r_1) + r_3 = m_k$$

Vertices with degree $k+2$ are happy

$$r_2 = 1 - r_1 - \frac{m_k}{k+3} \qquad r_3 = m_k - (k+2)(1-r_1)$$

$$r_1 = \frac{m_k}{2} - 1 \qquad r_4 = \frac{m_k}{3} - 1 \qquad m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$$

- $v \notin X$

$k \leq 3$: at most k neighbors of degree 2

$$\omega(v) \geq 2 + k(1 - r_1 - r_2) = 2 + m_k - \frac{3m_k}{k+3} > m_k$$

$k \geq 4$: at most $k+1$ nbrs of degree 2 and one 3-nbr.

$$\omega(v) \geq (k+1)(1-r_1-r_2) + 1 - r_4 = m_k - \frac{2m_k}{k+3} + 2 - \frac{m_k}{3} > m_k.$$

- $v \in X$

If every neighbor has degree 2, then

$$\omega(v) = (k+2)(1-r_1) + r_3 = m_k$$

If some neighbor has degree > 2, then $k \leq 3$ and

$$\omega(v) = (k+1)(1-r_1) + 1 > m_k$$

(since $(k+1)(2-\frac{m_k}{2}) + 1 = m_k - (k+3)\frac{m_k}{2} + 2k+3 > m_k$)

Cores are Happy

$$r_2 = 1 - r_1 - \frac{m_k}{k+3} \qquad r_3 = m_k - (k+2)(1 - r_1)$$

$$r_1 = \frac{m_k}{2} - 1 \qquad m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$$

Cores are Happy

$$r_2 = 1 - r_1 - \frac{m_k}{k+3} \qquad r_3 = m_k - (k+2)(1 - r_1)$$

$$r_1 = \frac{m_k}{2} - 1 \qquad m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$$

A core C is a tree; count vertices by their degree in C.

$V(C) \cap X$: n_1 with degree $k+1$ and n_2 with degree $k+2$.
$V(C) \cap Y$: n' with degree 2 and n_0 with degree 1.

Cores are Happy

$$r_2 = 1 - r_1 - \frac{m_k}{k+3} \qquad r_3 = m_k - (k+2)(1-r_1)$$

$$r_1 = \frac{m_k}{2} - 1 \qquad m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$$

A core C is a tree; count vertices by their degree in C.

$V(C) \cap X$: n_1 with degree $k+1$ and n_2 with degree $k+2$.
$V(C) \cap Y$: n' with degree 2 and n_0 with degree 1.

Degree sum:

$$(k+1)n_1 + (k+2)n_2 + 2n' + n_0 = 2(n_1+n_2+n'+n_0)-2.$$

Hence $n_0 = (k-1)n_2 + kn_2 + 2$.

Cores are Happy

$$r_2 = 1 - r_1 - \frac{m_k}{k+3} \qquad r_3 = m_k - (k+2)(1-r_1)$$

$$r_1 = \frac{m_k}{2} - 1 \qquad m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$$

A core C is a tree; count vertices by their degree in C.

$V(C) \cap X$: n_1 with degree $k+1$ and n_2 with degree $k+2$.
$V(C) \cap Y$: n' with degree 2 and n_0 with degree 1.

Degree sum:
$(k+1)n_1 + (k+2)n_2 + 2n' + n_0 = 2(n_1+n_2+n'+n_0)-2$.
Hence $n_0 = (k-1)n_2 + kn_2 + 2$.

C gets $n_0 r_2$ from big vertices $\notin C$; gives $n_2 r_3$ to $V(C) \cap X$.
Since $n_0 > kn_2$, it suffices to have $kr_2 \geq r_3$.

Cores are Happy

$$r_2 = 1 - r_1 - \frac{m_k}{k+3} \qquad r_3 = m_k - (k+2)(1-r_1)$$

$$r_1 = \frac{m_k}{2} - 1 \qquad m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$$

A core C is a tree; count vertices by their degree in C.

$V(C) \cap X$: n_1 with degree $k+1$ and n_2 with degree $k+2$.
$V(C) \cap Y$: n' with degree 2 and n_0 with degree 1.

Degree sum:

$$(k+1)n_1 + (k+2)n_2 + 2n' + n_0 = 2(n_1+n_2+n'+n_0)-2.$$

Hence $n_0 = (k-1)n_2 + kn_2 + 2$.

C gets $n_0 r_2$ from big vertices $\notin C$; gives $n_2 r_3$ to $V(C) \cap X$.

Since $n_0 > kn_2$, it suffices to have $kr_2 \geq r_3$.

$$kr_2 - r_3 = (2k+2)(1-r_1) - \frac{2k+3}{k+3}m_k = 4(k+1) - \frac{k^2+6k+6}{k+3}m_k = 0.$$

Cores are Happy

$$r_2 = 1 - r_1 - \frac{m_k}{k+3} \qquad r_3 = m_k - (k+2)(1-r_1)$$

$$r_1 = \frac{m_k}{2} - 1 \qquad m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$$

A core C is a tree; count vertices by their degree in C.

$V(C) \cap X$: n_1 with degree $k+1$ and n_2 with degree $k+2$.
$V(C) \cap Y$: n' with degree 2 and n_0 with degree 1.

Degree sum:
$$(k+1)n_1 + (k+2)n_2 + 2n' + n_0 = 2(n_1+n_2+n'+n_0)-2.$$
Hence $n_0 = (k-1)n_2 + kn_2 + 2$.

C gets $n_0 r_2$ from big vertices $\notin C$; gives $n_2 r_3$ to $V(C) \cap X$.
Since $n_0 > kn_2$, it suffices to have $kr_2 \geq r_3$.

$$kr_2 - r_3 = (2k+2)(1-r_1) - \frac{2k+3}{k+3} m_k = 4(k+1) - \frac{k^2+6k+6}{k+3} m_k = 0.$$

- m_k is maximized such that the last expression is ≥ 0.