Coloring and List Coloring of Graphs and Hypergraphs

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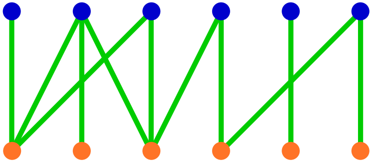

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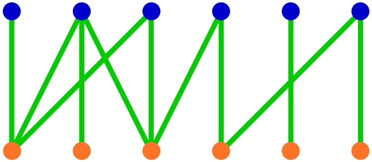

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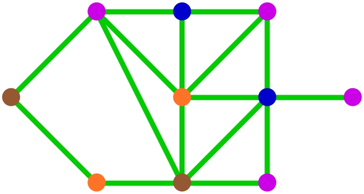

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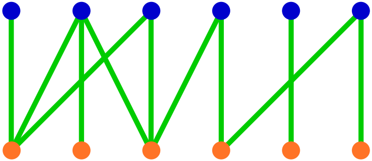

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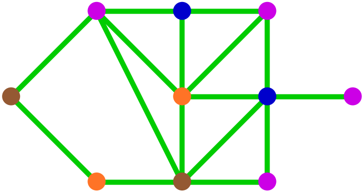
What do the colors mean?



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proper coloring c: $uv \in E(G) \Rightarrow c(u) \neq c(v)$.
G is k-colorable if $\exists$ proper coloring using $\leq k$ colors.
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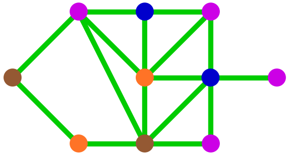
Here $\chi(G) = 4$



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Def. list assignment: $L(v)$ = color set available at v.
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k-choosable G: $\exists$ L-coloring whenever all $|L(v)| \geq k$.
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Prop. $\chi(G) \leq \chi_\ell(G) \leq \Delta(G) + 1$.

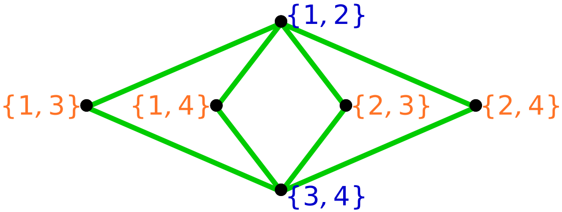
Pf. Lower bound: The lists may be identical.

Upper bound: Lists of size $\Delta(G) + 1$ $\Rightarrow$ a color is available for each successive vertex in any order. ■

- $d(v)$ = degree of vertex v; $\Delta(G) = \max_{v \in V(G)} d(v)$.

$\chi_\ell(G)$ May Exceed $\chi(G)$

Ex. $\chi(K_{2,4}) = 2$, but $\chi_\ell(K_{2,4}) > 2$.



- $K_{r,s}$ = complete bipartite graph, part-sizes r and s.

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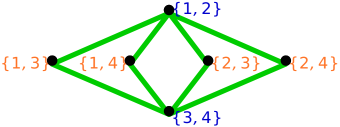

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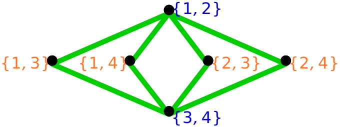

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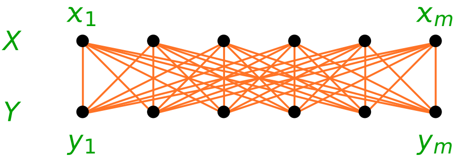

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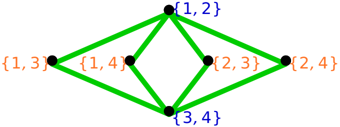

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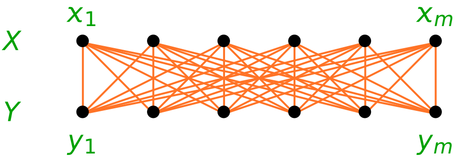

$< k$ colors on X $\Rightarrow$ some vertex of X left uncolored.

k colors on X $\Rightarrow$ vertex of Y w. that list is uncolorable. ■

Planar Graphs

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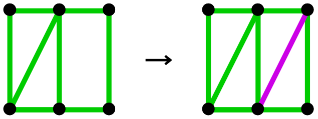

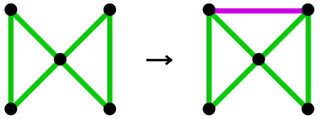

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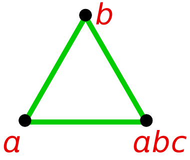
Basis step:



Induction Step

Let $[v_1, \ldots, v_p]$ be the outer cycle C in order, with $|L(v_1)| = |L(v_p)| = 1$.

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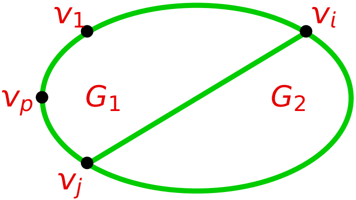

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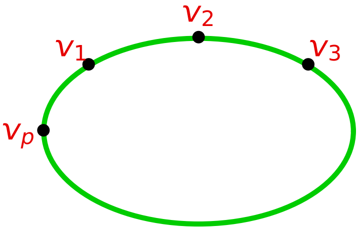

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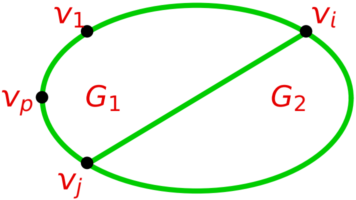

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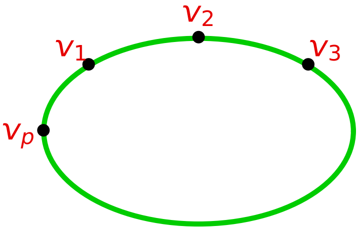

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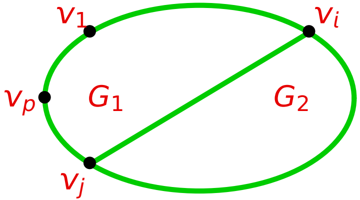

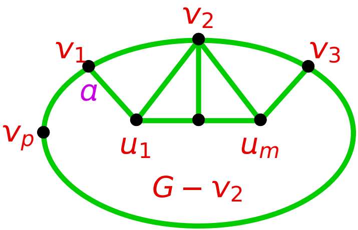

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Case 2: $L(v_1) = \{a\}$. Pick $x, y \in L(v_2) - \{a\}$. Delete x and y from each $L(u_i)$. Choose L-coloring on $G - v_2$. Extend to v_2 using x or y.

Alon–Tarsi: A tool for upper bounds on $\chi_\ell(G)$

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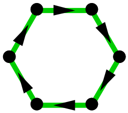

2 even, 0 odd

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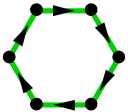

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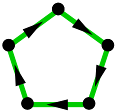
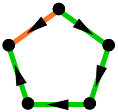
Ex. odd cycle



1 even, 1 odd

no info



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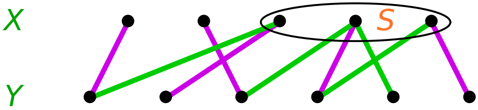
$\chi_\ell(C_{2k+1}) \le 3$

Hall's Theorem

Def. matching = a set of pairwise disjoint edges.
X, Y-bigraph = bipartite graph with partite sets X and Y.



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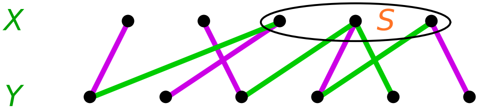

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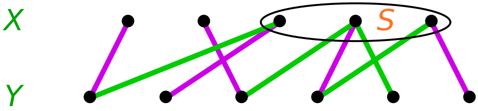

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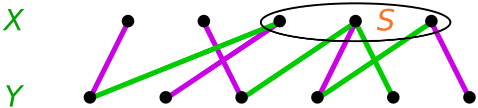

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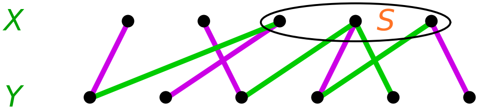

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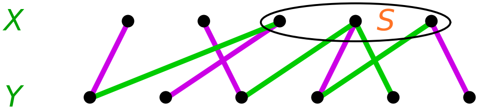

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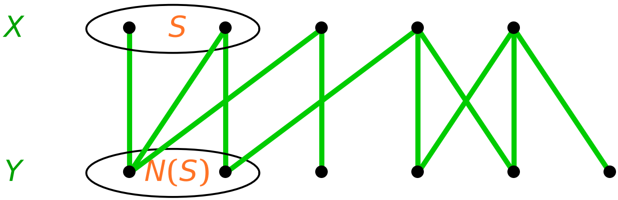

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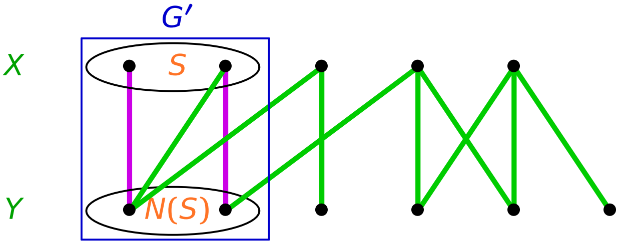

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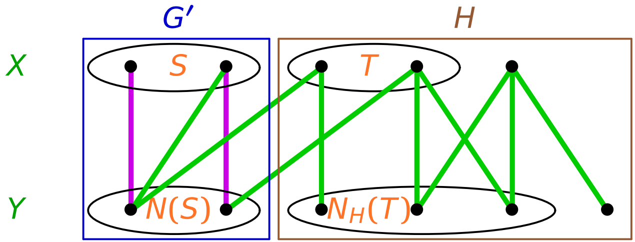

Let H = subgraph induced by $(X - S) \cup (Y - N(S))$.
To prove H satisfies H.C., compute

$$|N_H(T)| = |N(T \cup S)| - |N(S)| \geq |T \cup S| - |S| = |T|.$$

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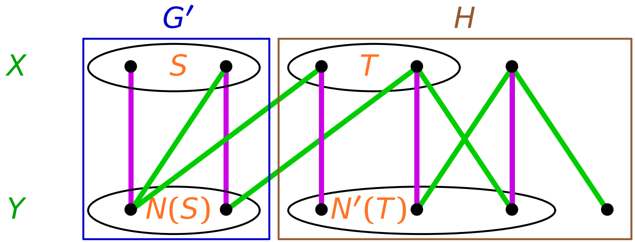
Let G' = subgraph induced by $S \cup N(S)$.
$S' \subseteq S \quad \Rightarrow \quad |N_{G'}(S')| = |N_G(S')| \geq |S'|$, so G' satisfies H.C.



Let H = subgraph induced by $(X - S) \cup (Y - N(S))$.
To prove H satisfies H.C., compute

$$|N_H(T)| = |N(T \cup S)| - |N(S)| \geq |T \cup S| - |S| = |T|.$$

Use matchings from G' and H (induction hypothesis). ∎

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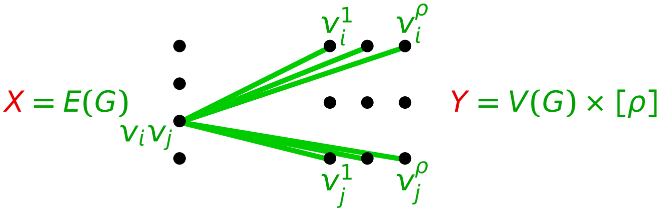

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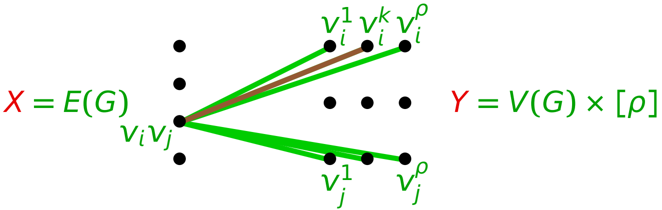

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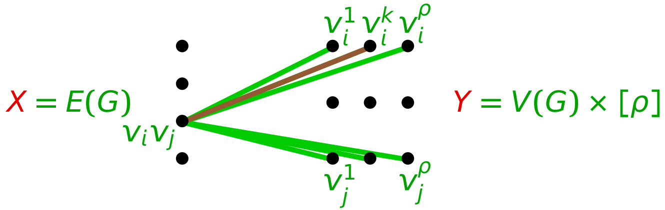

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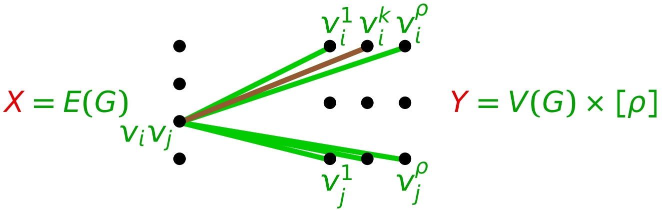

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$\therefore$ Hall's Theorem $\Rightarrow$ $\exists$ matching covering X. ■

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Def. For hypergraphs, the definitions of

coloring	list assignment
proper coloring	L-coloring
k-colorable	k-choosable
chromatic number	list chromatic number

are exactly the same as for the special case of graphs, except that we must rephrase one:

A proper coloring is a coloring with no monochromatic edge.

The Probabilistic Method

- Idea: The Existence Argument.

Build an object by a random experiment in which a desired property corresponds to some event A.

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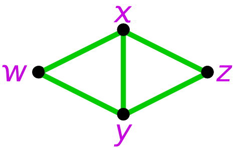
For each $v \in V(G)$, we must choose a color from $L(v)$.
When $v \in X$, choose a color restricted to X.
When $v \in Y$, choose a color restricted to Y. ■

Face Hypergraphs of Planar Graphs

Def. face hypergraph of a plane graph G = hypergr. H with $V(H) = V(G)$ and $E(H) = \{$vertex sets of faces of $G\}$



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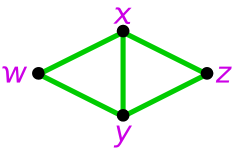

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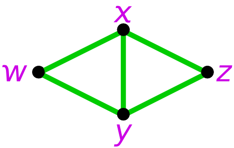

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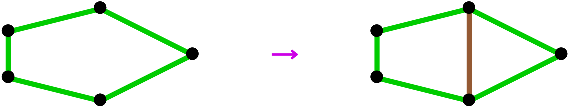

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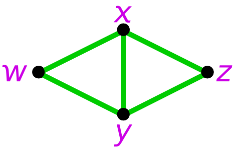

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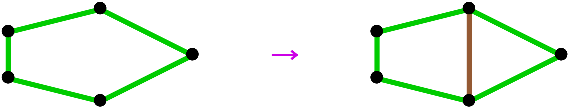

Given 3-lists at vertices, choose a proper coloring of H.

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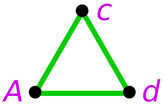

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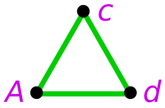

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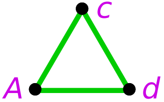

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Conj. (Ramamurthi) Face hypergraphs are 2-choosable.

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Lem. (Combinatorial Nullstellensatz; Alon [1999])
Let f be homogen. polynomial of degree m in n varbs.
If the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ is nonzero and each $|S_i| > t_i$, then $\exists\ s \in \prod_{i=1}^{n} S_i$ such that $f(s) \neq 0$.

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Bijection needed!

Concluding the Alon-Tarsi Theorem

Bijection: Given a fixed orientation D with outdegrees $d_1, \ldots, d_n$, each orientation D' with the same outdegrees as D corresponds to a circulation in D whose edges are those reversed in D'.

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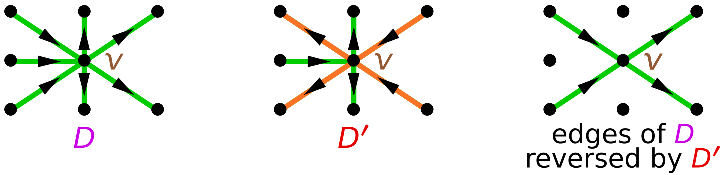

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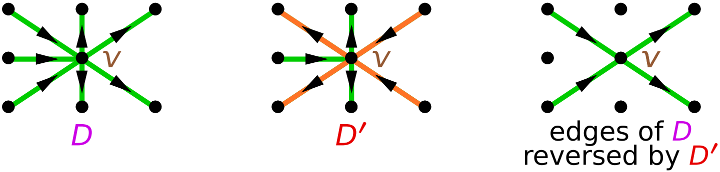

$$\begin{aligned}
\mathrm{diff}(D) \neq 0 \quad &\Rightarrow \quad \mathrm{coeff}(\textstyle\prod_i x_i^{d_i}) \neq 0 \\
&\Rightarrow \quad f(s) \neq 0 \text{ for some } s \in \textstyle\prod S_i \\
&\qquad\qquad \text{where each } S_i \text{ is fixed with } |S_i| > d_i \\
&\Rightarrow \quad \chi_\ell(G) \leq 1 + \Delta^+(D)
\end{aligned}$$

Alon–Tarsi for Hypergraphs

Plan of action (for a k-uniform hypergraph H):

0. Order the vertices $v_1, \ldots, v_n$.

1. Define a hypergraph polynomial f_H such that $f_H(x) = 0$ precisely when coloring v_i with x_i for all i gives the same number to all vertices of some edge.

2. Generalize graph orientation to hypergraphs to interpret exponents in monomials.

3. Nullstellensatz $\Rightarrow$ If lists are bigger than the corresponding exponents in a monomial term in f_H with nonzero coefficient, then H has an L-coloring.

4. Interpret coefficients in terms of some structure we can count to obtain a sufficient condition for an orientation of H to guarantee L-coloring.

The Hypergraph Polynomial (k prime)

Def. For a k-uniform hypergraph H with vertices $v_1, \ldots, v_n$, the hypergraph polynomial f_H is defined by

$$f_H(x_1, \ldots, x_n) = \prod_{v_{i_0} \cdots v_{i_{k-1}} \in E(H)} (x_{i_0} + \theta x_{i_1} + \cdots + \theta^{k-1} x_{i_{k-1}}),$$

where θ is a kth root of unity and the vertices of each edge are written in increasing order of indices.

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When numbers $x_1, \ldots, x_n$ are assigned to $v_1, \ldots, v_n$, $f_H(x_1, \ldots, x_n) = 0 \Leftrightarrow$ numbers are equal on some edge.

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Def. An orientation of a hypergraph chooses one source vertex in each edge.

Terms like $\theta^j \prod x_i^{d_i}$ in the expansion of $f_H(x)$ come from orientations with v_i chosen as the source in d_i edges.

The Hypergraph List Coloring Theorem

Thm. (Ramamurthi–West [2005]) For prime k, let D be an orientation of a k-uniform hypergraph H such that the number of balanced partitions with modular sum j is not the same for all j. If each $|L(v_i)|$ is larger than the number of edges in which D chooses v_i as the source, then H has an L-coloring.

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Pf. Follow the Plan!