

LINE DIGRAPHS AND COREFLEXIVE VERTEX SETS

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Abstract. The concept of coreflexive set is introduced to study the structure of digraphs. New characterizations of line digraphs and n th-order line digraphs are given. Correlative sets also lead to another natural way of forming an intersection digraph from a given digraph.

1. INTRODUCTION

In this paper we introduce new structural concepts about directed graphs. We use $V(D)$ and $E(D)$ for the vertex set and edge set of a digraph D . We allow loops but no multiple edges, unless specified otherwise. We write (u, v) or uv for an edge from u to v , with *tail* u and *head* v .

Let D be a digraph with multiple edges allowed. The *line digraph* of D is a digraph $L(D)$ (without multiple edges) such that $V(L(D)) = E(D)$, and $L(D)$ has an edge from e to f if and only if the head of e is the tail of f . Introduced in [4], line digraphs were studied also in [2,5,6,7] ([5] contains a survey). Beineke and Zamfirescu [1] studied the n th-order line digraph $L^n(D)$ (obtained by iterating the line digraph operation), and they characterized the second-order line digraphs. We obtain several characterizations of line digraphs and a characterization of n th-order line digraphs.

To study the structure of digraphs, we introduce the concept of “coreflexive set”. We use $\alpha(v)$ to denote the set of successors of a vertex v , also commonly written as $N^+(v)$ (the “out-neighbor” set). We use $\beta(v)$ to denote the set of predecessors of v , also commonly written as $N^-(v)$ (the “in-neighbor” set). Our use of α and β suggests the words “after” and “before”. We extend this notation to sets of vertices: $\alpha(S)$ is the set of vertices to which there is an edge from at least one vertex of S , and $\beta(S)$ is the set of vertices from which there is an edge to at least one vertex of S .

DEFINITION A *coreflexive set* in a digraph (henceforth *coreset*) is either 1) the collection of all sinks in the digraph, or 2) a minimal nonempty set U such that $U = \beta(\alpha(U))$.

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The collection of all sinks (vertices without successors) is the *trivial* coreset; the other coresets are *nontrivial*.

The definition immediately implies that no coreset contains another. In Fig. 1, the set $\{x_7\}$ is the trivial coreset, and the nontrivial coresets are $\{x_1, x_2, x_3, x_6\}$, $\{x_4\}$, and $\{x_5\}$.

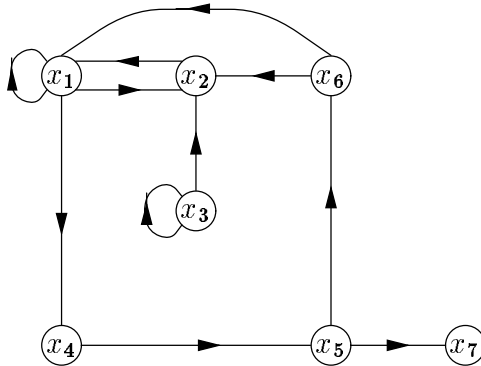


Fig. 1. A digraph to illustrate coresets.

Beineke and Zamfirescu [1] also introduced intersection digraphs, under the name “connection digraph”. As studied in [8], a digraph D is the *intersection digraph* of a collection of pairs of sets $\{(S_v, T_v) : v \in V(D)\}$ if $E(D)$ is the set $\{uv : S_u \cap T_v \neq \emptyset\}$. Informally, we call S_v and T_v the *source set* and *sink set* of v , and the edge uv occurs when the source set of u intersects the sink set of v . An alternative model was introduced in [3]. Various classes of intersection digraphs have been studied. Line digraphs are themselves intersection digraphs, arising by letting $(S_e, T_e) = (\{y\}, \{x\})$ for each edge $e = xy$. We study a special class of intersection digraphs that, like line digraphs, arise from arbitrary digraphs and capture some of their structure. Given a digraph D , the vertices of this new digraph will be the coresets of D .

2. PROPERTIES OF CORESETS

LEMMA 1. The coresets of a digraph are pairwise disjoint.

Proof: Since a sink is not a predecessor of any successor of itself, the trivial coreset intersects no others. Let U, W be distinct nontrivial coresets. Let $X = U \cap W$. Since U is a coreset, $\beta(\alpha(X))$ contains no element of $W - U$. Since W is a coreset, $\beta(\alpha(X))$ contains no element of $U - W$. Hence $\beta(\alpha(X)) = X$. Since U and W are minimal nonempty sets unchanged by $\beta \circ \alpha$, we have $X = \emptyset$, and U, W are disjoint. ■

The disjointness of distinct coresets implies that there is no edge from a coreset to a successor of another coreset. Equivalently, no two vertices in distinct coresets have a common successor.

For convenience, in a digraph we define $\alpha(\emptyset)$ to be the collection of source vertices and $\beta(\emptyset)$ to be the collection of sink vertices. In describing partitions of a vertex set, we allow sets of the partition to be empty.

THEOREM 2. For every digraph D , the coresets of D (including \emptyset) partition $V(D)$. The successor sets of the coresets (including the set $\alpha(\emptyset)$ of sources) also partition $V(D)$.

Proof: The first statement follows from Lemma 1 and the claim that every vertex v belongs to a coreset. We may assume that v is not a sink. Let U be a minimal set containing v such that $\beta(\alpha(U)) = U$. Such a set exists, since the collection of all non-sinks has this property. By the definition, U contains a coreset X . If $v \notin X$, then v cannot have a successor in $\alpha(X)$, since X is a coreset, and $\alpha(X)$ has no predecessor outside X . Thus $\beta(\alpha(U - X)) = U - X$, which contradicts the minimality of U . We conclude that v belongs to a coreset.

For the second statement, the observation that vertices of distinct coresets cannot have common successors implies that the successor sets of the coresets (and $\alpha(\emptyset)$) are pairwise disjoint. Furthermore, every non-source vertex is a successor of a vertex in some coreset, since the coresets partition $V(D)$. ■

In Fig. 1, the successor sets of the coresets listed earlier are \emptyset , $\{x_1, x_2, x_3, x_4\}$, $\{x_5\}$, $\{x_6, x_7\}$, respectively. This digraph has no source vertices. In general, we would list $U_0 = \emptyset$ in the partition into coresets in order to obtain the set $\alpha(\emptyset)$ of source vertices in the successor partition.

It is worth noting that the partitions in Theorem 2 are unchanged under reversal of all edges. Since $\beta(\alpha(U)) = U$ for a coreset U , the successor set $W = \alpha(U)$ of a coreset U satisfies $\alpha(\beta(W)) = W$. Thus the coresets of the reversal of D are the successor sets in D of the coresets in D .

Given a digraph D with $A, B \subseteq V(D)$, we use $[A, B]$ to denote the set of edges of D from A to B . A *decomposition* of a digraph D is a set of pairwise edge-disjoint subgraphs whose union is D .

THEOREM 3. Given a digraph D , let U_0, \dots, U_m be the partition of $V(D)$ into coresets, including $U_0 = \emptyset$. Let D_i be the subdigraph of D with vertex set $U_i \cup \alpha(U_i)$ and edge set $[U_i, \alpha(U_i)]$. The digraphs D_0, \dots, D_m decompose D .

Proof: By Theorem 2, every vertex and every edge appears in some D_i . If some edge uv appears in both D_i and D_j for $i \neq j$, then $\{u, v\} \subseteq V(D_i) \cap V(D_j)$. By Theorem 2, each of $\{u, v\}$ lies in $U_i \cap \alpha(U_j)$ or in $\alpha(U_i) \cap U_j$. Since no coreset has an edge to a successor of another, the two vertices cannot both lie in one of these sets. By symmetry, we may thus assume that $u \in U_i \cap \alpha(U_j)$ and $v \in \alpha(U_i) \cap U_j$, which requires $u \neq v$. Since $u \notin U_j$, we now have $uv \notin E(D_j)$, a contradiction. ■

We call the resulting decomposition the *core decomposition* of D ; the subgraphs D_0, \dots, D_m are the *core subgraphs*. Fig. 2 shows the core decomposition of the digraph in Fig. 1.

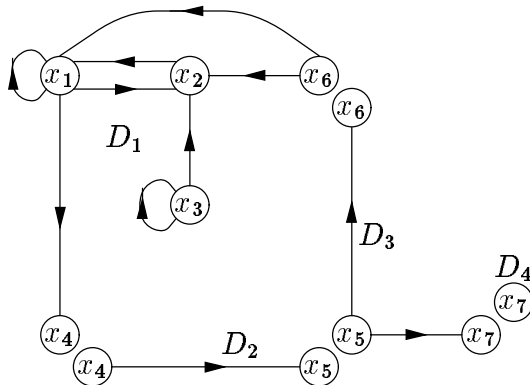


Fig. 2. A core decomposition.

PROPOSITION 4. If D' is a core subgraph of a digraph D , then the core decomposition of D' has only one nonempty subgraph, D' itself.

Proof: Let U be the coreset of D such that D' consists of all edges in D from U to $\alpha(U)$. We may assume that U is a nontrivial coreset of D , so that $\alpha(U) - U$ is the set of sinks in D' . Suppose U' is a nontrivial coreset in D' , so $\beta_{D'}(\alpha_{D'}(U')) = U'$. Because $\alpha(U)$ contains all successors in D of vertices of U' , we have $\alpha_{D'}(U') = \alpha(U')$. Similarly, since U is a coreset of D , U contains all predecessors of successors of vertices of U' , so $\beta_{D'}(\alpha_{D'}(U')) = \beta(\alpha(U'))$. We have proved that $\beta(\alpha(U')) = U'$. By the minimality property of coresets, $U' = U$. Thus the coresets of D' are U and $\alpha(U) - U$, and the only core subgraph is D' itself. ■

We can describe coresets in the language of adjacency matrices.

PROPOSITION 5. A nonempty vertex set U in a digraph D is a nontrivial coreset if and only if the rows of the adjacency matrix that correspond to U are nonzero, are orthogonal to the remaining rows, and have no nonempty subset with the same property.

Proof: The conditions state that the vertices of U are not sinks, that $\beta(\alpha(U)) \subseteq U$, and that U is a minimal nonempty set with these properties. Since $u \in \beta(\alpha(u))$ whenever u is not a sink, these are precisely the conditions for a nontrivial coreset. ■

In the language of adjacency matrices, Theorem 3 states that if the rows of D are grouped as U_0, \dots, U_m and the columns as $\alpha(U_0), \dots, \alpha(U_m)$, then the adjacency matrix $A(D)$ partitions into blocks $B_{i,j}$ with $0 \leq i, j \leq m$ such that $B_{i,i} = A(D_i)$ and all other blocks are zero matrices.

3. CHARACTERIZATION OF LINE DIGRAPHS

Geller and Harary [2] proved that a digraph D is a line digraph if and only if there exist partitions A_1, \dots, A_r and B_1, \dots, B_r of $V(D)$ (using possibly empty sets) such that $E(D) = \bigcup_{i=1}^r A_i \times B_i$. We give several related characterizations using the concept of

coreset, including the Geller-Harary characterization for completeness. A digraph with vertex set $A \cup B$ and edge set $A \times B$ is a *Cartesian product digraph*.

THEOREM 6. For a digraph D with core decomposition D_0, \dots, D_m , the following are equivalent:

- A) D is a line digraph.
- B) $V(D)$ admits partitions $\{A_i\}$ and $\{B_i\}$ such that $E(D) = \cup(A_i \times B_i)$.
- C) $\alpha(u) = \alpha(x)$ whenever u and x belong to the same coreset in D .
- D) For each $uv \in E(D)$, the set $\beta(v)$ is the coreset containing u .
- E) Each D_i is a Cartesian product digraph.
- F) Each D_i is a line digraph.

Proof: A \Rightarrow B (Geller-Harary [2]): Given $D = L(D')$, where $V(D') = \{w_1, \dots, w_r\}$, let A_i be the set of edges in D' with head w_i , and let B_i consist of those with tail w_i (when w_i is a source or sink, A_i or B_i is empty, respectively). These sets also partition $V(D)$, and the definition of line digraph yields $E(D) = \bigcup_{i=1}^r A_i \times B_i$.

B \Rightarrow C: Given the partitions as described, a vertex of A_i can have no successors outside B_i , and a vertex of B_i can have no predecessors outside A_i . Thus A_i is a coreset and B_i is its successor set, and $\alpha(x) = B_i$ for all $x \in A_i$.

C \Rightarrow D: Given an edge uv , Condition C implies that $\beta(v)$ contains the coreset that contains u ; by the definition of coreset, it cannot contain more.

D \Rightarrow E: Condition D implies that each vertex of a coreset U is a predecessor of each vertex of the successor set $\alpha(U)$, and thus $E(D_i) = U_i \times \alpha(U_i)$.

E \Rightarrow A: As a Cartesian product, we must have $E(D_i) = U_i \times \alpha(U_i)$. We must assume that the sink set is $\alpha(U_m)$ and the set U_0 is empty, so that $\alpha(U_0)$ is the source set. Construct a digraph D' with $V(D') = w_0, \dots, w_m$. Put an edge in D' for each $v \in V(D)$. The edge v in D' is $w_i w_j$ if v belongs to U_j in the coreset partition and to $\alpha(U_i)$ in the successor partition (D' may have multiple edges). By construction, $L(D') = D$.

E \Rightarrow F \Rightarrow D: Every Cartesian product digraph is explicitly a line digraph. Finally, F \Rightarrow D follows by applying Proposition 4 and the equivalence of A and D to each D_i . ■

The expression of Theorem 3 in the language of adjacency matrices allows us to interpret Theorem 6 in these terms also. In particular, a digraph is a line digraph if and only if its rows and columns can be permuted so that the 1's form rectangular blocks that do not share rows or columns. This immediately implies the characterization by Richards [7] that a 0,1-matrix is the adjacency matrix of a line digraph if and only if any two columns (or any two rows) are identical or orthogonal. We could also translate conditions C and D of Theorem 6 into conditions on the rows corresponding to vertices in a given coreset.

4. *n*th-ORDER LINE DIGRAPHS

Introduced in [1], the *n-order line digraph* $L^n(D)$ of a digraph D is the digraph obtained from D by iteratively applying the line digraph operator n times. Coresets enable us to characterize the *n*th-order line digraphs of digraphs with no sources or sinks. Note that the vertex set of $L^n(D)$ is the set of *n*-walks (walks of length n) in D , with an edge from an *n*-walk W_1 to an *n*-walk W_2 if deleting the first vertex and edge from W_1 yields the

same $n - 1$ -walk as deleting the last vertex and edge from W_2 . We confine our attention to digraphs without sources or sinks to avoid annoying technicalities about the nonexistence of walks of given lengths from a given vertex.

For the characterization, we introduce n -th order coresets. Given a digraph D , define the digraph D^n by $V(D^n) = V(D)$ and $E(D^n) = \{uv: D \text{ has a walk of length } n \text{ from } u \text{ to } v\}$. The successor operator in D^n is the n th iterate of the successor operator in D : $\alpha_{D^n}(u) = \alpha^n(u)$. Similarly $\beta_{D^n}(u) = \beta^n(u)$. The n th-order coresets of D are the coresets of D^n . Equivalently, the n th-order coresets of a digraph without sources or sinks are the minimal nonempty sets U such that $\beta^n(\alpha^n(U)) = U$. By applying Theorem 2 to D^n , we immediately obtain the following corollary.

COROLLARY 7. If D is a digraph without sources or sinks, then the n th-order coresets partition $V(D)$, as do the n th-order successor sets of these coresets. Furthermore, vertices of distinct n th-order coresets do not have a common n th-order successor. ■

LEMMA 8. If D is a digraph without sources or sinks, then U is an n th-order coreset in D if and only if W is an $(n + 1)$ th-order coreset in $L(D)$, where W is the set of edges in D with heads in U .

Proof: Observe first that an m th-order coreset W in $L(D)$ contains all edges of D that share heads with its members. If $uv \in W$ and $xv \in E(D)$, then xv can be substituted for uv as the first edge in an $m + 1$ -walk in D . Thus $xv \in \beta^m(\alpha^m(W)) = W$ in $L(D)$. As a result, if W is an $(n + 1)$ th-order coreset in $L(D)$ and U is the subset of $V(D)$ consisting of heads of elements of W , then W is obtained from U as defined above.

We prove that U is a coreset in D^n if and only if W is a coreset in $[L(D)]^{n+1}$. The digraph $[L(D)]^{n+1}$ has an edge from $uv \in E(D)$ to $xy \in E(D)$ if and only if $L(D)$ has a walk of length $n + 1$ from uv to xy . Such a walk exists if and only if D has an n -walk from v to x , which corresponds to existence of the edge vx in D^n .

Thus z is a predecessor of a successor of v in D^n if and only if there exist edges uv and tz in $E(D)$ such that tz is a predecessor of a successor of uv in $[L(D)]^{n+1}$. This implies that $\beta(\alpha(U)) = U$ in D^n if and only if $\beta(\alpha(W)) = W$ in $[L(D)]^{n+1}$. The equality holds for a proper subset of U if and only if it holds for a proper subset of W ; this completes the proof of the claim. ■

THEOREM 9. If D is a digraph without sources or sinks, then D is an n th-order line digraph if and only if, for every $1 \leq i \leq n$ and every i th-order coreset U , there is exactly one i -walk from each vertex of U to each vertex of $\alpha^i(U)$.

Proof: We call the desired condition for i the i -uniqueness condition in D . We first compare m -uniqueness in D with $(m + 1)$ -uniqueness in $L(D)$. As shown in Lemma 8, for each m th-order coreset U in D there is an $(m + 1)$ th-order coreset W in $L(D)$ (and vice versa) such that each vertex of W is an edge of D whose head is in U . Thus also each vertex of $\alpha_{L(D)}^{m+1}(W)$ is an edge of D whose tail is in $\alpha_D^m(U)$. Thus $(m + 1)$ -walks in $L(D)$ beginning in W correspond naturally to m -walks in D beginning in U . In particular, the m -uniqueness condition in D is equivalent to the $(m + 1)$ -uniqueness condition in $L(D)$.

The result now follows easily by induction on n . The case $n = 1$ is contained in Theorem 6. For the induction step, suppose that the characterization holds when $n = m$; we prove it for $n = m + 1$. For necessity, suppose that D is an m th-order line digraph, so $L(D)$ is an $(m + 1)$ th-order line graph. By the induction hypothesis, the i -uniqueness condition holds in D for $1 \leq i \leq m$, and hence the i -uniqueness condition holds in $L(D)$ for $2 \leq i \leq m + 1$. Since $L(D)$ is a line digraph, it also holds in $L(D)$ for $i = 1$.

Conversely, suppose that D satisfies the i -uniqueness condition for $1 \leq i \leq m + 1$. Since the claim holds for $n = 1$, we have $D = L(D')$ for some digraph D' . Now D' satisfies the i -uniqueness condition for $1 \leq i \leq m$. By the induction hypothesis, D' is an m th-order line digraph, and thus D is an $(m + 1)$ th-order line digraph. ■

5. THE CORESET DIGRAPH OF A DIGRAPH

By Theorem 2, the coresets of a digraph partition its vertex set. We define a special derived digraph of D with the elements of the coreset partition as the vertex set. Like the line digraph D , it is an intersection digraph where the sets used in the intersection representation are subsets of $V(D)$.

DEFINITION If D is a digraph with coreset partition U_0, \dots, U_m , then the *coreset digraph* $Y(D)$ is the digraph without multiple edges defined by $V(Y(D)) = U_0, \dots, U_m$ and $E(Y(D)) = \{U_i U_j : \alpha(U_i) \cap U_j \neq \emptyset\}$. We iterate this operation by defining $Y^1(D) = Y(D)$ and $Y^n(D) = Y(Y^{n-1}(D))$ for $n \geq 2$.

Fig. 3 shows the coreset digraph of the digraph in Fig. 1. Note that $Y(D)$ is the intersection digraph arising when the pair (S_i, T_i) assigned to the vertex U_i is $(\alpha(U_i), U_i)$. We give another interpretation of this operation and describe the effect of iterating it.

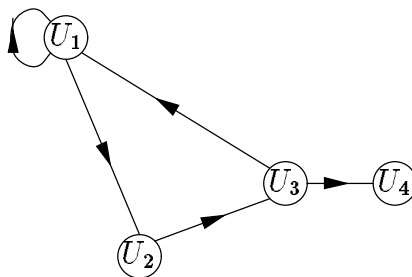


Fig. 3. A coreset digraph.

PROPOSITION 10. For a digraph D , $Y(D)$ is the digraph obtained by identifying the vertices of each coreset of D into a single vertex and deleting the resulting extra copies of edges. Furthermore, $Y(D) = D$ if and only if each nonempty coreset of D is a single vertex, and the sequence $Y^n(D)$ always converges to some digraph F as $n \rightarrow \infty$ (we write $Y^n(D) \rightarrow F$).

Proof: The coreset digraph of D has a edge from U_i to U_j precisely when some vertex of U_i has a successor in U_j ; this is achieved by identifying vertices within coresets. Thus the order of $Y(D)$ is less than that of D if and only some coreset has at least two elements, and the characterization of $Y(D) = D$ follows. Since the order of a digraph is an integer, the sequence $Y^n(D)$ converges because $Y(D) \neq D$ if and only if $Y(D)$ has fewer vertices than D . ■

COROLLARY 11. A digraph D is isomorphic to its coreset digraph if and only if it has at most one sink and has maximum indegree at most one. In particular, $Y(D) = D$ if and only if D is a path or is a digraph obtained by identifying a vertex of a cycle with the source of a path. Furthermore, $Y^n(D)$ always converges to such a graph.

Proof: If there is more than one sink or some vertex has more than one predecessor, then some coreset has size at least two. Conversely, if the condition holds, then each coreset has size one. The digraphs described are the only ones where the condition holds. Finally, $Y^n(D)$ always converges to a digraph F such that $Y(F) = F$. ■

One could measure the complexity of D by the value n where $Y^n(D)$ first reaches its limit.

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