Linear Discrepancy of Posets

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slides available on DBW preprint page

Joint work with
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The Problem

Tanenbaum–Trenk–Fishburn [2001]: Patients must be seen in a linear order, but "more urgent" is a poset P. We must treat x before y if $x < y$ in P. If $x \| y$ in P, then x and y should be treated not long apart.

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More generally, $\mathrm{ld}(P) \ge \mathrm{width}(P) - 1$.

An example

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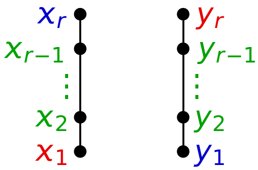

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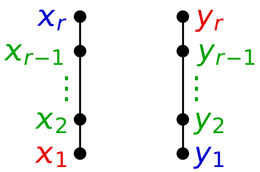

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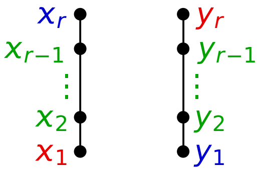

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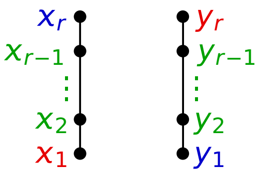

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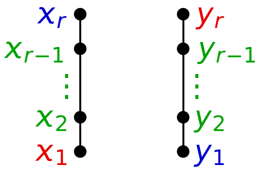

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• In this example, each element is incomparable to exactly r other elements.

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However, the conjecture in general is false (this talk).

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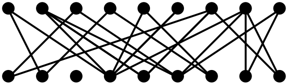

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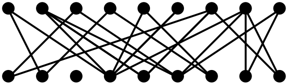

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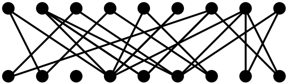

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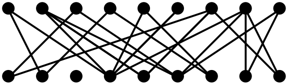

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Multiplied by $2n$ still $\to 0$, so $\mathbb{P}[r > n - 1 + 2pn] \to 0$.

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W.h.p., $r \leq n + 6\sqrt{n \ln n}$ and $\mathrm{ld}(P) \geq 2n - 2\sqrt{n \ln n}$,
so in this model almost always $\mathrm{ld}(P) \geq 2r - O(\sqrt{r \ln r})$. ■

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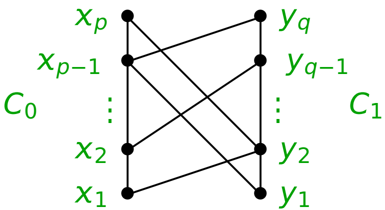

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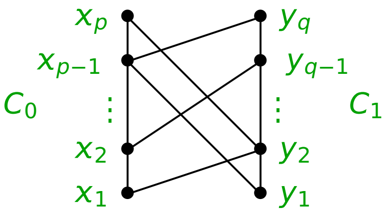

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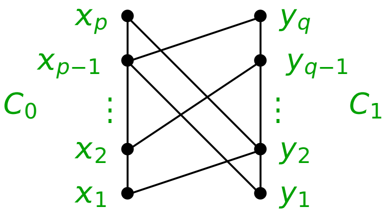

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Use these intervals to define the linear extension.

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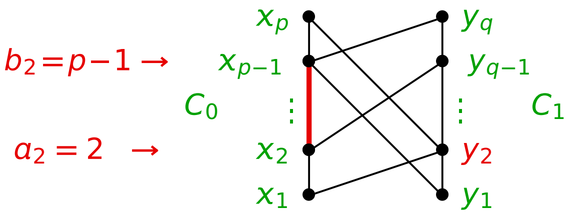

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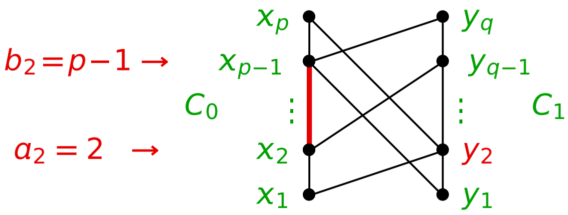

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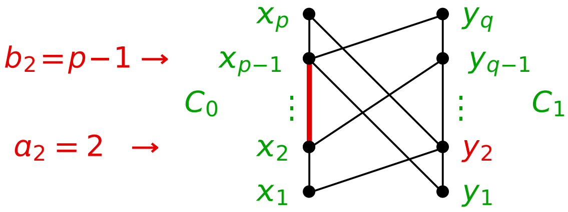

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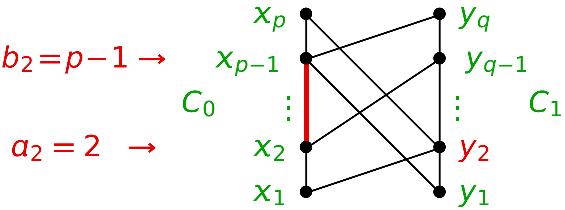

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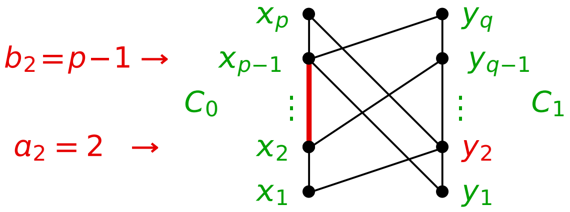

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It remains to show that if $x_i \| y_j$, then at most $\frac{3r-1}{2} - 1$ elements lie between them on L.

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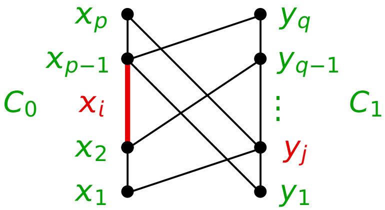

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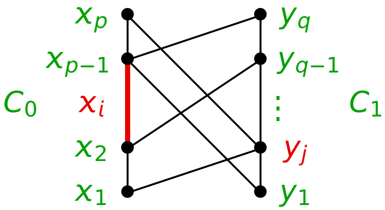

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Since $x_i \| y_j$, every element of C_1 between y_j and x_i is incomparable to x_i, as is y_j; hence $m_1 \leq r - 1$.

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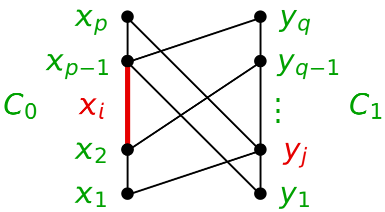

Fix $x_i \| y_j$. Let m_k be the number of elements of C_k between x_i and y_j on L; we want $m_0 + m_1 \leq \left\lfloor \frac{3r-1}{2} \right\rfloor - 1$.

Since $x_i \| y_j$, every element of C_1 between y_j and x_i is incomparable to x_i, as is y_j; hence $m_1 \leq r - 1$.

Since $x_i \in I(y_j)$, the placement of y_j just above x_{s_j} within C_0 guarantees $m_0 \leq \left\lfloor \frac{b_j - a_j}{2} \right\rfloor$.

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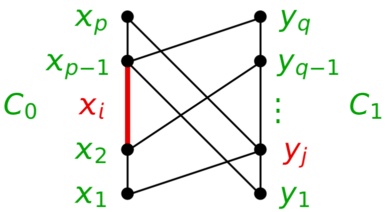

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$$m_0 + m_1 \leq \left\lfloor \frac{b_j - a_j}{2} \right\rfloor + r - 1 \leq \left\lfloor \frac{3(r-1)}{2} \right\rfloor .$$

■

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Thm. If L is any linear extension of P, then $t(L) \leq 3\mathrm{ld}(P)$, with inequality infinitely often.

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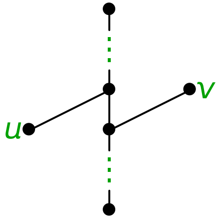

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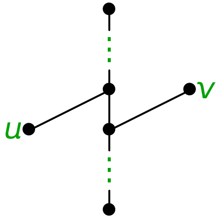

L	$u <$ chain $< v$	$(2k+1)$-tight
L'	$\frac{2k+1}{3} < u < \frac{2k-2}{3} < v < \frac{2k+1}{3}$	$\frac{2k+1}{3}$-tight

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$\therefore\ t = |P'| - 1 = |S_0| + |S_1| + |S_2| + 1 \leq 3t' \leq 3\mathrm{ld}(P)$. ∎

Products of Chains

Thm. (TTF [2001]) $\mathrm{ld}(\mathbf{2}^n) = 2^n - 3 \cdot 2^{n/2}$ for even n.
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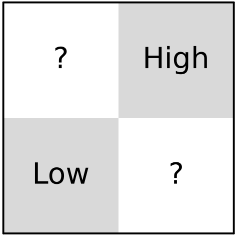

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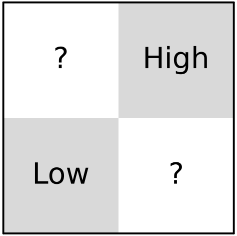

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Thm. (Choi-West) The general upper bound
$\mathrm{ld}(\mathbf{k}^d) \leq (1 - 2^{-d+1})k^d + O(k^{d-1})$ is sharp up to the lower-order term when $d = 4$.