Lichiardopol's conjecture on disjoint cycles in tournaments

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November 1, 2019

Abstract

In 2010, N. Lichiardopol conjectured for $q \geq 3$ and $k \geq 1$ that any tournament with minimum out-degree at least $(q-1)k-1$ contains k disjoint cycles of length q. We prove this conjecture for $q \geq 5$. Since it is already known to hold for $q \leq 4$, this completes the proof of the conjecture.

Keywords: Tournaments; Minimum out-degree; Disjoint cycles
AMS Subject Classification: 05C70, 05C38

1 Introduction

We consider cycles in digraphs (directed graphs); a cycle is a strongly connected digraph in which every vertex has indegree 1 and outdegree 1. The *length* of a cycle is the number of edges; a *q-cycle* is a cycle of length q. By "k disjoint cycles" we always mean k pairwise vertex disjoint cycles. A *tournament* is a digraph obtained from a complete graph by assigning a direction to each edge.

A famous conjecture of Bermond and Thomassen [5] for arbitrary digraphs asserts that large minimum outdegree guarantees many disjoint cycles.

Conjecture 1.1 ([5]). *If a digraph D has minimum outdegree at least $2k-1$, then D contains k disjoint cycles.*

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This conjecture is trivial for $k = 1$. It was proved for $k = 2$ by Thomassen [22] and for $k = 3$ by Lichiardopol, Pór, and Sereni [16]. Bang-Jensen, Bessy, and Thomassé [3] proved it for the special case of tournaments. Instead of considering special values of k or special classes of digraphs, one can also seek to reduce the minimum outdegree needed to guarantee k disjoint cycles. Alon [1] showed that $64k$ suffices, and later Bucić [7] reduced this to $18k$.

For general digraphs, Conjecture 1.1 remains open. In the special case of tournaments, stronger results are possible. Lichiardopol [17] conjectured that with large minimum outdegree, one can control not only the number of disjoint cycles but also their length.

Conjecture 1.2 ([17])**.** *If $q \geq 3$ and $k \geq 1$, then every tournament with minimum outdegree at least $(q-1)k-1$ contains k disjoint q-cycles.*

In a tournament, disjoint cycles of any length lead to disjoint 3-cycles by using chords. Hence the result of [3] yields the special case of Conjecture 1.2 for $q = 3$.

Theorem 1.3 ([3])**.** *Every tournament T with minimum outdegree at least $2k-1$ has k disjoint 3-cycles.*

Bessy, Lichiardopol, and Sereni [6] had earlier proved that every tournament with minimum indegree and outdegree both at least $2k-1$ has k disjoint cycles. In support of Conjecture 1.2, Lichiardopol [17] proved the two weaker theorems below, the first of which improves the result of [6].

Theorem 1.4 ([17])**.** *If $q \geq 3$ and $k \geq 1$, then every tournament with minimum outdegree and indegree both at least $(q-1)k-1$ contains k disjoint q-cycles.*

Ma and Yan [18] improved Theorem 1.4 by guaranteeing more than k disjoint cycles under the same conditions, so the conclusion of Theorem 1.4 is not sharp ([19] addressed the special case of 4-cycles in regular tournaments).

Theorem 1.5 ([17])**.** *If $q \geq 3$ and $k \geq 1$, then every tournament with minimum outdegree at least $(q-1)k-1$ contains $\lceil k-1-\frac{k-2}{q}\rceil$ disjoint q-cycles.*

The case $q = 4$ of Conjecture 1.2 was proved in the masters thesis of S. Zhu [26]. Hence our result in this paper completes the proof of Conjecture 1.2. Using some of our lemmas and similar methods, Wang, Ma, and Yan [25] gave an independent proof of the cases $q \leq 9$. Our result for $q \geq 5$ is self-contained and does not use any of their arguments.

Theorem 1.6. *For $q \geq 5$ and $k \geq 1$, every tournament with minimum outdegree at least $(q-1)k-1$ contains k disjoint q-cycles.*

For Conjecture 1.2 (and Theorem 1.6), the degree hypothesis is not known to be sharp. A trivial lower bound on the minimum outdegree needed to guarantee the conclusion is $qk/2$, since a tournament on $qk-1$ vertices in which every vertex has outdegree $qk/2-1$ does not have enough vertices to have k disjoint q-cycles. Given that the needed inequalities are easier to satisfy when q is large, we ask whether there is a positive constant ε such that minimum outdegree $(1-\varepsilon)qk$ suffices when q is sufficiently large.

Motivated by this problem on tournaments, one may wonder whether large outdegree can guarantee disjoint cycles of the same length in general digraphs, even without constraining which length it is. Our result guarantees this for tournaments. Thomassen [22] conjectured such a relationship for general digraphs, but Alon [1] showed that it cannot hold.

Theorem 1.7 ([1])**.** *For all $r \in \mathbb{N}$, some digraph with minimum outdegree r has no two edge-disjoint cycles of the same length (and hence also no disjoint cycles of the same length).*

Other papers on disjoint cycles in digraphs include [2] and [9]. Disjoint cycles have also been studied in undirected graphs, where the results are more plentiful. The Bermond–Thomassen Conjecture is in fact the directed analogue of the Corrádi–Hajnal Theorem [12].

Theorem 1.8 ([12])**.** *For $n, k \in \mathbb{N}$ with $n \geq 3k$, every n-vertex undirected graph with minimum degree at least $2k$ contains k disjoint cycles.*

There are many extensions and variations on this result. Those most similar to our work consider the lengths of disjoint cycles guaranteed by a threshold on the minimum degree. Let $\delta(G)$ denote the minimum degree of a graph G; also, the *order* of a graph or digraph is the number of vertices.

Thomassen [23] proved for $k \geq 2$ that every graph G with $\delta(G) \geq 3k+1$ and order at least some constant c_k contains k disjoint cycles of the same length. Thomassen conjectured that minimum degree $2k$ suffices, which had earlier been conjectured for $k=2$ by Häggkvist. Egawa [13] proved Thomassen's conjecture for $k \geq 3$ with a threshold of $|V(G)| \geq 17k+o(k)$. Verstraëte [24] later proved Thomassen's conjecture in full. Verstraëte also conjectured that order at least $4k$ is enough to guarantee k disjoint cycles of the same length when $\delta(G) \geq 2k$.

Chiba, Fujita, Kawarabayashi, and Sakuma [10] guaranteed for $k \in \mathbb{N}$ a constant c_k such that every graph with order at least c_k and minimum degree at least $2k$ contains k disjoint even cycles, with special exceptions. Other degree conditions for disjoint cycles in undirected graphs can be found in [4] and in the survey [11].

2 Structure of the Proof

To prove Theorem 1.6, we prove a theorem that was mainly inspired by the proof of the Bermond–Thomassen Conjecture for tournaments by Bang-Jensen, Bessy, and Thomassé [3].

Theorem 2.1. *Fix $k, q \in \mathbb{N}$ with $q \geq 5$ and $k \geq 2$, and let T be a tournament with minimum outdegree at least $(q-1)k-1$. For any family $\mathcal{F}$ of $k-1$ disjoint q-cycles in T, there is a family of k disjoint q-cycles in T using at most $3q-6$ vertices outside the cycles in $\mathcal{F}$.*

We show first that this suffices. For a path with vertices $v_1, \ldots, v_q$ in order, we use the notation $\langle v_1, \ldots, v_q \rangle$ (each edge in $\langle v_1, \ldots, v_q \rangle$ is oriented from v_i to v_{i+1}); we also call this a v_1, v_q-*path*. For a cycle with vertices $v_1, \ldots, v_q$ in order, we use the notation $[v_1, \ldots, v_q]$. The outdegree of a vertex v is $d^+(v)$, and the minimum outdegree in a digraph D is $\delta^+(D)$.

It is well-known that every tournament contains a spanning path (Rédei's Theorem [21]), and that a tournament is strongly connected if and only if it contains a spanning cycle (Camion's Theorem [8]), where a digraph is *strongly connected* or *strong* if it contains a u, v-path for any two vertices u and v. Moreover, a strong touranment (or the subtournament induced by any cycle) is *pancyclic*, meaning that it contains cycles of all lengths from 3 through through the number of vertices. When invoking this, we say "by pancyclicity". Finally, Moon [20] showed that every strong tournament with at least three vertices is *vertex pancyclic*, meaning that through any vertex there are cycles of all lengths.

Lemma 2.2. *Theorem 2.1 implies Theorem 1.6.*

Proof. Assuming that Theorem 2.1 holds, we prove Theorem 1.6 by induction on k. When $k = 1$, we are given a tournament T with $\delta^+(T) \geq q - 2$. Since every tournament has a spanning path, we may let $\langle v_1, \ldots, v_n \rangle$ be a spanning path of T. Since $d^+(v_n) \geq q - 2$ and v_{n-1} is not an outneighbor of v_n, the edge to the earliest outneighbor of v_n along P completes a cycle of length at least q in T. By pancyclicity, T contains a q-cycle.

For the induction step, suppose $k > 1$. We have $\delta^+(T) \geq (q-1)k - 1 > (q-1)(k-1) - 1$. By the induction hypothesis, T contains $k - 1$ disjoint q-cycles. From this family $\mathcal{F}$ of $k - 1$ disjoint q-cycles, Theorem 2.1 produces a family of k disjoint q-cycles. □

We will prove Theorem 2.1 by considering two cases. In Theorem 4.1, we will prove that the conclusion holds when $q \geq 5$ and $k \leq q$. In Theorem 5.1, we will prove by induction on k that the conclusion holds when $q \geq 5$ and $k > q$, using Theorem 4.1 as a basis. The restriction to using $3q - 6$ vertices outside $\mathcal{F}$ will be helpful in proving Theorem 5.1.

3 Cycles and Paths in Tournaments

Here we prove some structural lemmas about tournaments that will be useful in the proof of Theorem 4.1. We use $T[S]$ to denote the subtournament of T induced by the vertex set S. We also let $d^+(X, Y)$ denote the number of edges with tail in X and head in Y, where X and Y may be a set of vertices or a single vertex. When uv is an edge, we say that v is a *successor* of u and u is a *predecessor* of v.

Lemma 3.1. *If C is a cycle of length m in a tournament T, where $m \geq 4$, then $T[V(C)]$ contains a cycle C' of length $m - 1$ such that the omitted vertex u of C has at least two predecessors in $V(C')$.*

Proof. Since every strong tournament is pancyclic, $T[V(C)]$ contains a cycle C^* of length $m-1$. Let u' be the vertex of C not in C^*, and let v be the predecessor of u' on C. If u' has another predecessor on C^*, then let $u = u'$ and $C' = C^*$. Otherwise, let $\langle v, w, x\rangle$ be the portion of C^* leaving v (this exists since $m \geq 4$). Since u' has only one predecessor in $V(C)$, both $u'w$ and $u'x$ are edges. Now form C' by replacing w with u' in C^*, and let $u = w$. The omitted vertex u now has predecessors v and u' on C'. □

We believe in general that for $m \geq 2r$, a strong tournament with m vertices contains a cycle of length $m-1$ such that the omitted vertex has at least r predecessors in C'. We proved this for $r = 3$, but the proof is considerably longer than for $r = 2$, and we only need the result for $r = 2$. The threshold on m is sharp; a tournament on $2r-1$ vertices in which every vertex has $r-1$ predecessors and $r-1$ successors has no vertex with r predecessors.

Lemma 3.2. *If C is a cycle of length m in a tournament T, where $m \geq 5$, then $T[V(C)]$ contains a cycle C'' of length $m-2$ omitting vertices x and y with $xy \in E(T)$ such that y has at least one predecessor in $V(C'')$.*

Proof. We use Lemma 3.1 twice. First, it gives us a cycle C' in $T[V(C)]$, with length $m-1$, such that the remaining vertex u has at least two predecessors in $V(C')$. We then apply Lemma 3.1 using this C' as C; it gives us a cycle C'' in $T[V(C')]$, with length $m-2$, such that the remaining vertex v has at least two predecessors in $V(C'')$. From C, the vertices u and v have been omitted. Each has at least one predecessor in $V(C'')$. Hence we let x be the tail and y be the head of the edge joining u and v. □

We say that a vertex set X *dominates* a vertex set Y in a tournament if every edge joining X and Y is oriented from X to Y.

Lemma 3.3. *If C is a cycle of length m in a tournament T, where $m \geq 4$, and S is a 3-set in $V(C)$, then at least one vertex of S has at least two predecessors in $V(C)$.*

Proof. If the claim fails, then $T[S]$ is a 3-cycle. Now failure of the claim requires S to dominate $V(C) - S$, contradicting that C is a cycle. □

Lemma 3.4. *Let T be a tournament with minimum outdegree at least $(q-1)k-1$, where $2 \leq k \leq q$ and $q \geq 5$. Let $\mathcal{F}$ be a family of $k-1$ disjoint q-cycles in T, with vertex sets $V_1, \ldots, V_{k-1}$ inducing cycles $C_1, \ldots, C_{k-1}$, and let P be a path through the remaining vertices. If S_1 and S_2 partition $V(P)$ with S_2 dominating S_1 in T and $|S_1| \leq q$, then*

(a) $d^+(V^*, S_1) \leq \frac{|S_1|+1}{2q-2}|S_1|$ for some $V^* \in \{V_1, \ldots, V_{k-1}\}$.
(b) If $|S_1| \geq 2$, then $d^+(S_1, z) \geq 2$ for any $z \in V^*$.
(c) If $|S_1| \leq q-1$, then $d^+(u, V^*) \geq 3$ for any $u \in S_1$.
(d) If S_1 dominates $z \in V^*$, and z has at least r predecessors in V^*, then $d^+(z, S_2) \geq r$.

Proof. of (a): Since S_2 dominates S_1,

$$d^+(S_1, \bigcup V_i) \;\geq\; [(q-1)k-1]|S_1| - \frac{|S_1|(|S_1|-1)}{2} \;=\; \left[(q-1)(k-1) + \frac{2q-|S_1|-3}{2}\right]|S_1|.$$

By the pigeonhole principle, there exists $C_i \in \mathcal{F}$ such that $d^+(S_1, V_i) \geq (q-1+\frac{2q-|S_1|-3}{2(k-1)})|S_1|$. Since $|S_1| \leq q$ and $q \geq 3$, we have $2q - |S_1| - 3 \geq 0$. We then use $k \leq q$ to conclude $d^+(S_1, V_i) \geq (q - \frac{|S_1|+1}{2q-2})|S_1|$. Let C^* be this cycle C_i, with vertex set V^*.

Proof of (b): If $d^+(S_1, z) \leq 1$ for some $z \in V^*$, then $d^+(V^*, S_1) \geq d^+(z, S_1) \geq |S_1| - 1$. Using part (a), we obtain

$$|S_1| - 1 \leq \frac{|S_1|+1}{2q-2}|S_1|. \tag{1}$$

With $s = |S_1|$, the inequality can be rewritten as $2q - 2 - s(2q - 3 - s) \geq 0$. However, with $2 \leq s \leq q$, the left side of this is negative when $q \geq 5$.

Proof of (c): If $d^+(u, V^*) \leq 2$ for some $u \in S_1$. Now $d^+(V^*, S_1) \geq d^+(V^*, u) \geq q - 2$. Using (a) and $|S_1| \leq q - 1$, we obtain $q - 2 \leq \frac{q}{2q-2}(q-1)$, which requires $q \leq 4$.

Proof of (d): Since z is dominated by S_1 and has at least r predecessors in V^*,

$$d^+(z, S_2) = d^+(z) - d^+(z, \bigcup_i V_i) \geq (q-1)k - 1 - [q(k-1) - (r+1)] = q - k + r.$$

Since $k \leq q$, we obtain $d^+(z, S_2) \geq r$. □

4 The Case of Small k

In this section we prove Theorem 2.1 for $k \leq q$, stated as Theorem 4.1. Essentially, we provide an algorithm to produce the desired family $\mathcal{F}^*$ of k disjoint q-cycles by iteratively increasing the length of a cycle found outside the $k-1$ disjoint q-cycles. The subtournament T' induced by the vertices not in the given cycles has a spanning path P; let v be its last vertex. If v lies in a cycle of length at least q in T', then by pancyclicity there is a q-cycle in T', and we are done. Hence our approach, given a longest cycle through v in T' (in the first step the length may be 0), is to rearrange $\mathcal{F}$ to find a new family $\mathcal{F}'$ of $k-1$ disjoint q-cycles so that the vertex at the end of the resulting remaining path P' lies in a longer cycle.

Some of the claims in this argument are not valid when $q = 4$. Nevertheless, the same framework applies when $q = 4$, with additional more detailed reasoning.

Theorem 4.1. *Given $k, q \in \mathbb{N}$ with $q \geq 5$ and $k \leq q$, let T be a tournament with $\delta^+(T) \geq (q-1)k - 1$. For any family $\mathcal{F}$ of $k-1$ disjoint q-cycles in T, there is a family $\mathcal{F}^*$ of k disjoint q-cycles in T whose union has at most $3q - 6$ vertices outside the cycles in $\mathcal{F}$.*

Proof. These hypotheses are the same as those of Lemma 3.4 once we obtain a partition (S_2, S_1) of the vertices outside $\mathcal{F}$ such that S_2 dominates S_1. Given such a partition, let C^* with vertex set V^* be the cycle in $\mathcal{F}$ guaranteed by Lemma 3.4(a). Let P be a spanning path through the subtournament T' of vertices not used by $\mathcal{F}$, with last vertex v.

When T' has a cycle through v, let l be the maximum length of such a cycle; otherwise $l = 0$. If $l \geq q$, then pancyclicity of the subtournament spanned by this cycle provides a q-cycle to complete $\mathcal{F}^*$. Otherwise, we obtain a new family $\mathcal{F}'$ where l is larger, generally by replacing C^* with a new cycle $\widehat{C}$ and defining a new path P' through the vertices outside $\mathcal{F}'$. We will use at most two new vertices in $\widehat{C}$ at each step that increases l, except that the steps to reach $l \geq 4$ will use at most three new vertices. In addition, when a sufficiently long cycle appears, it uses at most $q-1$ new vertices, because the subtournament that was outside the $k-1$ given q-cycles entering that step did not contain a cycle of length at least q. Thus in total at most $3q-6$ new vertices are used.

Case 1: *$l = 0$, so T' has no cycle through v.* We seek $\widehat{C}$ and P' so that there is a cycle outside $\mathcal{F}'$ through the last vertex of P'. Let u be the predecessor of v on P.

Case 1a. *T' has no cycle through u* (see Figure 1). Let $S_1 = \{u, v\}$ and $S_2 = V(T') - S_1$. Since T' has no cycle through u or v, S_2 dominates S_1. By Lemma 3.4(a), there exists $C^* \in \mathcal{F}$ such that $d^+(V^*, S_1) \leq \frac{3}{2q-2} \cdot 2$, and $\frac{3}{2q-2} \cdot 2 < 1$ when $q \geq 5$. Thus S_1 dominates V^*.

Since $q \geq 5$, Lemma 3.2 implies that $T[V^*]$ contains a cycle C'' of length $q-2$ such that y has a predecessor in $V(C'')$, where xy is the edge in $T[V^*] - V(C'')$. Choose three vertices in $V(C'')$. By Lemma 3.3, among them is a vertex z with at least two predecessors in V^*. Since also z is dominated by S_1, Lemma 3.4(d) guarantees $d^+(z, S_2) \geq 2$. Let w be a successor of z in S_2, and let z' be the successor of z on C''. Replacing zz' in C'' with $\langle z, w, u, z' \rangle$ yields a q-cycle $\widehat{C}$ with two vertices outside $\mathcal{F}$. Replace C^* with $\widehat{C}$ to form $\mathcal{F}'$ from $\mathcal{F}$.

Since S_2 dominates S_1, we can form the P' outside $\mathcal{F}'$ by appending v, x, y in order to a spanning path of $T[S_2] - w$. Since S_1 dominates V^*, and y has at least x and a vertex of C'' as predecessors in V^*, Lemma 3.4(d) yields $d^+(y, S_2) \geq 2$. Thus y has a successor in S_2 other than w, so there is a cycle through y using vertices of P'.

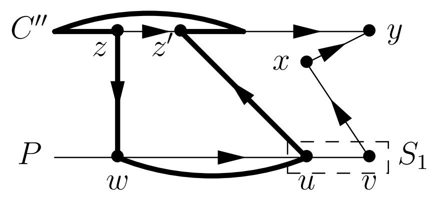

Figure 1: Case 1a of Theorem 4.1.

Case 1b. *T' has a cycle containing u.* Let B be a longest cycle containing u in T'. Let $S_1 = V(B) \cup \{v\}$ and $S_2 = V(T') - S_1$. Any edge from $V(B)$ to S_2 yields a larger strong tournament in T' containing $V(B)$, which contains a larger cycle containing u. Hence

S_2 dominates S_1. Also $|V(B)| \le q-1$, since otherwise we have the kth q-cycle. Hence $4 \le |S_1| \le q$, so Lemma 3.4 applies. Choose C^* with vertex set V^* as given by Lemma 3.4.

We will use the following "degree fact". If $\mathcal{F}'$ is a family of $k-1$ disjoint q-cycles, and the last vertex v' in a spanning path P' through the set S of remaining vertices has at least two predecessors used by $\mathcal{F}'$, then v' has a successor in S. The reason, using $k \le q$, is

$$[(q-1)k-1]-[q(k-1)-2] = q-k+1 \ge 1.$$

There is then a cycle in S through v', as desired.

The degree fact implies $d^+(V^*, v) \le 1$; otherwise already T' has a cycle through v. First suppose $d^+(V^*, v) = 1$ (see Figure 2). Let z be the predecessor of v in V^*, and let y be the successor of z on C^*. Form $\mathcal{F}'$ by replacing C^* with the cycle $\widehat{C}$ obtained from C^* by replacing y with v (the successor of y on C^* is a successor of v). Since $4 \le |S_1| \le q$, by Lemma 3.4(b) y has at least two predecessors in S_1, at least one in B (call this vertex w). Since S_2 dominates S_1, we can form P' by following P through S_2, then B from the successor of w on B to w, and finally the edge wy. Since v and z are predecessors of y, the degree fact yields a cycle outside $\mathcal{F}'$ through y. The only vertex used by $\mathcal{F}'$ and not by $\mathcal{F}$ is v.

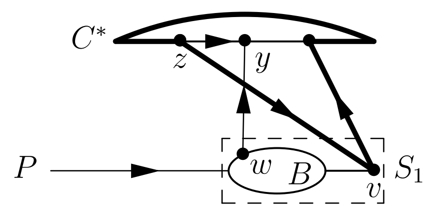

Figure 2: Case 1b of Theorem 4.1 when $d^+(V^*, v) = 1$.

Therefore, we may assume $d^+(V^*, v) = 0$, so v dominates V^* (see Figures 3 and 4). By Lemma 3.1, $T[V^*]$ contains a cycle C' of length $q-1$ such that the vertex y of $V^* - V(C')$ has at least two predecessors in C'. Let $V' = V(C')$.

If $T[V' \cup \{w\}]$ is strong for some $w \in V(B)$, then let $\widehat{C}$ be a spanning cycle in $T[V' \cup \{w\}]$ (outside $\mathcal{F}$ only w is used). Since $V(P) - \{v\}$ dominates v, we can let P' be a path through all of $V(P) - \{w, v\}$ followed by v and y. Since $d^+(V', y) \ge 2$, the degree fact applies to P'.

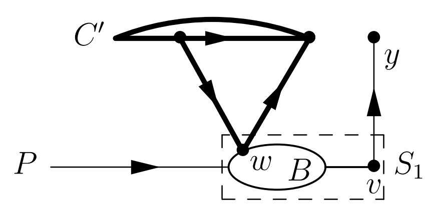

Figure 3: Case 1b of Theorem 4.1 when $d^+(V^*, v) = 0$ and $T[V' \cup \{w\}]$ is strong.

In the remaining case, every vertex of B dominates or is dominated by V'. If V' dominates some vertex of B, then $d^+(V^*, S_1) \geq q-1$, but $d^+(V^*, S_1) \leq \frac{|S_1|+1}{2q-2}|S_1|$ by Lemma 3.4(a). Since $q-1 \leq \frac{|S_1|+1}{2q-2}|S_1| \leq \frac{q^2+q}{2q-2}$ requires $q<5$, we conclude that $V(B)$ dominates V'.

Since also $d^+(V^*, v) = 0$, now S_1 dominates V' (see Figure 4). The remaining argument is similar to Case 1a. Since $q-1 \geq 4$, by Lemma 3.1 there is a cycle C'' of length $q-2$ in $T[V']$ such that the vertex y' of $V' - V(C'')$ has at least two predecessors in $V(C'')$. Now y and y' each have at at least two predecessors in V^*. Let zz' be the edge joining y and y'.

Since S_1 dominates V', any vertex of C'' has a successor in S_2, by Lemma 3.4(d); let w be one such successor. Now $T[V(C'') \cup \{w, v\}]$ is strong and has a spanning cycle $\widehat{C}$ of length q. Form $\mathcal{F}'$ from $\mathcal{F}$ by replacing C^* with $\widehat{C}$ (note that $\mathcal{F}'$ uses only w and v outside $\mathcal{F}$).

Using Lemma 3.4(b), $d^+(V(B), z) \geq 1$; let x be a predecessor of z in $V(B)$. Since S_2 dominates S_1, we can build a path P' that starts with all of $S_2 - \{w\}$ (in some order), then visits all of $V(B)$ ending with x, and finally follows $\langle x, z, z' \rangle$. Since v dominates V^*, vertex z' has at least two predecessors in $\widehat{C}$, and the degree fact applies.

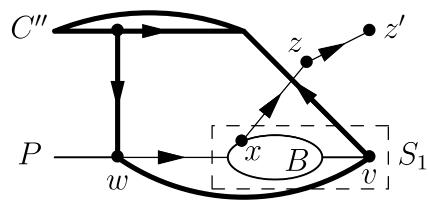

Figure 4: Case 1b of Theorem 4.1 when $d^+(V^*, v) = 0$ and $V(B)$ dominates V'.

Case 2: *$l > 0$, so T' has a cycle through v, the longest having length l.* Let C be such a cycle of length l. We find a new family $\mathcal{F}'$ and path P' outside it with a longer cycle through the last vertex of P'. We may assume $l < q$, since otherwise pancyclicity yields the desired q-cycle. Let $S_1 = V(C)$ and $S_2 = V(T') - S_1$. If there is an edge from S_1 to S_2, then S_1 and part of P induce a strong tournament, which has a longer spanning cycle (containing v). Hence S_2 dominates S_1, and Lemma 3.4 applies.

Let C^* with vertex set V^* be the cycle in $\mathcal{F}$ guaranteed by Lemma 3.4. Since $q \geq 5$, by Lemma 3.2 $T[V^*]$ contains a cycle C'' of length $q-2$ and an edge xy with $\{x, y\} = V^* - V(C'')$ such that y has at least one predecessor in $V(C'')$. Let $V'' = V(C'')$. By Lemma 3.4(a),

$$d^+(V'', S_1) \leq d^+(V^*, S_1) \leq \frac{|S_1|+1}{2q-2}|S_1| \leq \frac{q}{2q-2}(q-1) = \frac{q}{2}.$$

Since $q/2 < q-2$ when $q \geq 5$, in V'' there is a vertex z dominated by S_1. By Lemma 3.4(d), z has a successor w in S_2.

Case 2a. $|S_1| = 3$ (see Figure 5). Here $d^+(V^*, S_1) \leq \frac{4}{2q-2} \cdot 3 < 2$, by Lemma 3.4(a). If x or y has a successor in S_1, then let u' be this successor; otherwise choose $u' \in S_1$ arbitrarily.

Let the cycle C through S_1 be $[u', x', u]$, so $x'x \in E(T)$. Note that $T[V'' \cup \{w, u\}]$ is strong, with a spanning cycle $\widehat{C}$. Form $\mathcal{F}'$ from $\mathcal{F}$ by replacing C^* with $\widehat{C}$.

Let P' follow a spanning path through $S_2 - \{w\}$ and then $\langle u', x', x, y\rangle$. If $yu' \in E(T)$, then we have the cycle $[y, u', x', x]$ through y. Otherwise $d^+(y, S_1) = 0$, and by Lemma 3.4(d) y has at least two successors in S_2, which means it has one other than w. In this case y lies on a cycle of length more than l in $T[V(P')]$.

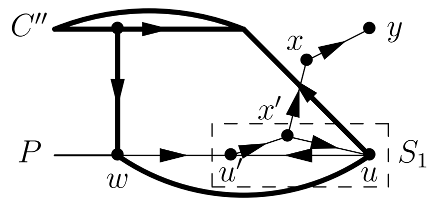

Figure 5: Case 2a of Theorem 4.1.

Case 2b. $|S_1| \geq 4$ (see Figure 6). By pancyclicity, $T[S_1]$ contains a cycle B omitting one vertex u of S_1. Since $l < q$, Lemma 3.4(c) implies $d^+(u, V^*) \geq 3$, yielding an edge uz' with $z' \in V''$. Using the edge zw from V'' to S_2 (obtained earlier), the path $\langle z, w, u, z'\rangle$ guarantees that $T[V'' \cup \{w, u\}]$ is strong and hence has a spanning q-cycle $\widehat{C}$. Form $\mathcal{F}'$ from $\mathcal{F}$ by replacing C^* with $\widehat{C}$.

By Lemma 3.4(b), x has a predecessor x' in $V(B)$. Build the path P' outside $\mathcal{F}'$ by visiting all of $S_2 - \{w\}$ (in some order), then all of $V(B)$ ending with x', and finally $\langle x', x, y\rangle$. Since y has at least two predecessors in $V'' \cup \{x\}$ and hence at most $q-3$ successors in V^*,

$$d^+(y, S_1 \cup S_2) \geq [(q-1)k - 1] - q(k-2) - (q-3) = q - k + 2 \geq 2.$$

If y has a successor outside $\{w, u\}$ in $S_1 \cup S_2$, then with $V(B) \cup \{x, y\}$ it induces a strong tournament of order at least $l+1$, yielding the desired cycle through the last vertex of P'.

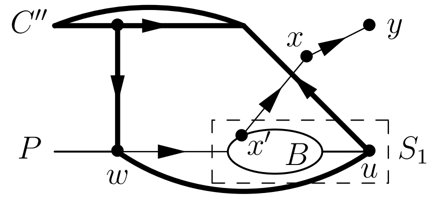

Figure 6: Case 2b of Theorem 4.1.

Hence w and u must be the only successors of y in $S_1 \cup S_2$. This forces $d^+(V'', y) = 1$. If any vertex of V'' has a successor in S_2 other than w, then we obtain $\mathcal{F}'$ as above using that vertex instead of w. Hence w must be the only successor of vertices in V'', so $d^+(V'', S_2) \leq q - 2$. We already computed $d^+(V^*, S_1) \leq q/2$, and one of the edges counted is yu.

For any $\alpha \in V''$, using $k \leq q$ we have

$$d^+(\alpha, S_1 \cup S_2) \geq [(q-1)k - 1] - q(k-2) - d^+(\alpha, V^*) \geq q - 1 - d^+(\alpha, V^*).$$

When we sum over all $\alpha \in V''$, the last term counts all edges in $T[V'']$, possibly $q-2$ edges from V'' to x, and one edge from V'' to y. Hence

$$d^+(V'', S_1 \cup S_2) \geq (q-2)(q-1) - \binom{q-2}{2} - (q-2) - 1 = (q-2)\frac{q-1}{2} - 1.$$

The upper and lower bounds on $d^+(V'', S_1 \cup S_2)$ require $(q-2)(q-1)/2 - 1 \leq 3q/2 - 3$, but this inequality requires $q < 5$. Hence we obtain the desired improvement $\mathcal{F}'$.

In all cases we have improved the family $\mathcal{F}$ as desired. □

5 The Case of Large k

For the proof of Theorem 5.1, we will use Theorem 4.1 as a basis for induction on k. The two theorems have the same conclusion.

Theorem 5.1. *Given $k, q \in \mathbb{N}$ with $q \geq 5$, let T be a tournament with $\delta^+(T) \geq (q-1)k - 1$. For any family $\mathcal{F}$ of $k-1$ disjoint q-cycles in T, there is a family of k disjoint q-cycles in T whose union has at most $3q-6$ vertices outside the cycles in $\mathcal{F}$.*

We first discuss the basic set-up for the argument, defining notation to be used throughout the proof. We call the desired family an *extension* of $\mathcal{F}$; finding it is *extending* $\mathcal{F}$. By Theorem 4.1, we may assume $k \geq q+1$. In the tournament T, consider a family $\mathcal{F}$ of $k-1$ disjoint q-cycles in T. We may assume that the tournament T' given by deleting the vertices covered by $\mathcal{F}$ contains no cycle with length at least q; otherwise we have the desired extension.

Let P be a spanning path in T', listed as $\langle u_l, \ldots, u_1 \rangle$. Since $\delta^+(T) \geq (q-1)k - 1$ implies $|V(T)| \geq 2(q-1)k - 1$, we have $|V(T')| \geq 2(q-1)k - 1 - q(k-1) = (q-2)(k+1) + 1$. Since $k \geq 3$, we conclude $l = |V(T')| \geq 4q - 7$.

Partition $V(P)$ into $\{U_1, S, U_2\}$ by letting $U_1 = \{u_1, \ldots, u_q\}$, $S = \{u_{q+1}, \ldots, u_{4q-11}\}$, and $U_2 = V(P) - (U_1 \cup S)$. All three sets are nonempty, with $|S| = 3q - 11$ and $|U_2| \geq 4$. Also, since T' contains no q-cycle, the edge joining u_j and u_i is oriented as $u_j u_i$ when $j - i \geq q - 1$. Hence U_2 dominates U_1.

We aim to find a value $t \in \{1, 2\}$ such that we can replace t cycles in $\mathcal{F}$ with $t+1$ cycles of length q using at most $3q-6$ vertices of T'. This will complete the proof.

Let X and Y be two disjoint sets of vertices in T. We say that there is an *r-matching from X to Y* if the set of edges with tail in X and head in Y contains r edges with no common endpoints. In order to guarantee the existence of desired matchings, we will use the famous König–Egerváry Theorem (König [15], Egerváry [14]), phrased for bipartite digraphs with all edges directed from one part to the other.

Lemma 5.2 ([14, 15])**.** *If there is no r-matching from X to Y, then $X \cup Y$ contains a set of at most $r-1$ vertices whose deletion eliminates all edges from X to Y.*

For the proof of Theorem 5.1, we need a number of additional lemmas. The first is a standard application of the König–Egerváry Theorem, which we will apply with various values of the parameters.

Lemma 5.3. *Let X and Y be disjoint vertex sets in T, with $s = \min\{|X|, |Y|\}$ and $t = \max\{|X|, |Y|\}$. If $d^+(X, Y) > (r-1)t$, where $1 \le r \le s$, then T contains an r-matching from X to Y.*

Proof. Since $r-1$ vertices cover at most $(r-1)t$ edges, the König–Egerváry Theorem implies that the desired matching exists. □

Lemma 5.4. *Let C be a q-cycle in $\mathcal{F}$. If there is a vertex $v \in V(C)$ with at least $3q-6$ successors in U_2, each having at least two successors in C, then there is an extension of $\mathcal{F}$.*

Proof. Let $W \subseteq U_2$ be such a set of $3q-6$ successors of v. Since $T[V(C)]$ is strong, by Moon's Theorem it has a $(q-1)$-cycle C' containing v, omitting one vertex u of C. Since each vertex $w \in W$ has a successor in C other than u, the subtournament $T[V(C') \cup \{w\}]$ is strong and has a spanning q-cycle (see Figure 7).

Let $T' = T - V(C')$ and $\mathcal{F}_0 = \mathcal{F} - \{C\}$. Since T' omits only $q-1$ vertices, $\delta^+(T') \ge (q-1)(k-1)-1$, and $\mathcal{F}_0$ is a family of $k-2$ cycles of length q in T'. Using the induction hypothesis, we can extend $\mathcal{F}_0$ to a family $\widehat{\mathcal{F}}$ of $k-1$ cycles of length q in T' using at most $3q-6$ new vertices. Since $|W \cup \{u\}| = 3q-5$, some vertex in $W \cup \{u\}$ is not used by $\widehat{\mathcal{F}}$.

If u is not used, then adding C to $\widehat{\mathcal{F}}$ completes the desired extension. If u is used, then at most $3q-7$ vertices not in $\mathcal{F}$ are used in $\widehat{\mathcal{F}}$. In this case, some vertex $w \in W$ is not used, and a spanning cycle in $T[V(C') \cup \{w\}]$ completes the extension $\mathcal{F}'$ using a total of at most $3q-6$ vertices not in $\mathcal{F}$. □

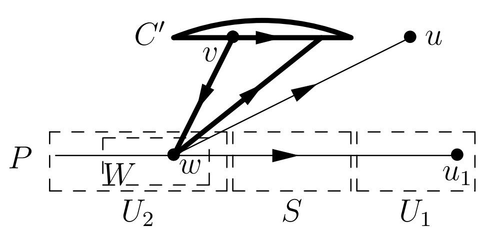

Figure 7: Figure for Lemma 5.4.

The need to find an unused vertex in $W \cup \{u\}$ in the preceding proof is the reason we limit the number of new vertices in Theorem 4.1 and in the induction hypothesis there.

We use Lemma 5.4 to prove the next two lemmas.

Lemma 5.5. *Let C be a q-cycle in $\mathcal{F}$. Suppose that $\mathcal{F}$ has no extension.*

(i) If $d^+(C, U_2) \geq 3q - 6$, then T contains a 2-matching from C to U_2.

(ii) If $d^+(C, U_2) \geq 6q - 13$, then T contains a 3-matching from C to U_2.

Proof. (i) If there is no such 2-matching, then by Lemma 5.2 one vertex covers all the edges from $V(C)$ to U_2. Such a vertex v can only be in C, and there is no other edge from $V(C)$ to U_2. Since $q \geq 3$, each of the $3q - 6$ successors of v in U_2 has at least two successors in $V(C)$. By Lemma 5.4, we have an extension of $\mathcal{F}$.

(ii) If there is no such 3-matching, then by Lemma 5.2 two vertices u and v cover all the edges from $V(C)$ to U_2, of which there are at least $6q - 13$. If u and v are both in $V(C)$, then one has at least $3q - 6$ successors in U_2, each of which has at least two successors in C, since $q \geq 4$. Otherwise, name u and v with $u \in U_2$ and $v \in V(C)$; now v has at least $5q - 14$ successors in U_2 other than u. These vertices have no predecessors in $V(C)$ other than v; hence they have at least two successors in $V(C)$. Since $5q - 14 \geq 3q - 6$ when $q \geq 4$, in either case Lemma 5.4 applies to guarantee an extension of $\mathcal{F}$. □

Lemma 5.6. *Let C be a q-cycle in $\mathcal{F}$. If T contains*

(i) a q-matching from U_1 to $V(C)$ and a 2-matching from $V(C)$ to U_2, or

(ii) a $(q-1)$-matching from U_1 to $V(C)$ and a 3-matching from $V(C)$ to U_2,

then there is an extension of $\mathcal{F}$.

Proof. We will obtain two disjoint cycles of length at least q in $T[V(C) \cup V(P)]$, where P is a spanning path through the vertices outside $\mathcal{F}$. By pancyclicity, the subtournament induced by the vertices of each such cycle contains a q-cycle. Since T' induces no cycle of length at least q, each of the two new cycles replacing C contains at most $q - 1$ new vertices.

Since $q \geq 5$, we have $3q - 11 \geq 2q - 6$. Hence $|S| \geq 2q - 6$. Since $V(P)$ induces no cycle of length at least q, any edge joining two vertices of P with at least $q - 2$ vertices between them on P is directed from the earlier to the later vertex.

Let M and M' be the given matchings from U_1 to $V(C)$ and from $V(C)$ to U_2, respectively. We prove (i) and (ii) together. In either case, let u be the last vertex of P matched into C from U_1 by M; note that $u \in \{u_2, u_1\}$. After following the edge from u to $V(C)$, let v be the vertex matched into U_2 by M' that is reached first when continuing along C, and let vw be this edge of M'. Let Q be the path thus followed, from u via M, along C, ending with vw.

Now choose $yz \in M' - \{vw\}$ so that y is also the head of an edge in M (under (ii), more than one edge of M' remains, but then at most one vertex of C is not covered by M.) Say that a vertex of U_1 *leads to y* if it is matched by M into the path along C that starts with the successor of v on C and ends at y. If z is closer to S than w along P, then let x be the highest-indexed vertex of U_1 (closest to S) that leads to y. Otherwise, let x be the lowest-indexed vertex of of U_1 that leads to y. Let R be the path leaving x via M, then along C to y, ending with yz. We want to form two cycles of length at least q in $T[V(C) \cup V(P)]$ by adding vertices of P to Q or R. Let $P[a, b]$ or $C[a, b]$ denote the a, b-path along P or C.

Case 1: *z is closer to S than w along P, so x is the highest-indexed vertex of U_1 leading to y* (see Figure 8). First consider $x = u_q$. Let one cycle be $R \cup P[z, x]$. Since $|S| \geq 2q - 6$ and this cycle contains $S \cup \{x, y, z\}$, it has length at least $2q-3$, which exceeds q. Meanwhile, P has at least $2q - 4$ vertices between w and u_{q-1}. Since $2q - 4 \geq q - 2$, the edge joining them is oriented as wu_{q-1}. Hence $Q \cup wu_{q-1} \cup P[u_{q-1}, u]$ is a cycle with at least q vertices.

When $x \neq u_q$, the edge joining u_{2q-2} and x has the desired orientation, because along P it skips u_q and $q - 3$ vertices of S. Hence $R \cup P[z, u_{2q-2}] \cup u_{2q-2}x$ is a cycle. Since $|S| \geq 2q-6$, it has at least $q-3$ vertices in S plus $\{x, y, z\}$. The other cycle is $Q \cup wu_{2q-3} \cup P[u_{2q-3}, u_{q+1}] \cup u_{q+1}u$. Since w is earlier than z along P, there are at least $q - 2$ vertices between w and u_{2q-3} along P; the same is true of u_{q+1} and u. Hence this is a cycle, and $\{u_{2q-3}, \ldots, u_{q+1}\} \cup \{u, v, w\}$ has at least q vertices.

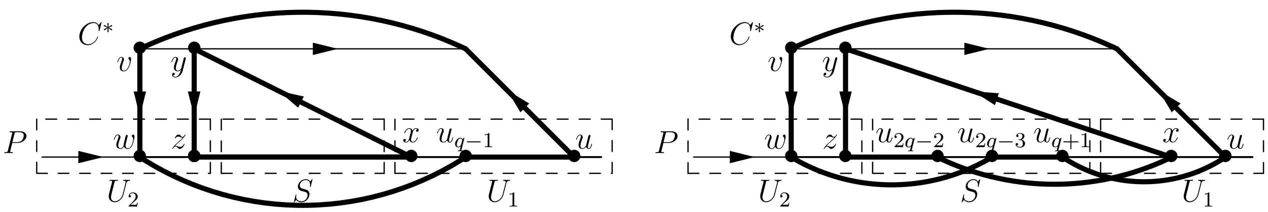

Figure 8: Case 1 of Lemma 5.6.

Case 2: *z is farther from S than w along P, so x is the lowest indexed vertex of U_1 leading to y.* Let $x = u_t$, and define the paths Q and R as in Case 1. Note that $t \leq q$. We want the two cycles to be $R \cup zu_{t+q-3} \cup P[u_{t+q-3}, x]$ and $[Q \cup P[w, u_{t+q-2}] \cup u_{t+q-2}u_{t-1} \cup P[u_{t-1}, u]$. The jumps along P must skip at least $q - 2$ vertices. This is explicit for $u_{t+q-2}u_{t-1}$. Since $(z, w) = (u_j, u_i)$ with $j > i > 4q - 11$, and $j - (t + q - 3) \geq 3q - 6 - t \geq 2q - 6$, the other construction is also a cycle if $2q - 6 \geq q - 1$, which holds when $q \geq 5$.

For length at least q, the first cycle has $q-3$ vertices along P plus at least $\{x, y, z\}$, and the second adds to $\{u, v, w\}$ all of $S \cup U_1$ except the $q-2$ vertices used by the first cycle and u_1 and maybe u_2. Since $4q - 11 - q \geq q - 3$ when $q \geq 4$, both cycles are long enough. ☐

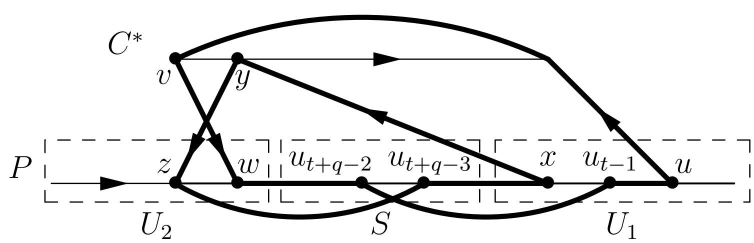

Figure 9: Case 2 of Lemma 5.6.

Lemma 5.7. *Let C and C' be two members of $\mathcal{F}$, with $W = V(C)$ and $W' = V(C')$. If T contains a q-matching from U_1 to W and a 3-matching from W' to U_2, and $d^+(W, W')) \geq q(q-1) + 3$, then there is an extension of $\mathcal{F}$.*

Proof. Again use the same notation. We may assume that $T[V(P)]$ contains no q-cycle. We will extend $\mathcal{F}$ by replacing C and C' in $\mathcal{F}$ with three q-cycles (except in one case). We must ensure that they introduce at most $3q-6$ new vertices. The other members of $\mathcal{F}$ remain.

Since $d^+(W,W') \ge q(q-1)+3$, the set W has at least three vertices that each dominate W' (otherwise, $d^+(W,W') \le 2q+(q-2)(q-1) = q(q-1)+2$). Label these as x_1, x_2, x_3 so that with w_1x_1, w_2x_2, w_3x_3 being edges in the q-matching from U_1 to W, the vertices w_1, w_2, w_3 occur in that order along P through U_1 (w_1 is closest to S). See Figure 10.

Let the edges of the given 3-matching from W' to U_2 be y_1z_1, y_2z_2, y_3z_3, indexed so that z_3, z_1, z_2 occur in that order along P (z_2 is closest to S). Since each vertex in $\{x_1, x_2, x_3\}$ dominates W', we now have three disjoint paths from U_1 to U_2; the ith path R_i is $\langle w_i, x_i, y_i, z_i \rangle$.

We complete these three paths to disjoint cycles by adding vertices along the path P. Recall that P contains the path $\langle u_{4q-11}, \dots, u_{q+1} \rangle$ through S between U_2 and U_1. Along this path of $3q-11$ vertices define three disjoint paths, each having $q-4$ vertices (one vertex of S is not needed); call them Q_2, Q_3, Q_1 in order along P.

Let B_i be the cycle formed by combining R_i and Q_i; add the edges from the end of each of R_i and Q_i to the beginning of the other, except that between z_2 and Q_2 in B_2 we follow P to the end of U_2, and between Q_1 and w_1 in B_1 we follow P through the beginning of U_1. The edges from z_1 to Q_1, from z_3 to Q_3, from Q_3 to w_3, and from Q_2 to w_2, are oriented in the desired direction because they skip at least $q-2$ vertices along P and hence would complete cycles of length at least q if oriented in the other direction.

Note that B_3 has exactly q vertices. Possibly B_1 or B_2 has more vertices due to picking up extras at the beginning of U_1 or the end of U_2. However, we can shorten the cycles to length q by omitting vertices at the beginning of Q_1 and/or the end of Q_2. This only makes the jumps along P longer, so the edges make the jumps still have the desired orientation. The resulting cycle B_i' is a q-cycle using exactly two vertices used in $\mathcal{F}$ (x_i and y_i). Hence $\{B_1', B_2', B_3\}$ replaces $\{C, C'\}$ to yield an extension using $3q-6$ vertices not used by $\mathcal{F}$. $\square$

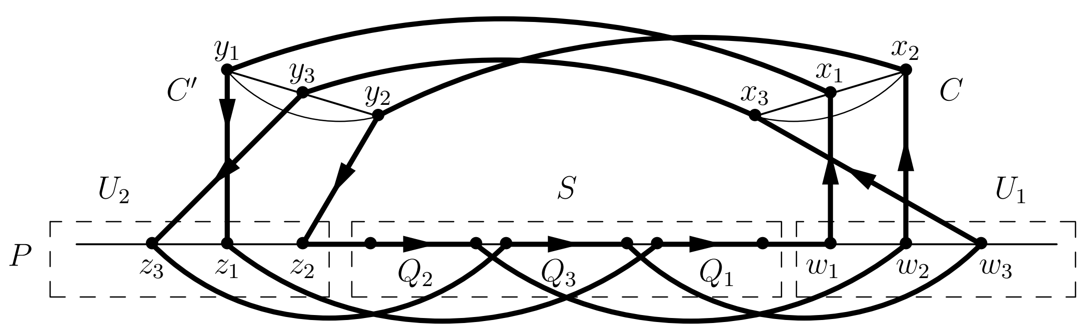

Figure 10: Figure for Lemma 5.7.

Finally we are ready to complete the main proof.

Proof of Theorem 5.1. We will obtain bounds on the sizes of various sets of edges under the assumption that $\mathcal{F}$ has no extension. These will lead to a contradiction.

Let $\mathcal{L}$ consist of those q-cycles in $\mathcal{F}$ receiving at least $q(q-1)+1$ edges from U_1 (thereby guaranteeing a q-matching from U_1, by Lemma 5.3). Let $\mathcal{M}$ consist of those q-cycles in $\mathcal{F}$ sending at least $6q-13$ edges to U_2 (thereby guaranteeing a 3-matching to U_2, by Lemma 5.5(ii)). Let $\mathcal{R} = \mathcal{F} - (\mathcal{L} \cup \mathcal{M})$. Let l, m, and r, respectively, denote the sizes of $\mathcal{L}, \mathcal{M}$ and $\mathcal{R}$. By Lemma 5.6, $\mathcal{L} \cap \mathcal{M} = \emptyset$. Hence $\{\mathcal{L}, \mathcal{M}, \mathcal{R}\}$ is a partition of $\mathcal{F}$, and

$$l + m + r = k - 1. \tag{2}$$

Now consider $d^+(U_1, S \cup U_2)$. If this is nonzero, then let u_{q+s} be the highest-indexed (earliest) vertex of P having a predecessor in U_1, and let u_{q+1-t} be the lowest-indexed vertex of P having a successor in $S \cup U_2$. Since P gives a path from u_{q+s} to u_{q+1-t}, the tournament induced by these $s+t$ vertices is strong and has a spanning cycle. Hence $s+t < q$. Also $d^+(U_1, S \cup U_2) \le st$. With $s+t \le q-1$, we have $d^+(U_1, S \cup U_2) \le (q-1)^2/4$.

Next we obtain upper and lower bounds on $d^+(U_1, \mathcal{F})$ in order to obtain an inequality involving l and m. Using the computation above,

$$d^+(U_1, \mathcal{F}) \ge q[(q-1)k - 1] - \binom{q}{2} - \frac{(q-1)^2}{4}. \tag{3}$$

To avoid obtaining an extension of $\mathcal{F}$ via Lemma 5.6, each cycle in $\mathcal{M}$ must avoid a $(q-1)$-matching from U_1 and hence receives at most $q(q-2)$ edges from U_1. By definition, each cycle in $\mathcal{L}$ or $\mathcal{R}$ receives at most q^2 or $q(q-1)$ edges from U_1, respectively (the latter because otherwise it would be in $\mathcal{L}$). Thus

$$d^+(U_1, \mathcal{F}) \le q^2 l + q(q-2)m + q(q-1)r.$$

After dividing by q, we obtain the following inequality:

$$(q-1)k - 1 - \frac{q-1}{2} - \frac{q-2}{4} - \frac{1}{4q} \le ql + (q-2)m + (q-1)r.$$

Since $l + m + r = k - 1$, we can rewrite the right side as $(q-1)(k-1) + l - m$. Thus

$$l - m \ge \frac{q}{4} - 1 - \frac{1}{4q}, \tag{4}$$

and we can drop the $-1/(4q)$ term since l and m are integers.

Finally, we will obtain upper and lower bounds on the number of edges leaving $\mathcal{L} \cup \mathcal{R}$ in order to obtain an inequality that cannot be satisfied. First, since $\delta^+(T) \ge (q-1)k - 1$,

$$d^+(\mathcal{L} \cup \mathcal{R}, \overline{\mathcal{L} \cup \mathcal{R}}) \ge q(l+r)[(q-1)k - 1] - \binom{q(l+r)}{2}. \tag{5}$$

The absence of extensions imposes bounds on the number of edges leaving $\mathcal{L}$. Since every cycle in $\mathcal{L}$ receives a q-matching from U_1, Lemma 5.5(i) and Lemma 5.6(i) imply $d^+(\mathcal{L}, U_2) \le l(3q-7)$. By Lemma 5.7, $d^+(\mathcal{L}, \mathcal{M}) \le lm[q(q-1)+2]$. Also, $d^+(\mathcal{R}, \mathcal{M}) \le q^2mr$. Each cycle in $\mathcal{R}$ sends at most $6q-14$ edges to $|U_2|$ (otherwise it would be placed in $\mathcal{M}$), so $d^+(\mathcal{R}, U_2) \le (3q-7)2r$. Also $d^+(\mathcal{L}\cup\mathcal{R}, S) \le (3q-11)q(l+r)$, since $|S| = 3q-11$.

The lower bound $d^+(U_1, \mathcal{F}) \ge q((q-1)k-1) - \binom{q}{2} - (q-1)^2/4$ is from (3). Since every cycle in $\mathcal{M}$ has a 3-matching to U_2, by Lemma 5.6 there is no $(q-1)$-matching from U_1 to a cycle in $\mathcal{M}$; hence $d^+(U_1, \mathcal{M}) \le mq(q-2)$. Thus

$$d^+(U_1, \mathcal{L}\cup\mathcal{R}) \ge q[(q-1)k-1] - \binom{q}{2} - \frac{(q-1)^2}{4} - q(q-2)m.$$

Let $\alpha = \binom{q}{2} + (q-1)^2/4$. Using (2), we conclude

$$\begin{aligned} d^+(\mathcal{L}\cup\mathcal{R}, U_1) &\le q^2(l+r) - q[(q-1)k-1] + q(q-2)m + \alpha \\ &= q^2(l+r) - q^2k + kq + q + q^2m - 2qm + \alpha \\ &= q^2(l{+}r{+}m{-}k{+}1) - q^2 + kq + q - 2qm + \alpha = q(l{+}r{+}2{-}m{-}q) + \alpha. \end{aligned}$$

Collecting the bounds proved above (with some rearrangement), we have

$$\begin{aligned} &d^+(\mathcal{L}\cup\mathcal{R}, \overline{\mathcal{L}\cup\mathcal{R}}) \\ &\quad= \quad d^+(\mathcal{L}\cup\mathcal{R}, \mathcal{M}) \quad + d^+(\mathcal{L}\cup\mathcal{R}, U_2) \quad + d^+(\mathcal{L}\cup\mathcal{R}, S) \quad + d^+(\mathcal{L}\cup\mathcal{R}, U_1) \qquad (6) \\ &\quad\le m[q^2(l{+}r) - l(q{-}2)] + (3q{-}7)(l{+}2r) + (3q{-}11)q(l{+}r) + q(l{+}r{+}2{-}m{-}q) + \alpha. \end{aligned}$$

Combining (5) and (6) and collecting the terms involving $q(l+r)$ yields

$$\begin{aligned} &q(l+r)\Big[(q-1)k - 1 - \frac{q(l+r)-1}{2} - mq - 3q + 7 + \frac{7}{q}\Big] \\ &\qquad\qquad\qquad \le -ml(q-2) + (3q-7)r - q(m+q-2) + \alpha. \qquad (7) \end{aligned}$$

Using (2), we simplify the last factor on the left:

$$\begin{aligned} (q-1)k - 1 - &\frac{q(l+r)-1}{2} - mq - 3q + 7 + \frac{7}{q} \\ &= (q-1)k - \frac{q(l+r+m+1)}{2} - \frac{mq}{2} - \frac{5q}{2} + \frac{13}{2} + \frac{7}{q} \\ &= \left(\frac{q}{2} - 1\right)(k-5) - \frac{mq}{2} + \frac{3}{2} + \frac{7}{q}. \qquad (8) \end{aligned}$$

On the right side of (7), we compute $\alpha - q(q-2) = \frac{-(q-1)(q-3)}{4} + 1$. On the left side, we replace $l+r$ with $k-1-m$, and on the right we replace r with $k-1-m-l$. The inequality is now

$$q(k-1-m)\Big[\Big(\frac{q}{2}-1\Big)(k-5)-\frac{mq}{2}+\frac{3}{2}+\frac{7}{q}\Big]$$
$$\leq -ml(q-2)+(3q-7)(k-1-m-l)-qm-\frac{(q-1)(q-3)}{4}+1.$$

The coefficients of l in its only appearances are negative. Hence for given q,k,m, the inequality can hold only if it holds when l takes its smallest allowed numerical value. By (4), we have $l \geq m-1+(q/4)$, which yields $l \geq m+1$ when $q \geq 5$ since $l \in \mathbb{N}$. Setting $l = m+1$, we now have a quadratic inequality for m in terms of k and q:

$$q(k-1-m)\Big[\Big(\frac{q}{2}-1\Big)(k-5)-\frac{mq}{2}+\frac{3}{2}+\frac{7}{q}\Big] \tag{9}$$
$$\leq -m(m+1)(q-2)+(3q-7)(k-2m-2)-qm-\frac{(q-1)(q-3)}{4}+1.$$

We first collect terms to write this as a quadratic inequality for m:

$$\left(\frac{q^2}{2}+q-2\right)m^2+\left[(q-q^2)k+3q^2+\frac{3}{2}q-23\right]m+c \leq 0, \tag{10}$$

where c depends only on k and q.

The inequality $l \geq m+1$ also yields $k-1 = l+m+r \geq 2m+1$, and hence $m \leq \lfloor k/2 \rfloor -1$. We thus want to show that (10) cannot be satisfied when $k \geq q+1 \geq 6$ and $0 \leq m \leq \lfloor k/2 \rfloor -1$.

In order to obtain the desired contradiction, it suffices to show that the left side of (10) is positive at its lowest allowed point. Since the coefficient of the quadratic term is positive, the quadratic polynomial is minimized where its derivative is 0. This occurs when

$$(q^2+2q-4)m = (q^2-q)k-(3q^2+\frac{3}{2}q-23). \tag{11}$$

The analysis simplifies if the lowest value of the polynomial in (10) among allowed values for m occurs at the highest allowed value, $\lfloor k/2 \rfloor - 1$. Since the graph of a quadratic polynomial is symmetric around the minimum, when k is even this holds if the minimizing point is at least $k/2-3/2$. When k is odd this also suffices, due to the floor function.

Thus we want the solution for m in (11) to be at least $(k-3)/2$. This holds unless $(q^2+2q-4)\frac{k-3}{2} > (q^2-q)k-(3q^2+\frac{3}{2}q-23)$. Solving for k yields

$$k < \frac{3q^2-3q-34}{q^2-4q+4}. \tag{12}$$

The right side of (12) is less than 5 for all q (since $5(q^2-4q+4)-(3q^2-3q-34)$ has no root). Hence in the case $k \geq q+1 \geq 5$ we have the desired reduction.

Hence it suffices to show that (10) or equivalently (9) cannot hold when $m = \lfloor k/2 \rfloor - 1$. When k is even, we set $m = k/2 - 1$ in (9). This simplifies the expression, since now $k - 2m - 2 = 0$ and $k - 1 - m = k/2$. We require

$$q\frac{k}{2}\left[\left(\frac{q}{2}-1\right)(k-5) - \frac{(k-2)q}{4}+\frac{3}{2}+\frac{7}{q}\right] \leq -\frac{(k-2)k}{4}(q-2)-q\frac{k-2}{2}-\frac{(q-1)(q-3)}{4}+1. \quad (13)$$

The right side is negative and decreases as k increases. The coefficient on k in the third factor on the left is $q/4 - 1$, which is positive, so the factor increases as k increases. Hence it suffices to show that the inequality cannot hold when k takes its least allowed value, $q + 1$. The inequality then simplifies to

$$q\frac{q+1}{2}\left[\left(\frac{q}{2}-1\right)(q-4) - \frac{(q-1)q}{4}+\frac{3}{2}+\frac{7}{q}\right] \leq -(q-1)\left[\frac{(q+1)(q-2)}{4}+\frac{q}{2}+\frac{(q-3)}{4}\right]+1.$$

The left side increases with q, and the right side decreases with q, so it suffices to show that the inequality fails when $q = 4$. The left side is then $10/4$ and the right side is $-53/4$.

When k is odd, we instead set $m = (k-3)/2$. In this case $k - 2m - 2 = 1$ and $k - 1 - m = (k+1)/2$, so (9) becomes

$$\begin{aligned} q\frac{k+1}{2}\left[\left(\frac{q}{2}-1\right)(k-5) - \frac{(k-3)q}{4}+\frac{3}{2}+\frac{7}{q}\right] \quad (14) \\ \leq -\frac{(k-3)(k-1)(q-2)}{4} + (3q-7) - \frac{(k-3)q}{2} - \frac{(q-1)(q-3)}{4} + 1. \end{aligned}$$

Again the last factor on the left increases with k, so again it suffices to consider the smallest allowed value for k, which is $q + 1$. We require

$$\begin{aligned} q\frac{q+2}{2}\left[\left(\frac{q}{2}-1\right)(q-4) - \frac{(q-2)q}{4}+\frac{3}{2}+\frac{7}{q}\right] \quad (15) \\ \leq -\frac{(q-2)q(q-2)}{4} + (3q-7) - \frac{(q-2)q}{2} - \frac{(q-1)(q-3)}{4} + 1. \end{aligned}$$

Again the left side increases with q and the right side decreases with q, so it suffices to show that the inequality cannot hold when $q = 4$. Then the left side is 15 and the right is $-11/4$.

Thus the inequality cannot hold for any allowed values of the parameters, and an extension must exist. ■

Although the computation at the end of the proof works for $q = 4$, other difficulties arise when seeking this extension. First, instead of $l \geq m + 1$ we must also consider $l = m$. Also, although most cases in the proofs of Theorem 4.1 and Lemmas 5.6 and 5.7 extend to $q = 4$ (sometimes with additional case analysis), the very special part of Case 2 in Lemma 5.6 when $(u, x, w, z) = (u_2, u_4, u_6, u_7)$ does not work. This can be fixed by changing $|S|$ from $3q - 11$ to $3q - 10$, but then the term $3/2$ on the left side of the numerical inequality beomes $1/2$, and the desired contradiction fails to occur in the one special case $(q, k, l, m) = (4, 5, 1, 1)$. Further analysis for that case could complete the proof for $q = 4$.

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