

On reconstruction of graphs from the multiset of subgraphs obtained by deleting ℓ vertices

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Dedicated to the memory of Vladimir Levenshtein

Abstract

The Reconstruction Conjecture of Ulam asserts that, for $n \geq 3$, every n -vertex graph is determined by the multiset of its induced subgraphs with $n - 1$ vertices. The conjecture is known to hold for various special classes of graphs but remains wide open. We survey results on the more general conjecture by Kelly from 1957 that for every positive integer ℓ there exists M_ℓ (with $M_1 = 3$) such that when $n \geq M_\ell$ every n -vertex graph is determined by the multiset of its induced subgraphs with $n - \ell$ vertices.

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1 Introduction

Among the basic problems in combinatorics are so-called *reconstruction problems*, where we try to identify an object knowing only partial information about it. A classic example is the famous Graph Reconstruction Conjecture of Ulam first posed in the thesis of Kelly in 1942:

Conjecture 1.1 (The Reconstruction Conjecture; Kelly [18, 19], Ulam [52]). *For $n \geq 3$, every n -vertex graph is determined by the multiset of its $(n - 1)$ -vertex induced subgraphs.*

The multiset of $(n - 1)$ -vertex induced subgraphs is called the *deck* of the graph, with each such subgraph being a *card* in the deck. The vertices in the cards are unlabeled, meaning that only the isomorphism class of each card is given. The restriction $n \geq 3$ is needed because the two graphs with two vertices have the same deck. A graph is *reconstructible* if it is determined by its deck, meaning that no other graph has the same deck. In this

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terminology, the Reconstruction Conjecture asserts that every graph with at least three vertices is reconstructible.

The conjecture has attracted a lot of attention. Graphs in many families are known to be reconstructible; these include disconnected graphs, trees, regular graphs, and perfect graphs. Surveys on graph reconstruction include [3, 4, 23, 24, 30].

Various parameters have been introduced to measure the difficulty of reconstructing a graph. Harary and Plantholt [17] defined the *reconstruction number* of a graph to be the minimum number of cards from its deck that suffice to determine it, meaning that no other graph has this multiset of cards in its deck (surveyed in [1]). All trees with at least five vertices have reconstruction number 3 (Myrvold [37]), and almost all graphs have reconstruction number 3 (Bollobás [2]). No graphs have reconstruction number 2, since two cards cannot determine whether the two vertices deleted to form the cards are adjacent.

Let K_{t_1, \dots, t_r} denote the complete r -partite graph with part-sizes t_1, \dots, t_r . Since $K_{r,r}$ and $K_{r+1, r-1}$ have $r + 1$ common cards, the reconstruction number of an n -vertex graph can be as large as $\frac{n}{2} + 2$ (Myrvold [36]). (Here $K_{r,s}$ is the complete bipartite graph with parts of sizes r and s .) Harary and Plantholt [17] strengthened the Reconstruction Conjecture by conjecturing that when $n \geq 3$ every n -vertex graph has reconstruction number at most $\frac{n}{2} + 2$, with equality only for $K_{n/2, n/2}$ and $2K_{n/2}$ in general, plus P_4 . Kocay and Kreher [20] constructed n -graphs with reconstruction number $\frac{n}{2} + 1$ when $n = 4q - 4$ and q is a prime power congruent to 1 modulo 4.

We can also study the reconstruction number of graph properties. Myrvold [35] and Bowler et al. [9] showed that any $\lfloor n/2 \rfloor + 2$ cards determine whether an n -vertex graph is connected. Much effort went into reducing the number of cards needed to determine m , the number of edges. Myrvold [38] showed that m and in fact also the degree list are determined by any $n - 1$ cards when $n \geq 7$ (this is sharp). Monikandan and Balakumar [33] showed that m is determined within 1 by any $n - 2$ cards (strengthening [45]). Woodall [53] proved for $n \geq \max\{34, 3p^2 + 1\}$ that m is determined within $p - 2$ by $n - p$ cards. Brown and Fenner [10] proved that m is determined by any $n - 2$ cards when $n \geq 29$, and they presented two 8-vertex graphs with six common cards whose numbers of edges differ by 1. Groenland, Guggiari, and Scott [16] proved that in fact m is determined by any $n - \sqrt{n}/20$ cards when n is sufficiently large.

These results concern not so much the reconstruction number as the *adversary reconstruction number* [35] or *universal reconstruction number* [8], which is the minimum t such that any t cards determine the graph (or a particular property). Bowler, Brown, and Fenner [8] presented infinite families of pairs of graphs sharing $2 \lfloor (n - 1)/3 \rfloor$ common cards, improving Myrvold [35, 37]. They conjectured that when n is sufficiently large, every n -vertex graph is determined by any $2 \lfloor (n - 1)/3 \rfloor + 1$ of its cards ($n > 12$ is needed).

Kelly [19] took another direction, considering cards obtained by deleting more vertices.

Definition 1.2. A k -card of a graph is an induced subgraph having k vertices. The k -deck of G , denoted $\mathcal{D}_k(G)$, is the multiset of all k -cards (given as isomorphism classes). A graph G is *determined by its k -deck* if $\mathcal{D}_k(H) = \mathcal{D}_k(G)$ implies $H \cong G$. A graph G (or a graph invariant) is ℓ -reconstructible if it is determined by $\mathcal{D}_{|V(G)|-\ell}(G)$ (agreeing on all graphs having that deck). The *maximum reconstructibility* of a graph G is the maximum ℓ such that G is ℓ -reconstructible.

Study of reconstruction from the k -deck was begun by Manvel [31]. There followed several papers by Nýdl, including surveys ([40, 44]) of the early results. Nýdl studied the least k (as a function of n) such that every n -vertex graph (or every n -vertex graph in a restricted family such as trees) is determined by its k -deck.

For an n -vertex graph, “determined by its k -deck” and “ ℓ -reconstructible” have the same meaning when $k + \ell = n$. The motivation for defining the maximum reconstructibility as a measure of the ease of reconstructing a graph is the following elementary observation.

Observation 1.3. For any graph G , the k -deck $\mathcal{D}_k(G)$ determines the $(k-1)$ -deck $\mathcal{D}_{k-1}(G)$.

The proof is that each card in $\mathcal{D}_{k-1}(G)$ appears in $|V(G)| - k + 1$ cards in \mathcal{D}_k . By Observation 1.3, information that is determined by the k -deck is also determined by the j -deck when $j > k$. This leads to a stronger version of the Reconstruction Conjecture.

Conjecture 1.4 (Kelly [19]). *For $\ell \in \mathbb{N}$, there is an integer M_ℓ such that every graph with at least M_ℓ vertices is ℓ -reconstructible.*

The original Reconstruction Conjecture is the claim $M_1 = 3$. Having checked by computer that every graph with at least six and at most nine vertices is 2-reconstructible (there are 5-vertex graphs that are not), McMullen and Radziszowski [32] asked whether $M_2 = 6$. With computations up to nine vertices, Rivshin and Radziszowski [47] conjectured $M_\ell \leq 3\ell$. Nýdl [43] disproved this, showing that M_ℓ must grow at least superlinearly in ℓ ; that is, $M_\ell/\ell \rightarrow \infty$. He proved that for any $n_0 \in \mathbb{N}$ and $0 < q < 1$, there are nonisomorphic n -vertex graphs for some n larger than n_0 having the same $\lfloor qn \rfloor$ -deck.

For more detailed understanding, it is natural to study the threshold number of vertices for ℓ -reconstructibility for graphs in a given family, which may be smaller than for the family of all graphs. We may write this as $M_\ell(\mathbf{G})$ for a family \mathbf{G} . For example, Spinoza and West [48] (see Section 4) proved that the path $P_{2\ell}$ and the graph $C_{\ell+1} + P_{\ell-1}$ have the same ℓ -deck (here “+” denotes disjoint union of graphs, while C_n, P_n, K_n respectively denote the cycle, path, and complete graph on n vertices). Thus $M_\ell(\mathbf{G}_2) \geq 2\ell + 1$ for $\ell \geq 3$, where \mathbf{G}_d is the family of graphs with maximum degree at most d . In fact, they proved $M_\ell(\mathbf{G}_2) = 2\ell + 1$.

For the family \mathbf{T} of trees, the same lower bound is known and is sharp for $\ell = 2$ (Giles [14]), but equality is open for $\ell \geq 3$. Let $S_{a,b,c}$ be the subdivision of $K_{1,3}$ consisting of paths of lengths a, b , and c with one common endpoint (in general, a tree consisting of

paths with one common endpoint is called a “spider”). Nýdl [39] observed that $S_{k-1,k-1,1}$ and $S_{k,k-2,1}$ are spiders with $2k$ vertices having the same k -deck. We will give a short proof of this using the results on common k -decks for graphs in \mathbf{G}_2 that are discussed in Section 4. The result implies $M_\ell(\mathbf{T}) \geq 2\ell + 1$, and Nýdl [39] conjectured that equality holds. One can generalize this question to the family \mathbf{T}_r of connected graphs G such that $|E(G)| - |V(G)| + 1 \leq r$; that is, $\mathbf{T} = \mathbf{T}_0$. Nýdl [39] constructed two graphs with $3k + 9$ vertices and $3k + 12$ edges having the same $2k$ -deck, thus yielding $M_\ell(\mathbf{T}_4) \geq 3\ell - 18$.

As with ordinary reconstruction, proving that the graphs in a family \mathbf{G} are ℓ -reconstructible may involve two steps. One is to show that the family is ℓ -recognizable, meaning that whether $G \in \mathbf{G}$ holds is determined by $\mathcal{D}_{|V(G)|-\ell}$. That is, every graph having the same deck as a graph in \mathbf{G} is also in \mathbf{G} . For example, Manvel [31] showed that when $|V(G)| = n \geq 6$, the $(n - 2)$ -deck determines whether G is connected, acyclic, unicyclic, regular, or bipartite. That is, these properties are 2-reconstructible when $n \geq 6$.

The pair $\{P_{2\ell}, C_{\ell+1} + P_{\ell-1}\}$ mentioned earlier shows for n -vertex graphs that guaranteeing ℓ -reconstructibility of the property of connectedness (or ℓ -recognition of the family of connected graphs) requires $n \geq 2\ell + 1$. The correct general threshold remains open.

On the other hand, the fraction of n -vertex graphs whose maximum reconstructibility is at least $(1 - o(1))n/2$ tends to 1 (see Section 8). This was observed originally by Müller [34]. In particular, there is surprisingly small difference between the maximum reconstructibility of almost all graphs and the failure of reconstructibility of the property of connectedness. Spinoza and West [48] showed that in fact in this setting only $\binom{\ell+2}{2}$ cards are needed, generalizing the concept of reconstruction number to ℓ -reconstruction number.

For some easily reconstructed families it is natural to fix the number of vertices kept in each card. The 2-deck of G determines only $|E(G)|$ and $|V(G)|$. The 3-deck determines also the number of edge incidences, whether G is triangle-free, and whether G belongs to the family of complete multipartite graphs, since that is true if and only if $P_2 + P_1$ is not an induced subgraph.

Results on ℓ -reconstructibility are known for degree lists, connectedness, trees, graphs with maximum degree 2, random graphs, and graphs that are disconnected, complete multipartite, or regular. We describe these results in the subsequent sections, and we include a few new results about these classes. In Section 5 we offer a new short proof of Nýdl’s result that $M_\ell(\mathbf{T}) \geq 2\ell + 1$. In Section 6 we show that n -vertex graphs whose components have at most $n - \ell$ vertices are ℓ -reconstructible, while graphs with components having more vertices are guaranteed to be ℓ -reconstructible only if the original Reconstruction Conjecture is true. In Section 7 we show that r -regular graphs with connectivity 1 are $(r + 1)$ -reconstructible.

We mention two other models of reconstruction. Levenshtein et al. [27] considered a local version of reconstruction in which the vertices of a graph are labeled and for an n -vertex graph G we have only n cards: for each vertex v , we are given the set $B_2(v)$ of the vertices at distance at most 2 from v (but do not know which of them are adjacent to v). It was

proved in [27] that every connected graph whose girth is at least 7 and whose diameter and minimum degree are at least 2 is reconstructible in this model. The authors also provided a graph with girth 6 that is not reconstructible. Levenshtein [25, 26] posed a more general problem where we know the sets $B_t(v)$ instead of $B_2(v)$ and presented partial results on it.

In Section 9, we describe the main results on a model that uses the term “ k -reconstructible” with a different meaning. In that model, due to Fraïssé [13], we are reconstructing digraphs (viewed as general binary relations), and we are told the identities of the deleted vertices. Our aim in describing these alternative models is to avoid future confusion.

2 Degree Lists

The *degree list* (also called *degree sequence*) of a graph is the multiset of degrees of the vertices in the graph. When studying the Reconstruction Problem, the first thing one learns is that the degree list of a graph is 1-reconstructible. It suffices to find the number m of edges, because the degree of each vertex v is the difference between m and the number of edges in the card $G - v$. The number m is the information provided by the 2-deck (known by Observation 1.3); one can also compute m directly from the $(n - 1)$ -deck by $m = \sum_v |E(G - v)| / (n - 2)$.

For decks with smaller cards, reconstruction of anything becomes more difficult. The pairs of graphs with the same k -decks mentioned in the introduction have the same degree list, but it is easy to construct examples with different lists.

Example 2.1. *For any positive a, b, c with $a + b + c = t \geq 4$, the graphs $C_t + P_1$ and $S_{a,b,c}$ with $t + 1$ vertices all have the same 3-deck. Note that $\Delta(C_t + P_1) = 2$ and $\Delta(S_{a,b,c}) = 3$. All these graphs have t copies of P_3 and $t(t - 3)$ copies of $P_2 + P_1$ as induced subgraphs (this involves a few cases for $S_{a,b,c}$), and the remaining 3-vertex induced subgraphs all have no edges. Hence $\mathcal{D}_3(C_t + P_1) = \mathcal{D}_3(S_{a,b,c})$, for all such (a, b, c) .*

Concerning thresholds, this easy example shows that guaranteeing ℓ -reconstructibility of the degree list or the maximum degree requires $n > \ell + 3$. In this example, we considered $\mathcal{D}_k(G)$ with $k \in \{\Delta(G), \Delta(G) + 1\}$. Manvel observed that having one more vertex in the cards prevents such examples.

Theorem 2.2 (Manvel [31]). *The degree list of a graph G is determined by $\mathcal{D}_{\Delta(G)+2}(G)$. It is also determined by $\mathcal{D}_{\delta(G)-1}(G)$.*

The result for $\Delta(G)$ is easy since all induced subgraphs with at most k vertices are visible in the k -deck. Hence $\mathcal{D}_{\Delta(G)+2}(G)$ shows that G has a vertex of degree $\Delta(G)$ and none larger. To determine the degree list, one can then proceed by induction on r to count the vertices of degree $\Delta(G) - r$ using the following tool observed originally by Manvel [31].

Lemma 2.3. *Let G be an n -vertex graph. The sum, over all cards in the k -deck $\mathcal{D}_k(G)$, of the number of vertices of degree j in the card, equals $\sum_{i=j}^{j+n-k} a_i \binom{i}{j} \binom{n-1-i}{k-1-j}$, where a_i is the number of vertices having degree i in G .*

The lemma holds because a vertex v appears and has degree j in a card in the k -deck if and only if the card is formed by choosing v along with j of its neighbors and $k-1-j$ of its nonneighbors. The lemma yields the following corollary by solving for a_i through successively smaller i .

Corollary 2.4 (Manvel [31]). *The degree list of a graph G is determined when both $\mathcal{D}_k(G)$ and the numbers of vertices with degree i for all i at least k are known.*

Example 2.5. For sharpness of Theorem 2.2, Manvel [31] showed that the maximum degree itself is not always determined by $\mathcal{D}_{\Delta(G)+1}(G)$. He constructed graphs G and H such that $\Delta(G) = k$, $\Delta(H) = k-1$, and $\mathcal{D}_k(G) = \mathcal{D}_k(H)$. Both graphs are forests of stars. However, in this construction the number of vertices is $(k+2)2^{k-2}$, exponential in k . In particular, $G = \sum_i \binom{k}{2i} K_{1,k-2i}$ and $H = \sum_i \binom{k}{2i+1} K_{1,k-1-2i}$. From this Manvel concluded that for all k there exist nonisomorphic graphs with the same k -deck.

Question 2.6. What is the smallest value of n such that n -vertex graphs G and H with maximum vertex degrees k and $k-1$ exist having the same k -deck?

Lemma 2.3 and Corollary 2.4 are used as tools in reconstruction of degree lists.

Theorem 2.7 (Chernyak [11]). *When $n \geq 6$, the degree list of every n -vertex graph is 2-reconstructible.*

This result is sharp, because the 5-vertex graphs $C_4 + P_1$ and $S_{2,1,1}$ from Example 2.1 have the same 3-deck but different degree lists. Exact results were pushed one step further.

Theorem 2.8 (Kostochka–Nahvi–West–Zirlin [21]). *For $n \geq 7$, the degree list of every n -vertex graph is 3-reconstructible.*

Theorem 2.8 is sharp: the 6-vertex graphs $C_5 + P_1$ and $S_{2,2,1}$ and $S_{3,1,1}$ from Example 2.1 all have the same 3-deck. Note that since the $(n-2)$ -deck determines the $(n-3)$ -deck, Theorem 2.8 combined with an analysis of 6-vertex graphs implies Theorem 2.7.

By making more thorough use of Lemma 2.3, Taylor [51] obtained a surprisingly small general threshold on the number of vertices for ℓ -reconstructibility of the degree list.

Theorem 2.9 (Taylor [51]). *For $n \geq g(\ell)$, the degree list of every n -vertex graph is ℓ -reconstructible, where*

$$g(\ell) = (\ell - \log \ell + 1) \left(e + \frac{e \log \ell + e + 1}{(\ell - 1) \log \ell - 1} \right) + 1.$$

Here e denotes the base of the natural logarithm.

This result also shows that the degree list of an n -vertex graph is reconstructible from the k -deck when n is not too much larger than k , regardless of the value of the maximum degree. In particular, $n \geq \frac{1+o(1)}{1-1/e}k$ suffices. This theorem seems rather strong about reconstructibility but perhaps does not say much about Question 2.6, giving only a linear lower bound.

Theorem 2.9 is strong but likely not sharp; answering the next question would improve it and generalize Theorem 2.8. By Theorem 2.9 the threshold is asymptotically at most $e\ell$. One could begin by seeking the largest graphs whose degree lists are not 4-reconstructible.

Question 2.10. For fixed $\ell \in \mathbb{N}$, what is the least threshold n_ℓ such that the degree list of every graph with at least n_ℓ vertices is ℓ -reconstructible?

3 Connectedness

For graphs with at least three vertices, connectedness is 1-reconstructible, because an n -vertex connected graph has at least two connected $(n-1)$ -cards, while a disconnected graph has at most one connected $(n-1)$ -card (when $n \geq 3$). Manvel [31] strengthened this result.

Theorem 3.1 (Manvel [31]). *For $n \geq 6$, the connectedness of an n -vertex graph is 2-reconstructible, and the threshold for n is sharp.*

The threshold is sharp because the 5-vertex graphs $C_4 + P_1$ and $S_{2,1,1}$ have the same 3-deck (Example 2.1). In fact, these graphs and their complements are the only 5-vertex graphs that are not 2-reconstructible [32].

All results that obtain a function $f(\ell)$ such that some property or class of graphs is ℓ -reconstructible for graphs with at least $f(\ell)$ vertices provide support for Kelly's Conjecture. For general ℓ , this is known for connectedness.

Theorem 3.2 (Spinoza–West [48]). *For $\ell \in \mathbb{N}$, the connectedness of every n -vertex graph is ℓ -reconstructible when $n > 2^{\ell(\ell+1)^2}$.*

The threshold in Theorem 3.2 is not sharp.

Conjecture 3.3 (Spinoza–West [48]). *For $n \geq 2\ell + 2$, the connectedness of an n -vertex graph is ℓ -reconstructible, and the threshold for n is sharp.*

For $\ell = 2$, the 5-vertex graphs $C_4 + P_1$ and $S_{2,1,1}$ again show that the conjecture is sharp. For larger ℓ the right answer may be $2\ell + 1$, which is needed due to the example $\{P_{2\ell}, C_{\ell+1} + P_{\ell-1}\}$ (see Section 4). There are three sets of two 7-vertex graphs that have the same 4-deck; none consists of a connected and a disconnected graph. Nýdl's graphs showing that M_ℓ grows superlinearly are all connected. We believe that connectedness of an n -vertex graph is ℓ -reconstructible whenever $\ell < \lceil n/2 \rceil$ (except for $\{C_4 + P_1, S_{2,1,1}\}$).

For $\ell = 3$, Spinoza and West [48] improved the threshold in Theorem 3.2 to $n \geq 25$. The exact answer was found later.

Theorem 3.4 (Kostochka–Nahvi–West–Zirlin [21]). *For every graph with at least seven vertices, connectedness is 3-reconstructible.*

Theorem 3.4 is sharp due to $\{C_5 + P_1, S_{2,2,1}, S_{3,1,1}\}$ (Example 2.1). When combined with a short analysis of 6-vertex graphs, Theorem 3.4 implies Theorem 3.1. The proof of Theorem 3.4 uses Theorem 2.8 to reduce the problem to graphs with exactly two vertices of degree 1 and none of degree 0.

Toward Conjecture 3.3, it would be interesting to find a substantial improvement of the threshold in Theorem 3.2 or to find the largest two graphs, one connected and one disconnected, that have the same deck of subgraphs with four vertices deleted.

4 Graphs with Maximum Degree 2

In Problem 11898 of the American Mathematical Monthly, Stanley posed a question related to reconstructing 2-regular graphs from their k -decks.

Problem 4.1 (Stanley [49]). *Let n and k be integers, with $n \geq k \geq 2$. Let G be a graph with n vertices whose components are cycles of length greater than k . Let $i_k(G)$ be the number of k -element independent sets of vertices of G . Show that $i_k(G)$ depends only on k and n .*

Let $s(G, H)$ denote the number of induced subgraphs of G isomorphic to H . Stanley's problem asserts $s(G, \overline{K}_k) = s(G', \overline{K}_k)$ for n -vertex 2-regular graphs G and G' whose components have length greater than k (here \overline{H} denotes the complement of H). Stanley's proposed solution of Problem 4.1 used generating functions.

Independent sets are just one type of k -vertex induced subgraph. In a graph with maximum degree 2 whose cycles have more than k vertices, all k -vertex induced subgraphs are *linear forests*, meaning disjoint unions of paths. By looking at a larger class of graphs, Spinoza and West gave a bijective proof by induction on k that proves the same conclusion for the number of subgraphs isomorphic to any k -vertex linear forest and thereby proves that the graphs with the stated property all have the same k -deck.

Theorem 4.2 (Spinoza–West [48]). *Let G and G' be graphs with maximum degree 2 having the same number of vertices and the same number of edges. If every component in each graph is a cycle with at least $k + 1$ vertices or a path with at least $k - 1$ vertices, then $\mathcal{D}_k(G) = \mathcal{D}_k(G')$.*

Important cases of the theorem, and indeed its proof, are captured by the following three statements, among which the third is the key, proved inductively.

Claim 4.3. $\mathcal{D}_k(C_{q+r}) = \mathcal{D}_k(C_q + C_r)$ if $q, r \geq k + 1$,
 $\mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$, and
 $\mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1})$ if $q, r \geq k$.

The proof of Theorem 4.2 reduces to Claim 4.3 by the following natural lemma.

Lemma 4.4. *If G , G' , and H are graphs, then $\mathcal{D}_k(G) = \mathcal{D}_k(G')$ if and only if $\mathcal{D}_k(G + H) = \mathcal{D}_k(G' + H)$.*

For every graph G with maximum degree 2, Theorem 4.2 provides a lower bound for the value k such that $\mathcal{D}_k(G)$ determines G and hence an upper bound on the maximum reconstructibility. Except for some small instances, these bounds turn out to be sharp. Without giving the complete details of the statement, the result is the following.

Theorem 4.5. *Let G be a graph with maximum degree 2. If m is the maximum number of vertices in a component, F is a component with m vertices, and m' is the maximum number of vertices in a component other than F , then G is k -deck reconstructible if and only if $k \geq \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}$, where $\epsilon = 1$ if F is a path (otherwise $\epsilon = 0$), and $\epsilon' \in \{0, 1, 2\}$.*

We omit the technical definition of ϵ' that incorporates the small exceptions to the general formula. In particular, for a 2-regular n -vertex graph G , the formula for k simplifies to $\max\{\lfloor m/2 \rfloor, m'\}$. Thus the maximum reconstructibility of the cycle C_n is $\lceil n/2 \rceil$, and no graph in this class has smaller reconstructibility.

That is, every 2-regular n -vertex graph is $\lceil n/2 \rceil$ -reconstructible, and in fact graphs with maximum degree 2 are $\lceil n/2 \rceil$ -reconstructible. For maximum degree 3 or 3-regular graphs, discussed in Section 7, much less is known. It is only known that 3-regular graphs are 2-reconstructible, with no nontrivial upper bounds known on the maximum reconstructibility.

5 Trees

Trees have played a prominent role in the study of reconstruction. The original 1957 paper of Kelly [19] showed that trees are 1-reconstructible. Giles [14] showed in 1976 that trees with at least five vertices are 2-reconstructible (P_4 and $K_{1,3}$ have the same 2-deck). According to Nýdl [44], the survey of Bondy and Hemminger [4] reported the existence of a preprint by Giles proving that sufficiently large trees are ℓ -reconstructible, but this was apparently never published and seems to remain open.

Nýdl gave a lower bound for the threshold M'_ℓ such that for $n \geq M'_\ell$, no two n -vertex trees have the same $(n - \ell)$ -deck. Since that paper is somewhat inaccessible (and does not present a full proof), we give a new short proof here using the third statement of Claim 4.3. Recall that $S_{a,b,c}$ is the spider with $a + b + c + 1$ vertices consisting of paths of lengths a , b , and c with a common endpoint.

Theorem 5.1 (Nýdl [42]). *The two trees $S_{k-1,k-1,1}$ and $S_{k,k-2,1}$ have the same k -deck.*

Proof. Let $G = S_{k-1,k-1,1}$ and $H = S_{k,k-2,1}$; both G and H have $2k$ vertices. We partition the k -decks according to the usage of the non-peripheral leaf, which we call v in each graph. The portions of the k -deck in which v does not appear are the same, since both equal $\mathcal{D}_k(P_{2k-1})$. The portions in which v appears and its neighbor does not are also the same, since they are the $(k-1)$ -decks (plus an isolated vertex) of $P_{k-1} + P_{k-1}$ and $P_k + P_{k-2}$, which by Claim 4.3 are the same.

In the remainder of the cards in the decks, v appears in a nontrivial component that is a spider $S_{a,b,1}$. Consider those cards where this spider takes a vertices from the first leg and b vertices from the second leg in the original specification of the host trees. Since $a, b \leq k-2$, such a spider exists in H if and only if it exists in G .

For each choice of (a, b) , the cards in this portion of the k -deck of G consist of the disjoint union of $S_{a,b,1}$ with cards in the $(k-a-b-2)$ -deck of $P_{k-1-a} + P_{k-1-b}$. Similarly, in the k -deck of H we have the disjoint union of $S_{a,b,1}$ with cards in the $(k-a-b-2)$ -deck of $P_{k-a} + P_{k-2-b}$. Since $k-a \geq k-a-b-2$ and $k-b-1 \geq k-a-b-2$, Claim 4.3 applies to guarantee that these portions of the two k -decks are the same. \square

Thus $M'_\ell \geq 2\ell + 1$.

Conjecture 5.2 (Nýdl [42]). $M'_\ell = 2\ell + 1$.

Note that Conjecture 5.2 is not quite the same as $M_\ell(\mathbf{T}) = 2\ell + 1$. Nýdl required only that no two n -vertex trees have the same $(n-\ell)$ -deck, but for ℓ -reconstructibility there is also the matter of showing that all possible reconstructions from the deck are trees; that is, showing that the family of trees is ℓ -recognizable when $n \geq 2\ell + 1$.

An n -vertex graph is a tree if and only if it has $n-1$ edges and is connected (or has $n-1$ edges and no cycles). From the 2-deck, we know the number of edges. If Conjecture 3.3 is true in the stronger form replacing $2\ell + 2$ with $2\ell + 1$ for $\ell \geq 3$, which Theorem 3.4 proves for $\ell = 3$, then combined with Conjecture 5.2 it would imply $M_\ell(\mathbf{T}) = 2\ell + 1$. Indeed, the full strength of Conjecture 3.3 probably is not needed; we only need to know from the $\lfloor n/2 \rfloor$ -deck whether an n -vertex graph with $n-1$ edges has a cycle of length at least $n/2$.

6 Disconnected and Complete Multipartite Graphs

One of the earliest results on reconstruction, by Kelly [19], is that disconnected graphs are 1-reconstructible. Manvel [31] discussed the ℓ -reconstructibility of disconnected graphs. We expand on this discussion to obtain a sharp threshold on the size of components that makes ℓ -reconstructibility easy.

We first prove what might be called the “negative” result.

Proposition 6.1. *If graphs with at least $\ell + 2$ vertices consisting of a connected graph and $\ell - 1$ isolated vertices are ℓ -reconstructible, then the original Reconstruction Conjecture holds.*

Proof. We need to prove 1-reconstructibility when $n \geq 3$. Kelly [19] proved this for disconnected graphs, so consider a connected n -vertex graph G . We are given $\mathcal{D}_{n-1}(G)$. By Observation 1.3, we also know $\mathcal{D}_{n-i}(G)$ for $2 \leq i \leq \ell$. Let $G' = G + (\ell - 1)K_1$; note that G' has at least $\ell + 2$ vertices. For $2 \leq i \leq \ell$, let \mathcal{D}'_i consist of $\binom{\ell-1}{\ell-i}$ copies of $C + (i - 1)K_1$ for each occurrence of C in $\mathcal{D}_{n-i}(G)$. Note that

$$\mathcal{D}_{n-\ell}(G') = \mathcal{D}_{n-1}(G) \cup (\mathcal{D}'_2 \cup \cdots \cup \mathcal{D}'_\ell).$$

Thus if $G + (\ell - 1)K_1$ is ℓ -reconstructible, then we have determined G from $\mathcal{D}_{n-1}(G)$. \square

Now consider n -vertex graphs whose components all have at most $n - \ell$ vertices. Manvel [31] observed that IF it is known that G is a such a graph, then G is ℓ -reconstructible. In fact, we show that such graphs can be recognized from the $(n - \ell)$ -deck. With Manvel's observation, this implies that these graphs are in fact ℓ -reconstructible.

The argument generalizes a proof of the 1-reconstructibility of disconnected graphs, involving a counting argument for ordinary graph reconstruction that was applied by Bondy and Hemminger [5] and originated with Greenwell and Hemminger [15]. A similar argument to that given here appears in the ‘‘Main Lemma’’ of Nýdl [44].

We need the basic idea of Kelly's Lemma [19], which counts the copies of a given graph F that appear in G by dividing the total number of appearances of F in the deck by the number of times each copy appears. We use an analogue for induced subgraphs and generalize to the $(n - \ell)$ -deck.

Lemma 6.2. *If G is an n -vertex graph, and F is a graph with at most $n - \ell$ vertices, then the number $s_F(G)$ of occurrences of F as an induced subgraph of G is ℓ -reconstructible.*

Proof. Let $p = |V(F)|$. Each induced copy of F appears in $\binom{n-p}{\ell}$ cards in $\mathcal{D}_{n-\ell}(G)$. Letting t be the total count of all appearances of F as an induced subgraph in all cards in $\mathcal{D}_{n-\ell}(G)$, we have $s_F(G) = t / \binom{n-p}{\ell}$. \square

Theorem 6.3. *If every connected subgraph of G has at most $n - \ell$ vertices, then G is ℓ -reconstructible.*

Proof. It suffices to determine, for every connected graph F , the number $c_F(G)$ of components of such a graph G that are isomorphic to F .

Let an *induced chain* of length r be a list F_0, \dots, F_r of connected induced subgraphs of G such that F_i an induced subgraph of F_{i+1} for $0 \leq i < r$. For any connected induced subgraph F of G , let the *depth* of F be the maximum r such that F is the first subgraph in an induced chain of length r .

Since every connected subgraph of G has at most $n - \ell$ vertices, all connected induced subgraphs of G appear in the deck. Since we know all these subgraphs, we can determine all the induced chains, and hence we know the depth of each connected induced subgraph.

If F has depth 0, then every induced copy of F in G is a component, so $c_F(G) = s_F(G)$. For larger depth, group the induced copies of F by the unique component of G containing that copy. Summing over all components of G , we obtain

$$s_F(G) = \sum_H s_F(H)c_H(G). \quad (1)$$

When $s_F(H) \neq 0$ and $F \neq H$, every induced chain starting at H can be augmented by adding F at the beginning, so H has smaller depth than F . Using Lemma 6.2 to compute $s_F(G)$ and applying the induction hypothesis to compute values of $c_H(G)$, we now know every quantity in (1) other than $c_F(G)$ and can solve for $c_F(G)$. \square

Here we used the property that for the family \mathcal{F} of connected graphs, every member of \mathcal{F} contained in G has at most $n - \ell$ vertices and belongs to a unique maximal member of \mathcal{F} contained in G , namely a component. The argument applies also to other families \mathcal{F} that have this property.

For a very special class of disconnected graphs, much stronger results about reconstructibility are known. Note first a simple observation, using the fact that the cards in the k -deck determine their complements.

Observation 6.4. A graph G is determined by its k -deck if and only if its complement \overline{G} is determined by its k -deck.

Hence when we discuss ℓ -reconstructibility of graphs whose components are complete graphs, we are also discussing ℓ -reconstructibility of complete multipartite graphs.

Let G be a disjoint union of complete graphs. Membership in this family is determined by the 3-deck, since a graph is a disjoint union of complete graphs if and only if it does not have P_3 as an induced subgraph. When the largest component has at most k vertices, Theorem 6.3 implies that the graph is determined by its k -deck (Spinoza and West [48] had observed that the $(k + 1)$ -deck suffices).

More interesting is the situation when we bound the number of parts rather than the size of the parts.

Theorem 6.5 (Spinoza–West [48]). *Every complete r -partite graph G is determined by its $(r + 1)$ -deck (as are disjoint unions of r complete graphs).*

The proof of Theorem 6.5 is actually algebraic. For $r \geq 2$, the 3-deck tells us that G is complete multipartite, and the absence of K_{r+1} in the deck makes it r -partite. Letting the part-sizes be q_1, \dots, q_r , form the polynomial $\prod_{i=1}^r (x - q_i)$. The coefficient of $(-1)^j x^{r-j}$ in the expansion is the number of complete cards in $\mathcal{D}_i(G)$. Since $\mathcal{D}_i(G)$ is determined by $\mathcal{D}_{r+1}(G)$, we know the polynomial and can find the roots q_1, \dots, q_r .

Theorem 6.5 is sharp for $r \leq 2$ and all n . It is immediate that complete bipartite graphs are not 2-deck reconstructible, since they are not determined by their numbers of edges and vertices (the 3-vertex cards are not given). For complete tripartite graphs, the 3-deck determines that the graph is a complete multipartite graph, but the following example shows that that is not sufficient.

Example 6.6 (Spinoza–West [48]). The complete multipartite graphs $K_{7,4,3}$ and $K_{6,6,1,1}$ have the same 3-deck. It consists of 84 copies of K_3 , 240 copies of P_3 , and 40 copies of \overline{K}_3 .

We expect that diligence will yield more general examples.

Question 6.7. Is it true for all $r \in \mathbb{N}$ with $r \geq 3$ that there are a complete r -partite graph and a complete $(r + 1)$ -partite graph having the same r -deck?

Finally, Nýdl considered the reconstructibility of disjoint unions of complete graphs where neither the number of components nor the sizes of the components are restricted. We can still recognize from the 3-deck that our graph is in this class.

Theorem 6.8 (Nýdl [41]). *Let G be an n -vertex graph that is a disjoint union of complete graphs. If $n < k \ln(k/2)$, then G is determined by its k -deck. If $n = (k + 1)2^{k-1}$, then there is such a graph G that is not determined by its k -deck.*

These bounds are quite far apart, and neither says much about the threshold number of vertices for ℓ -reconstructibility of disjoint unions of complete graphs (or, equivalently, complete multipartite graphs). The extremal problem is the following.

Problem 6.9. *Determine the maximum n such that every n -vertex complete multipartite graph is determined by its k -deck.*

7 Regular graphs

As noted above, 1-reconstructibility of disconnected graphs is easy, but 2-reconstructibility of all disconnected graphs implies the Reconstruction Conjecture.

Similarly, 1-reconstructibility of regular graphs is easy using the 1-reconstructibility of the degree list. Motivated by this, at a meeting in Sanya in 2019 Bojan Mohar asked whether regular graphs are 2-reconstructible.

Since 1-regular graphs are determined by their degree lists, they are determined by their 3-decks and hence are $(n - 3)$ -reconstructible. The results of Spinoza and West [48] described in Section 4 imply that 2-regular graphs are $\lfloor n/2 \rfloor$ -reconstructible. Both thresholds are sharp.

For $r \geq 3$, 2-reconstructibility of r -regular graphs is not immediate, even though the degree list is 2-reconstructible, because we must determine which of the deficient vertices in a card is adjacent to which of the two missing vertices. Nevertheless, the question has been answered for $r = 3$.

Theorem 7.1 (Kostochka–Nahvi–West–Zirlin [22]). *Every 3-regular graph is 2-reconstructible.*

Although this result takes considerable effort, it is (we hope) just the beginning of study in this area. It would be interesting both to answer Mohar’s question and to determine the maximum reconstructibility for 3-regular graphs or for graphs with maximum degree 3, extending the results discussed earlier.

Problem 7.2. *For each $r \in \mathbb{N}$ with $r \geq 2$, prove that every r -regular graph is 2-reconstructible.*

Problem 7.3. *Show that for each $\ell \geq 1$ there is a threshold n_ℓ such that every 3-regular graph with at least n_ℓ vertices is ℓ -reconstructible.*

Although we do not know whether all r -regular graphs are 2-reconstructible, for those that are not 2-connected we can say something much stronger. Note that we are *deleting* $r + 1$ vertices; the cards have $n - r - 1$ vertices.

Theorem 7.4. *Every r -regular graph G that is not 2-connected is $(r + 1)$ -reconstructible.*

Proof. If G is disconnected, then every component has at least $r + 1$ and hence at most $n - (r + 1)$ vertices. Thus Theorem 6.3 applies to make it $(r + 1)$ -reconstructible.

Now suppose that G has a cut-vertex. A subgraph of an r -regular graph is *near r -regular* if it has exactly one vertex with degree less than r . In every leaf block of G , only the cut-vertex of G has degree less than r . Hence every leaf block is near r -regular; furthermore, every near r -regular subgraph of G having no cut-vertex is a leaf block.

Besides the cut-vertex, a leaf block must have at least $r + 1$ other vertices; if only r , then G would be K_{r+1} . Since G has at least two leaf blocks, G has at least $2r + 3$ vertices. Hence the cards in the $(n - r - 1)$ -deck have at least $r + 2$ vertices, so by Manvel’s result (Theorem 2.2) we can reconstruct the degree list.

A 2-connected r -regular graph cannot have a near r -regular subgraph H with more than one vertex. If such H exists, let x be a vertex having degree r in H , and let y be a vertex of G not in H . Since G is 2-connected, by Menger’s Theorem it has internally disjoint paths from x to y . Such paths must leave H at distinct vertices having degree less than r in H , contradicting that H is near r -regular. Hence we have shown that the class of r -regular graphs with connectivity 1 is $(r + 1)$ -recognizable.

Since every leaf block omits at least the $r + 1$ non-cut-vertices of some other leaf block, every leaf block has at most $n - (r + 1)$ vertices. By Observation 1.3, we know all the subgraphs of G having at most $n - (r + 1)$ vertices, with their multiplicities. The near r -regular ones without cut-vertices are the leaf blocks. Hence we know all the leaf blocks, with their multiplicities.

Let B be a leaf block with fewest vertices, and let $s = |V(B)|$. In $\mathcal{D}_{n-s+1}(G)$ there is a card that has as (leaf) blocks all the leaf blocks of G other than B , and one less leaf block isomorphic to B than G has. This card H is near r -regular. Reconstruct G by attaching B at the vertex of H with degree less than r . \square

8 Almost All Graphs

Using cards not much larger than those that fail to determine connectedness, we can almost always reconstruct a graph. Chinn [12] and Bollobás [2] proved that almost all graphs are 1-reconstructible. In fact, this holds also for ℓ -reconstructibility, as observed earlier by Müller [34]. The needed tool is that for almost all graphs, the induced subgraphs with many vertices are pairwise nonisomorphic and have no nontrivial automorphisms (precise statement below). We say that a property *holds for almost all graphs* if the fraction of graphs with vertex set $\{1, \dots, n\}$ for which the property holds tends to 1 as n tends to ∞ .

For 1-reconstructibility, Chinn proved the following (in a stronger form):

Theorem 8.1 (Chinn [12]). *If the subgraphs of a graph G obtained by deleting two vertices are pairwise nonisomorphic, then G is reconstructible.*

When the subgraphs satisfy this hypothesis, vertex u is identifiable in $G - w$ because it is the only vertex in $G - w$ whose deletion yields a subgraph obtainable from $G - u$ by deleting one vertex. From $G - v$ and $G - w$, one can similarly identify v in $G - w$. Now one can check whether u and v are adjacent in G by checking whether u and v are adjacent in $G - w$. However, since we used both $G - u$ and $G - v$ to determine whether u and v are adjacent, we used all the cards.

Theorem 8.2 (Bollobás [2]). *For almost every graph, any three cards determine G .*

Under the same hypothesis as in Theorem 8.1, Bollobás gave a more careful argument to reconstruct all of G from $G - u$ and $G - v$ except for determining whether u and v are adjacent. For that he consulted a third card, invoking the uniqueness of the graphs in $\mathcal{D}_{n-3}(G)$ to identify u and v in $G - w$. However, it seems that uniqueness in $\mathcal{D}_{n-2}(G)$ suffices to identify u and v in $G - w$ as discussed above.

Theorem 8.2 is stronger than saying that *some* three cards determine G , which is the meaning of reconstruction number 3 (two cards can never determine whether the two deleted vertices are adjacent). The needed tool is the next lemma.

Lemma 8.3 (Müller [34]). *Let ϵ be a small positive real number. For almost every graph G , the induced subgraphs with at least k vertices have no nontrivial automorphisms and are pairwise nonisomorphic, where $k = (1 + \epsilon) \frac{|V(G)|}{2}$.*

Via counting arguments, Müller showed that graphs with this property are reconstructible from smaller cards. Spinoza and West more directly generalized the combinatorial argument of Bollobás, thereby reconstructing the graph from a small set of cards in the $(k + 1)$ -deck. However, one step in their construction does not work when $\ell = 1$.

Theorem 8.4 (Spinoza–West [48]). *For $\ell > 1$, if the subgraphs of G obtained by deleting $\ell + 1$ vertices have no nontrivial automorphisms and are pairwise nonisomorphic, then G is ℓ -reconstructible, using just $\binom{\ell+2}{2}$ cards from the $(|V(G)| - \ell)$ -deck.*

This not only shows ℓ -reconstructibility; it also places a bound on the natural generalization of reconstruction number to the $(n - \ell)$ -deck. Furthermore, asymptotically at least $\binom{n}{\ell+1}(n - \ell - 1)\binom{\ell+1}{2}$ sets of $\binom{\ell+2}{2}$ cards from $\mathcal{D}_{n-\ell}(G)$ determine G . The cards are chosen by specifying a fixed set S of $\ell + 1$ vertices in G and taking all cards that delete ℓ of them, plus for each pair $u, v \in S$ one card obtained by deleting $S - \{u, v\}$ and one vertex outside S .

9 Another Model of Reconstruction

As mentioned in the introduction, the term “ k -reconstructible” is also used in another model of reconstruction with different definitions. Here we explain the difference in order to reduce confusion.

We use “digraph” to mean a general binary relation (no repeated edges). Two digraphs D and D' on an n -element vertex set V are *k -isomorphic* if for every k -element subset $X \subseteq V$, the subdigraphs of D and D' induced by X are isomorphic. They are *$(\leq k)$ -isomorphic* if they are k' -isomorphic for all k' with $1 \leq k' \leq k$. They are *$(-k)$ -isomorphic* if they are $(n - k)$ -isomorphic. A digraph D is *α -reconstructible*, where $\alpha \in \{k, \leq k, -k\}$, if every digraph α -isomorphic to D is isomorphic to D .

These notions were introduced by Fraïssé [13], who conjectured that for sufficiently large k every digraph is $(\leq k)$ -reconstructible (and analogously for m -ary relations, for each m). The difference between Fraïssé’s model and that of Kelly and Ulam is that in Fraïssé’s problem we are told the identities of the missing vertices, but in the problem of Kelly and Ulam we are given only the multiset of isomorphism types. The notions coincide for the original conjecture: a graph (that is, a symmetric digraph) is reconstructible (in the Kelly–Ulam sense) if and only if it is (-1) -reconstructible (in the Fraïssé sense). Stockmeyer [50] showed that general digraphs (in fact, orientations of complete graphs) are not (-1) -reconstructible.

The difference is clear when $k = 2$. Only graphs with at most one edge (and their complements) are reconstructible from their 2-decks, but every symmetric digraph is 2-reconstructible, since we are told which pairs are adjacent. This does not hold for general digraphs; any two orientations of a complete graph are 2-isomorphic.

Fraïssé’s conjecture was proved for digraphs (that is, binary relations) by Lopez [28, 29], who proved that every digraph is (≤ 6) -reconstructible (this is sharp). The theorem was proved independently by Reid and Thomassen [46], and it also follows from the later characterization of the non- $(\leq k)$ -reconstructible digraphs by Boudabbous and Lopez [7]. A history of the topic appears in [6].

Analogously to Observation 1.3, Pouzet showed that if two n -vertex digraphs are p -isomorphic, then they are also q -isomorphic whenever $1 \leq q \leq \min\{p, n - p\}$. With Lopez's Theorem, this implies that every digraph with at least 11 vertices is 6-reconstructible, and every digraph with at least 12 vertices is (-6) -reconstructible.

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