Reconstruction from *k*-vertex Induced Subgraphs

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slides and paper on preprint page

Joint work with
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and
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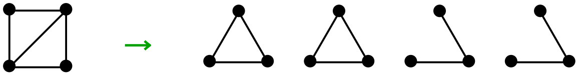
The Classical Problem

Def. A card of a graph G is an induced subgraph $G - v$.
The deck of a graph is the multiset of its cards.



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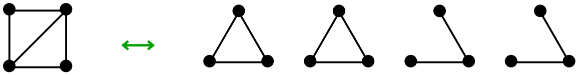

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Any graph with ≥ 3 vertices is determined by its deck.

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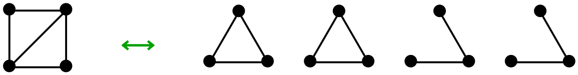

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Posed earlier in Kelly's thesis, 1942.

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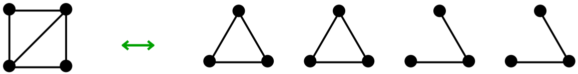

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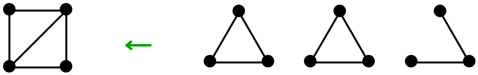

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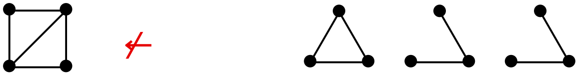

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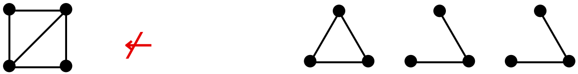

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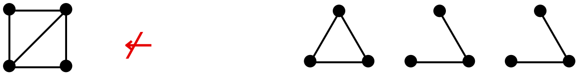

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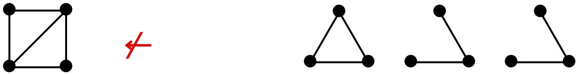

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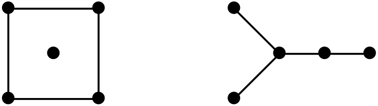

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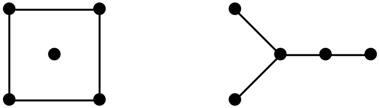

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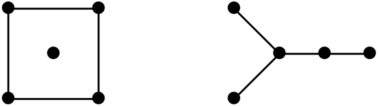

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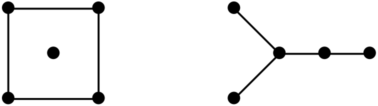

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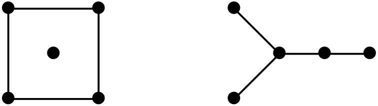

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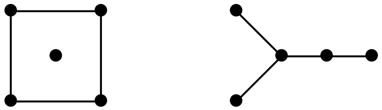

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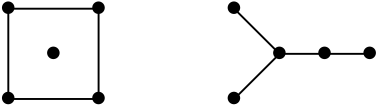

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- Another way to ask how hard it is to reconstruct G.

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(The previous theorem provides the upper bounds.)

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Cor. Among n-vertex graphs, the fraction that are reconstructible from the subgraphs obtained by deleting $(1-\epsilon)\frac{n}{2}$ vertices tends to 1 as $n \to \infty$.

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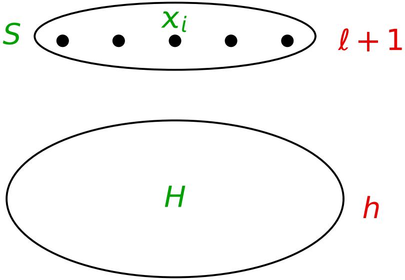
Pf. Let $n = |V(G)|$. Fix $S = \{x_1, \ldots, x_{\ell+1}\} \subseteq V(G)$.
Let $H = G - S$ and $h = |V(H)| = n - \ell - 1$.



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Thm. $\mathcal{D}_{n-\ell-1}$ good $\Rightarrow$ $\mathcal{D}_{n-\ell}$ determines G.

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Let $H = G - S$ and $h = |V(H)| = n - \ell - 1$.

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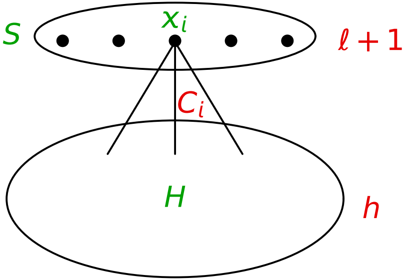

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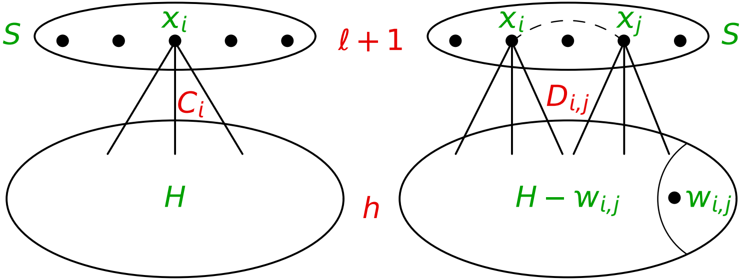

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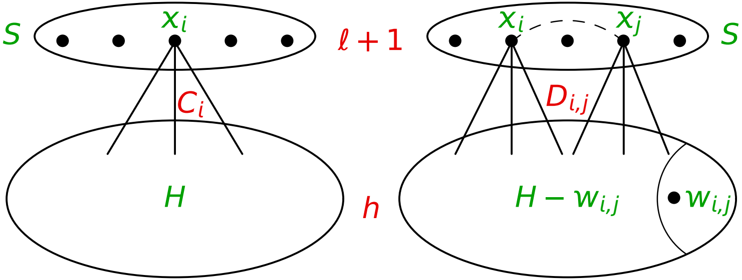

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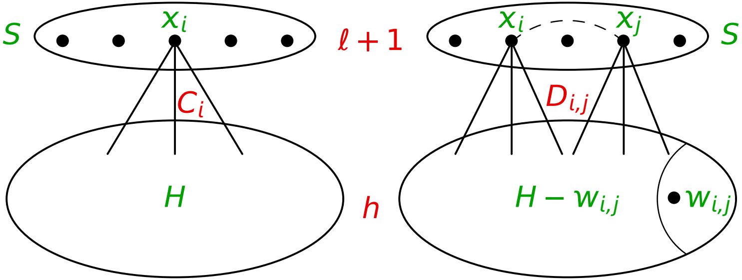

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If $|V(H') \cap S| \geq 3$, then H' appears in no card in $C \cap D$.

If $V(H') \cap S = \{x_i, x_j\}$, then H' appears only in $D_{i,j}$.

If $V(H') \cap S = \{x_i\}$, then H' is in one card in C and can be in cards $D_{i,j}$ as $H' = D_{i,j} - x_j = G[V(H) + x_i - w_{i,j}]$.
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Note $H = C_i - x_i$. For $w \in V(H)$, a card $D' \in D$ contains both $C_i - w$ and $C_j - w$ only when $D' = D_{i,j}$ and $w = w_{i,j}$.
This identifies $D_{i,j}$, used to check whether $x_i x_j \in E(G)$. ∎

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For details, see the appendix at the end of the slides.

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Thm. All $c_i = 0$ when $n \geq 7$ and $k = n - 3$.

Set $k = n - 3$

Using $j = n - 4$ and $j = n - 5$, we have

$$c_{n-4} + (n-3)c_{n-3} + \binom{n-2}{2}c_{n-2} + \binom{n-1}{3}c_{n-1} = 0$$

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- Yields 2-reconstructibility result; just check $n = 6$.

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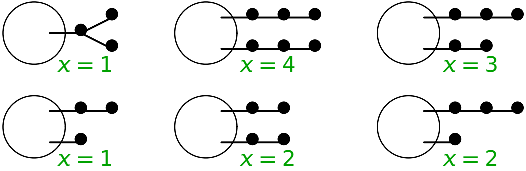

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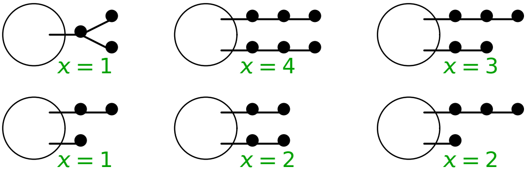

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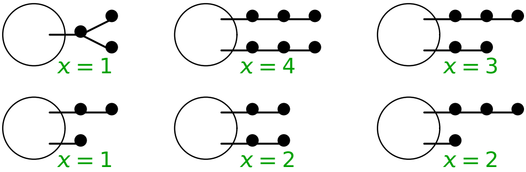

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Lem. If G, G', and H are graphs, then $\mathcal{D}_k(G) = \mathcal{D}_k(G')$ if and only if $\mathcal{D}_k(G + H) = \mathcal{D}_k(G' + H)$.

Sharpness and Key Idea

Thm. If $\Delta(G) = 2$, and two largest components have m and m' vertices, then G is k-deck reconstructible iff $k \geq \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}$, where $\epsilon \in \{0, 1\}$ and $\epsilon' \in \{0, 1, 2\}$. ($\epsilon = 1$ if largest component is P_m.)

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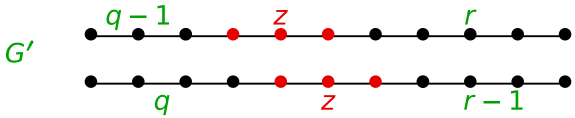

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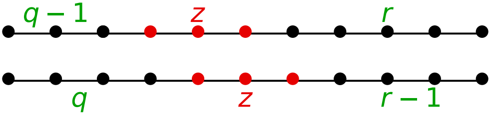

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- $s'(P_n, H)$ is indep of z when z is far enough from ends.

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Lem. Let L be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with k vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \le h \le n+1-k$, the value $s'(P_n, L)$ is the same.

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$s'(C_n, L)$ omits copies of L in P_n using w_1 and w_n.
$s'(C_n, L)$ counts unwanted subgraphs using $w_n w_1$.

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Lem. Let L be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with k vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \le h \le n+1-k$, the value $s'(P_n, L)$ is the same.

Pf. Induction on k. For $k=1$, any w_h is in one P_1.

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w_h is far enough from the ends to use induction hyp. ■

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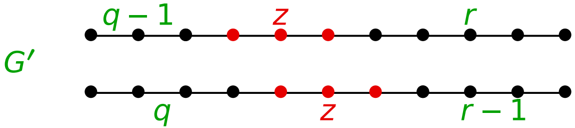
(3) $\mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1})$ if $q, r \geq k$.



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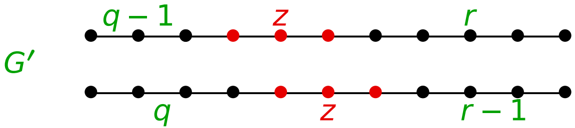

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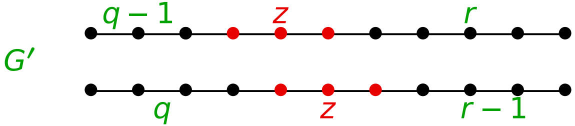

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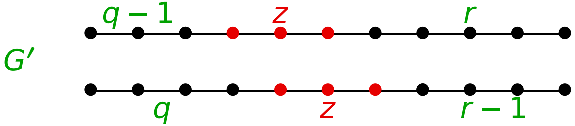

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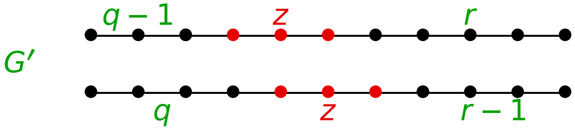

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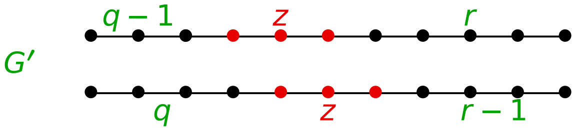

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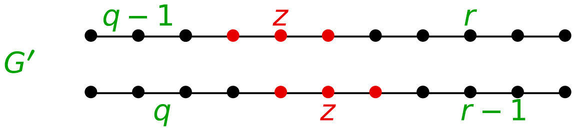

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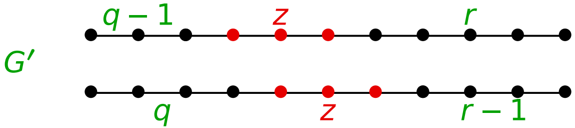

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Pf. $s(G,P_k)$ and $s(G,C_k)$ just count cards. Each copy of P_{k-1} is in $n-k+1$ cards, so $s(G,P_{k-1}) = \frac{\sum_{Q \in \mathcal{D}_k(G)} s(Q,P_{k-1})}{n-k+1}$. ■

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This completes the proof except for small k. ■

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Ques. Complete r-partite graphs are determined by $\mathcal{D}_{r+1}$. Are there complete r-partite and $(r+1)$-partite graphs with the same r-deck? ($\mathcal{D}_3(K_{7,4,3}) = \mathcal{D}_3(K_{6,6,1,1})$.)

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Let $|V(C)| = n - p$, so $H \Rightarrow c(\mathcal{D}) \leq \binom{n-p}{\ell-p} \leq \binom{n-1}{\ell-1}$.

(Keep only vertices from C, discarding $\ell - p$ of them.)

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(Keep only vertices from C, discarding $\ell-p$ of them.)

Also $H \Rightarrow \hat{c}(\mathcal{D}) \ge \binom{n-1}{\ell}$, where $\hat{c}(\mathcal{D}) =$ #cards having a component of order $\le \ell$. (Keep a vertex x outside C.)

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Idea: From G get lower bd on $c(\mathcal{D})$ & upper bd on $\hat{c}(\mathcal{D})$, leading to contradiction when n is large.

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Get lower bd on $c(\mathcal{D}')$ & upper bd on $\hat{c}(\mathcal{D}')$ instead.

Cards in $\mathcal{D}'$

Let t be the number of leaves in T.

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Requires $n < 2\ell^{(\ell+1)^2}$, roughly $\ell > \left(\frac{2\log n}{\log\log n}\right)^{1/2}$. ■

Counting Small Subtrees

Thm. If T is an n-vertex tree with t leaves, and $j \le \ell$, and b_j is #subtrees F with $|V(F)| \le \ell$ and exactly j outside nbrs, then $b_j \le \binom{t}{j}\binom{\ell+j-1}{j}$

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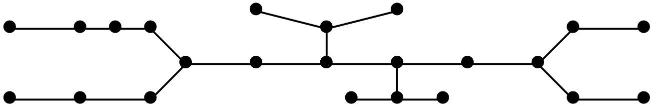
$t = 8$
$\ell = 11$
$j = 4$



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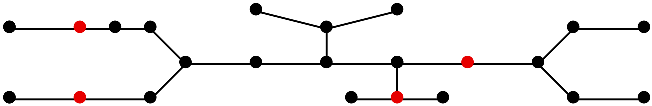

Pf. $j \ge 3$: Let S be the set of outside vertices with nbrs in F.

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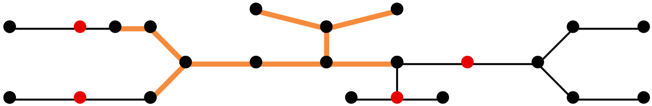

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F = component of $T - S$ having vertices between those of S.

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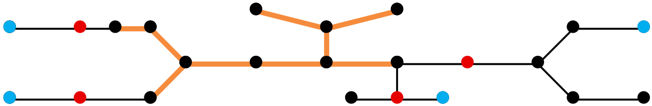

Pf. $j \ge 3$: Let S be the set of outside vertices with nbrs in F.
F = component of $T - S$ having vertices between those of S.
Paths from F through S reach leaves S' of T.

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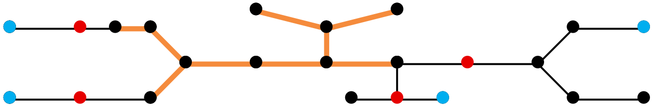

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Bound #subgraphs F generating fixed S' of size j.

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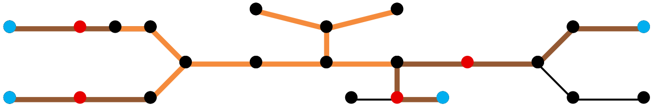

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Given S' (in $\binom{t}{j}$ ways), let T' be the tree generated by S'.

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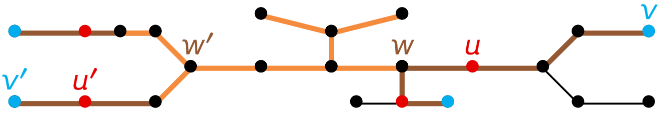

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The vertex $u \in S$ generating $v \in S'$ is on the path from v to the nearest branch vertex w in T'. Note $w \in V(F)$. Between w and u are fewer than ℓ vertices.

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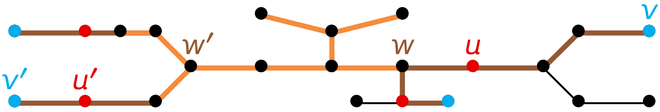

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#ways to place the break vertices $u \in S$ is at most #solutions to $x_1 + \cdots + x_j \le \ell - 1$, which equals $\binom{\ell+j-1}{j}$.

Smaller cases

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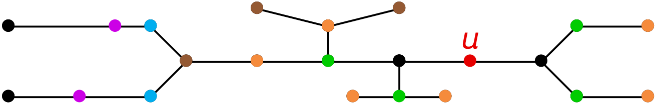
$i = 3$



$j = 2$: Since T has t leaves, from each vertex u there are at most t vertices at distance i, for $2 \leq i \leq \ell + 1$.

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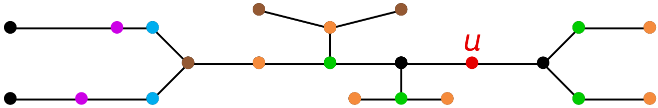

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Hence each vertex belongs to at most $t\ell$ sets S of size 2 that can cut off desired subtrees; the bound is $nt\ell/2$.

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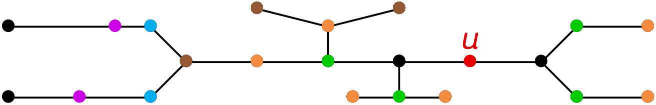

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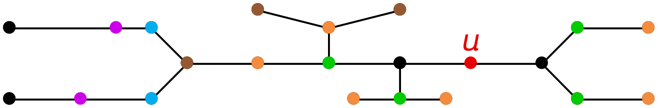

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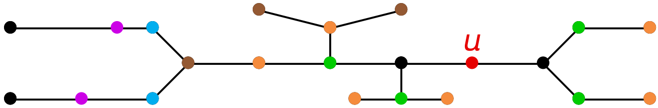

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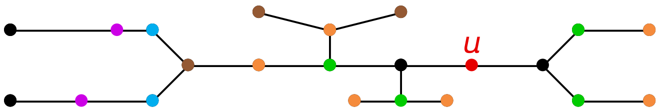

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- For $\ell = 3$, these computations imply that connectedness is 3-reconstructible for $n > 86{,}000{,}000$.

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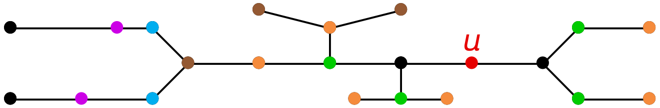
$t = 8$
$\ell = 5$
$i = 3$



$j = 2$: Since T has t leaves, from each vertex u there are at most t vertices at distance i, for $2 \le i \le \ell + 1$.

Hence each vertex belongs to at most $t\ell$ sets S of size 2 that can cut off desired subtrees; the bound is $nt\ell/2$.

$j = 1$: From a leaf move toward the centroid at most ℓ steps to place the vertex u cutting off F; bound is $t\ell$. ■

- $b_j\binom{n}{\ell-j} \le \binom{t}{j}\binom{\ell+j-1}{j}\binom{n}{\ell-j} \le \frac{\ell}{2}n^{\ell-1}t$ (biggest when $j = 1$)

- For $\ell = 3$, these computations imply that connectedness is 3-reconstructible for $n > 86{,}000{,}000$.

For $\ell = 3$, Spinoza–West [2019] reduced this to $n \ge 25$. The improvement to $n \ge 7$ uses different methods.