Reconstruction from k-vertex Induced Subgraphs

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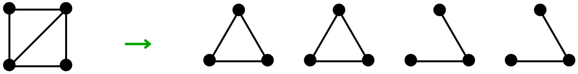
slides and paper on preprint page

Joint work with
Hannah Spinoza

The Classical Problem

Def. A card of a graph G is an induced subgraph $G - v$.
The deck of a graph is the multiset of its cards.



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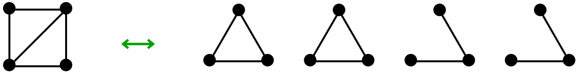

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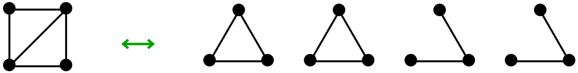

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Posed earlier in Kelly’s thesis, 1942.

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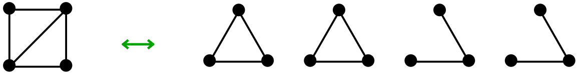

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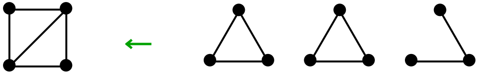

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Ex. K_4^- is determined by three of its cards.

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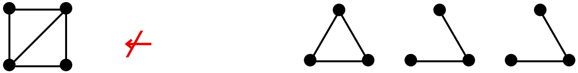

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Ex. K_4^- is determined by three cards. Which three?

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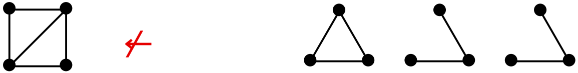

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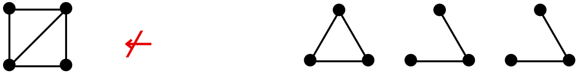

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Def. (Myrvold [1988]) The adversary reconstruction # $\alpha\mathrm{rn}(G)$ is the least k such that any k cards determine G.

Reconstruction numbers

- (Myrvold [1989]) $\mathrm{rn}(G) = 3$ for disconnected graphs with (at least) two nonisomorphic components.

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- (Kocay–Kreher [2014]) When q is a prime power with $q \equiv 1 \mod 4$, and $n = 4q - 4$, there exist (constructively) two connected (complementary) n-vertex graphs G with $\text{rn}(G) = \frac{n}{2} + 2$.

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Conj. (Kelly [1957]) For $\ell \in \mathbb{N}$, $\exists\, M_\ell \in \mathbb{N}$ such that $|V(G)| \geq M_\ell \;\Rightarrow\; G$ is reconstructible from the deck obtained by deleting ℓ vertices.

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This refinement asks how hard to it is to reconstruct G (in a different way from the reconstruction number).

Results (Spinoza–West [2016+])

Thm. P_n and $C_{\lceil n/2\rceil+1}+P_{\lfloor n/2\rfloor-1}$ have same $\lfloor n/2\rfloor$-deck.
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Thm. Connectedness is ℓ-reconstructible for $n>\ell^{(\ell+1)^2}$.

Thm. If $\ell\le(1-o(1))n/2$, then almost all graphs are reconstructible from some (many) sets of $\binom{\ell+2}{2}$ subgraphs obtained by deleting ℓ vertices.

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$k = 2$: only K_n, K_n^-, and their complements.

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Note $f(x)=\sum_{i=0}^{r}(-1)^i s_i x^{r-i}$, where s_i is the sum of products of i choices from $q_1,\ldots,q_r$.

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With $\Delta(G) < m$, by Manvel's result we can reconstruct the degree list, which determines G for such G. ∎

Thm. Complete r-partite is $(r+1)$-deck reconstructible

Pf. $P_2+P_1 \notin \mathcal{D}_3(G)$ & $K_{r+1} \notin \mathcal{D}_{r+1} \Rightarrow$ complete r-partite.
With part-sizes $q_1,\dots,q_r$, let $f(x)=\prod_{i=1}^r (x-q_i)$.
Note $f(x)=\sum_{i=0}^r (-1)^i s_i x^{r-i}$, where s_i is the sum of products of i choices from $q_1,\dots,q_r$.
Also $s_i = \#i$-cards that are K_i, so $\mathcal{D}_i(G)$ determines s_i.

k-deck Reconstruction for Small k

$k=2$: only K_n, K_n^-, and their complements.
(Manvel [1974]) $\mathcal{D}_{\Delta(G)+2}(G)$ determines the degree list.

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Knowing f, we find $q_1, \ldots, q_r$. ■

Almost All Graphs

Lem. (Müller [1976]) Fix $\epsilon > 0$. For almost every graph G, the induced subgraphs with at least $(1+\epsilon)\frac{|V(G)|}{2}$ vertices are good, meaning they have no nontrival automorphisms and are pairwise nonisomorphic.

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Cor. Among n-vertex graphs, the fraction that are reconstructible from the subgraphs obtained by deleting $(1-\epsilon)\frac{n}{2}$ vertices tends to 1 as $n \to \infty$.

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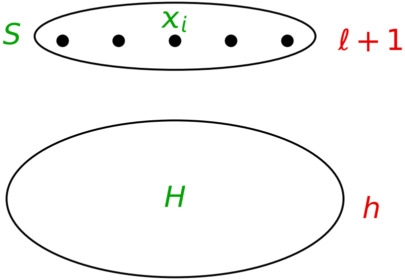

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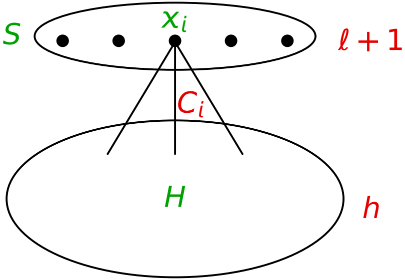

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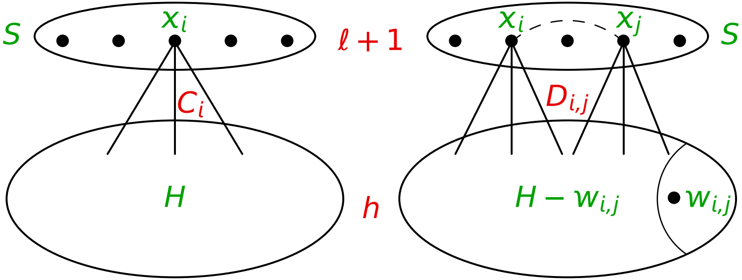

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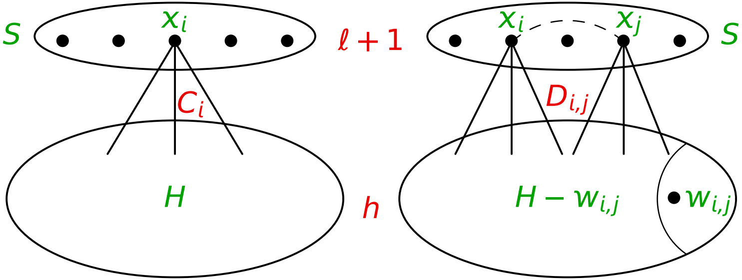

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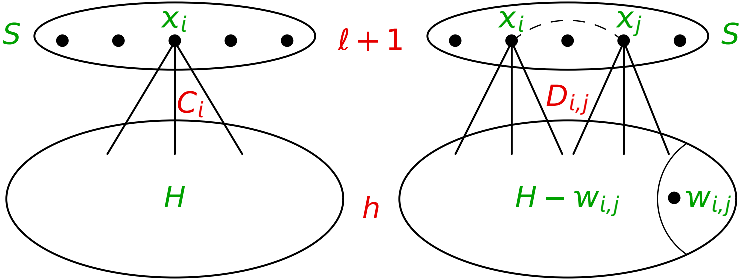

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If $V(H') \cap S = \{x_i, x_j\}$, then H' appears only in $D_{i,j}$.

If $V(H') \cap S = \{x_i\}$, then H' is in one card in C and can be in cards $D_{i,j}$ as $H' = D_{i,j} - x_j = G[V(H) + x_i - w_{i,j}]$.
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Note $H = C_i - x_i$. For $w \in V(H)$, a card $D' \in D$ contains both $C_i - w$ and $C_j - w$ only when $D' = D_{i,j}$ and $w = w_{i,j}$.
This identifies $D_{i,j}$, used to check whether $x_i x_j \in E(G)$. ■

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Requires $n < 2\ell^{(\ell+1)^2}$, roughly $\ell > \left(\frac{2\log n}{\log\log n}\right)^{1/2}$. ■

Counting Small Subtrees

Thm. If T is an n-vertex tree with t leaves, and $j \le \ell$, and b_j is #subtrees F with $|V(F)| \le \ell$ and exactly j outside nbrs, then $b_j \le \binom{t}{j}\binom{\ell+j-1}{j}$

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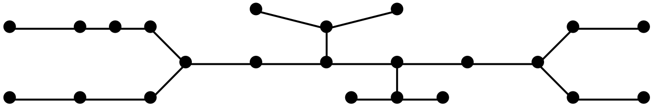

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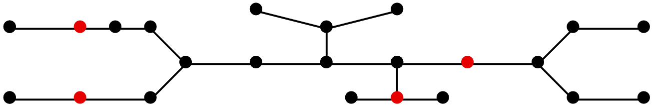

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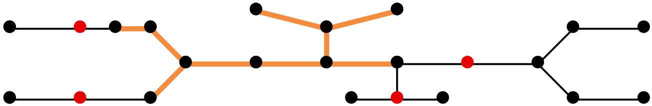

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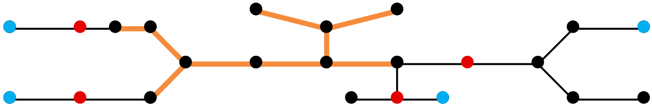

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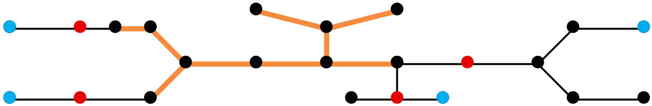

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Bound #subgraphs F generating fixed S' of size j.

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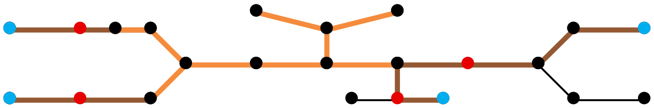

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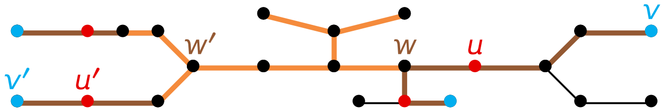

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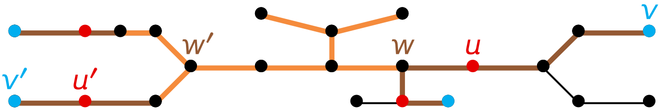

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#ways to place the break vertices $u \in S$ is at most #solutions to $x_1 + \cdots + x_j \le \ell - 1$, which equals $\binom{\ell+j-1}{j}$.

Smaller cases

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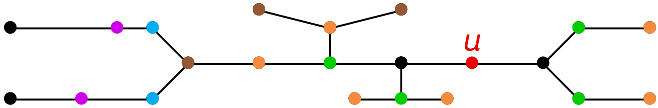

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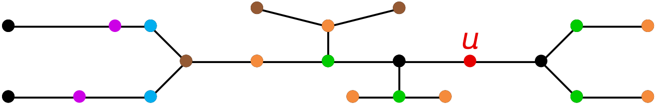

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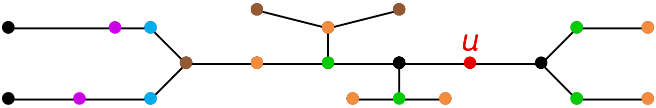

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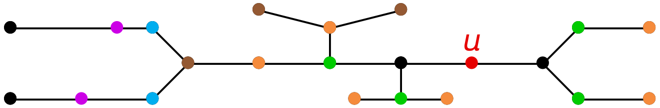

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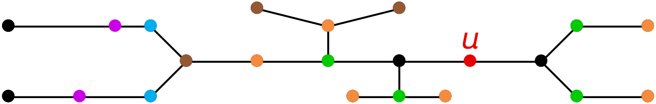

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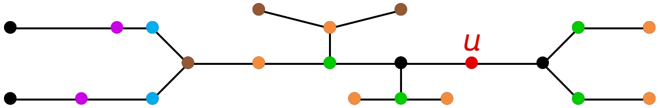

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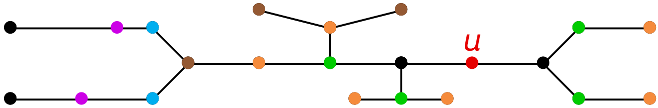

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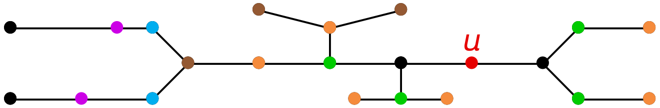

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For the ideas, see the appendix at the end of the slides.

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Lem. If G, G', and H are graphs, then $\mathcal{D}_k(G) = \mathcal{D}_k(G')$ if and only if $\mathcal{D}_k(G+H) = \mathcal{D}_k(G'+H)$.

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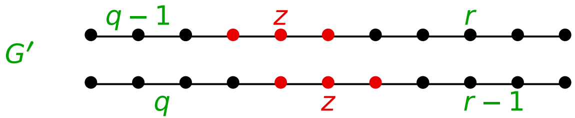

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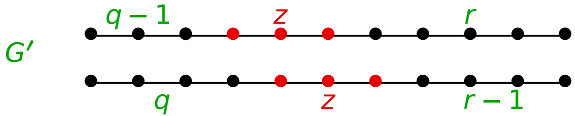

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Lem. Let L^m be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with k vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n+1-k$, the value $s'(P_n, L^m)$ is the same.

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Compare $s'(P_n, L^m)$ to $s'(C_n, L^m)$ with edge $w_n w_1$.

By symmetry, $s'(C_n, L^m)$ is independent of h.

$s'(C_n, L^m)$ omits copies of L^m in P_n using w_1 and w_n.
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w_h is far enough from the ends to use induction hyp. ∎

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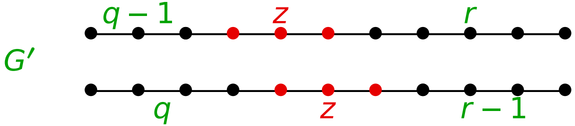

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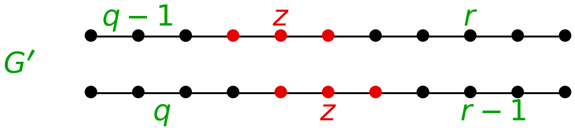

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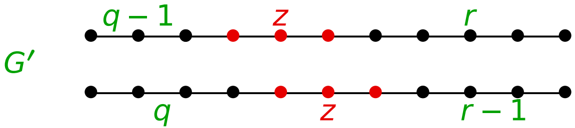

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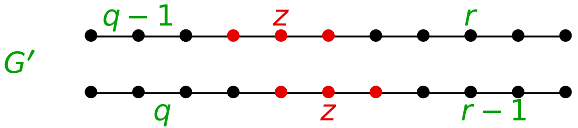

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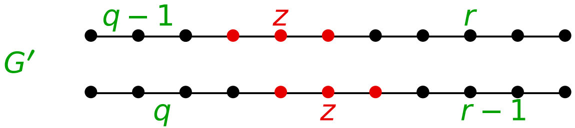

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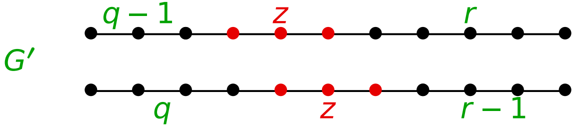

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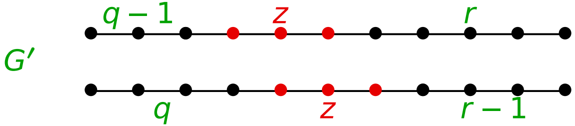

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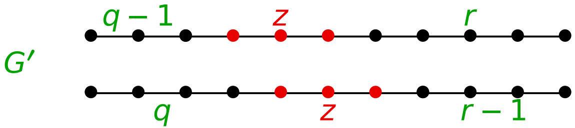

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Same idea, reducing to equalities given by (2). ■

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Pf. Each such path contributes 1 to $s(G,P_{k-1})-s(G,P_k)$.
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Pf. $s(G,P_k)$ and $s(G,C_k)$ just count cards. Each copy of P_{k-1} is in $n-k+1$ cards, so $s(G,P_{k-1})=\frac{\sum_{Q\in\mathcal{D}_k(G)} s(Q,P_{k-1})}{n-k+1}$. ∎

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This completes the proof except for small k. ∎

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Ques. Do there exist a complete r-partite graph and complete $(r+1)$-partite graph with the same r-deck? (Yes for $r \le 3$: $\mathcal{D}_3(K_{7,4,3}) = \mathcal{D}_3(K_{6,6,1,1})$.)

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Idea: From G get lower bound on $c(\mathcal{D})$ and
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leading to a contradiction when $n \geq 25$.

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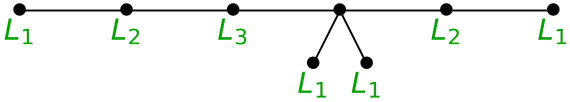

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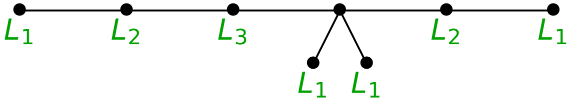

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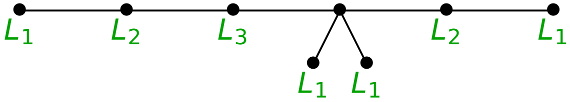

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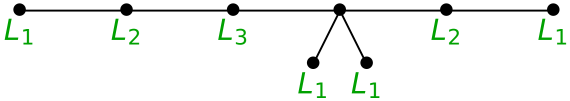

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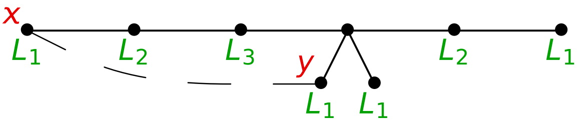

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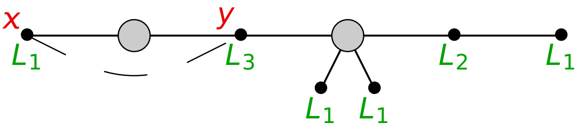

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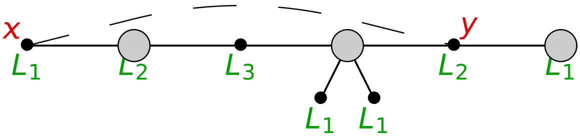

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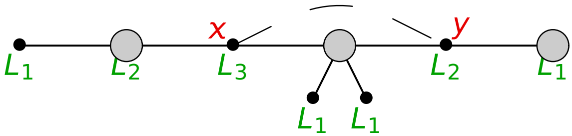

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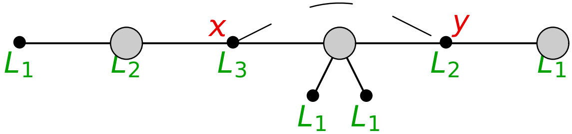

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With $l_2 \leq l_1 \leq 5$, the inequality

$$\binom{n-2}{3} \leq l_2\binom{n-3}{2} + (n-3)(n-4) + l_1(n-5) + 2 + n - 1$$

cannot hold for $n \geq 25$. ■