Reconstruction from $k$-vertex Induced Subgraphs

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slides and paper on preprint page

Joint work with
Hannah Spinoza
The Classical Problem

**Def.** A card of a graph $G$ is an induced subgraph $G - v$. The deck of a graph is the multiset of its cards.
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![Graphs](image)

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- **Surveys:** Bondy-Hemminger ['77], Lauri ['87], Ellingham ['88], Manvel ['88], Bondy ['91], Lauri ['97], Maccari-Rueda-Viazzi ['02], Asciak-Francalanza-Lauri-Myrvold['10]
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**Ex.** $K_4^-$ is determined by three of its cards.
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**Ex.** $K_4^-$ is determined by three cards. Which three?
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**Def.** (Harary-Plantholt [1985]) The reconstruction number $\text{rn}(G)$ is the least number of cards that determine $G$. 
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**Def.** (Harary-Plantholt [1985]) The reconstruction number \( \text{rn}(G) \) is the least number of cards that determine \( G \).

**Def.** (Myrvold [1988]) The adversary reconstruction number \( \#\text{arn}(G) \) is the least \( k \) such that any \( k \) cards determine \( G \).
Reconstruction numbers

- (Myrvold [1989]) $\text{rn}(G) = 3$ for disconnected graphs with (at least) two nonisomorphic components.

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$K_{n/2, n/2}$ shares $\frac{n}{2} + 1$ cards with $K_{n/2+1, n/2-1}$.

**Conj.** (Harary–Plantholt [1985]) $\text{rn}(G) \leq \frac{n}{2} + 2$, with equality only for $K_{n/2, n/2}$ and $2K_{n/2}$ when $n > 4$. 
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- (Kocay–Kreher [2014]) When $q$ is a prime power with $q \equiv 1 \mod 4$, and $n = 4q - 4$, there exist (constructively) two connected (complementary) $n$-vertex graphs $G$ with $\text{rn}(G) = \frac{n}{2} + 2$. 
Another Direction

**Conj.** (Kelly [1957]) For \( \ell \in \mathbb{N} \), \( \exists M_\ell \in \mathbb{N} \) such that \( |V(G)| \geq M_\ell \Rightarrow G \) is reconstructible from the deck obtained by deleting \( \ell \) vertices.
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**Obs.** $\mathcal{D}_k(G)$ determines $\mathcal{D}_{k-1}(G)$. 
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**Aim:** Find the least $k$ s.t. $G$ is $k$-deck reconstructible.
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**Aim:** Find the least $k$ s.t. $G$ is $k$-deck reconstructible.
(Same as $\ell$-reconstructible when $k + \ell = |V(G)|$.)
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**Conj.** (Kelly [1957]) For \( l \in \mathbb{N} \), \( \exists M_l \in \mathbb{N} \) such that \( |V(G)| \geq M_l \Rightarrow G \) is reconstructible from the deck obtained by deleting \( l \) vertices. “\( l \)-reconstructible”

**RC:** \( M_1 = 3 \). \( M_2 = 6 \)? (McMullen–Radziszowski [2007]) (\( C_4+K_1 \) and the 5-vertex tree \( K'_{1,3} \) are not 2-reconstructible.)

**Def.** \( k \)-deck \( D_k(G) \) = set of \( k \)-vertex induced subgrps.

**Obs.** \( D_k(G) \) determines \( D_{k-1}(G) \).

**Pf.** Each graph in \( D_{k-1} \) arises \( n - k + 1 \) times by deleting one vertex from a graph in \( D_k(G) \).

**Aim:** Find the least \( k \) s.t. \( G \) is \( k \)-deck reconstructible. (Same as \( l \)-reconstructible when \( k + l = |V(G)| \).

This refinement asks how hard it is to reconstruct \( G \) (in a different way from the reconstruction number).
Results (Spinoza–West [2016+])

**Thm.** \( P_n \) and \( C_{\lceil n/2 \rceil + 1} + P_{\lfloor n/2 \rfloor - 1} \) have same \( \lfloor n/2 \rfloor \)-deck. \( C_n \) and \( C_{\lceil n/2 \rceil} + C_{\lfloor n/2 \rfloor} \) have same \( \lfloor n/2 \rfloor - 1 \)-deck.
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**Thm.** $P_n$ and $C_{\lceil n/2 \rceil} + 1 + P_{\lfloor n/2 \rfloor - 1}$ have same $\lfloor n/2 \rfloor$-deck. $C_n$ and $C_{\lceil n/2 \rceil} + C_{\lfloor n/2 \rfloor}$ have same $\lfloor n/2 \rfloor - 1$-deck. Sharp!

**Thm.** For all $G$ with $\Delta(G) = 2$, we determine the least $k$ such that $G$ is $k$-deck reconstructible.
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**Thm.** For all $G$ with $\Delta(G) = 2$, we determine the least $k$ such that $G$ is $k$-deck reconstructible.

**Cor.** $\mathcal{D}_{\lfloor n/2 \rfloor}(G)$ may not determine connectedness.
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(Manvel [1974])  $D_{n-2}(G)$ does connectedness for $n \geq 6$.  

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(Manvel [1974]) $\mathcal{D}_{n-2}(G)$ does connectedness for $n \geq 6$.

($C_4 + K_1$ and the tree $K_{1,3}'$ have the same 3-deck.)
Results (Spinoza–West [2016+])

**Thm.** $P_n$ and $C_{\lceil n/2 \rceil + 1} + P_{\lfloor n/2 \rfloor - 1}$ have same $\lfloor n/2 \rfloor$-deck. $C_n$ and $C_{\lceil n/2 \rceil} + C_{\lfloor n/2 \rfloor}$ have same $\lfloor n/2 \rfloor - 1$-deck. Sharp!

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We show connectedness is $3$-reconstructible for $n \geq 25$. 
Results (Spinoza–West [2016+])

**Thm.** $P_n$ and $C_{\lceil n/2 \rceil} + P_{\lfloor n/2 \rfloor - 1}$ have same $\lfloor n/2 \rfloor$-deck. $C_n$ and $C_{\lceil n/2 \rceil} + C_{\lfloor n/2 \rfloor}$ have same $\lfloor n/2 \rfloor - 1$-deck. Sharp!

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**Thm.** Connectedness is $\ell$-reconstructible for $n > \ell^{(\ell+1)^2}$. 
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**Thm.** $P_n$ and $C_{\lceil n/2 \rceil} + P_{\lfloor n/2 \rfloor} - 1$ have same $\lfloor n/2 \rfloor$-deck. $C_n$ and $C_{\lfloor n/2 \rfloor} + C_{\lfloor n/2 \rfloor}$ have same $\lfloor n/2 \rfloor - 1$-deck. Sharp!

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($C_4 + K_1$ and the tree $K_{1,3}$ have the same 3-deck.)

We show connectedness is 3-reconstructible for $n \geq 25$.

**Thm.** Connectedness is $\ell$-reconstructible for $n > \ell^{(\ell+1)^2}$.

**Thm.** If $\ell \leq (1 - o(1))n/2$, then almost all graphs are reconstructible from some (many) sets of $\binom{\ell+2}{2}$ subgraphs obtained by deleting $\ell$ vertices.
$k$-deck Reconstruction for Small $k$

$k = 2$: only $K_n$, $K_n^-$, and their complements.
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$k = 2$: only $K_n$, $K^-_n$, and their complements.
(Manvel [1974]) $D_{\Delta(G) + 2}(G)$ determines the degree list.
k-deck Reconstruction for Small k

\[ k = 2: \text{ only } K_n, K_n^-, \text{ and their complements.} \]

(Manvel [1974]) \( \mathcal{D}_{\Delta(G)+2}(G) \) determines the degree list.

**Prop.** \( \sum K_{n_i} \) with all \( n_i \leq m \) is \((m+1)\)-deck reconstructible
**k-deck Reconstruction for Small $k$**

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(Manvel [1974]) $D_{\Delta(G)+2}(G)$ determines the degree list.

**Prop.** $\sum K_{n_i}$ with all $n_i \leq m$ is $(m+1)$-deck reconstructible

**Pf.** $P_3 \notin D_3(G) \iff G$ has the form $\sum K_{n_i}$. 
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**Pf.** \( P_3 \notin \mathcal{D}_3(G) \iff G \) has the form \( \sum K_{n_i} \).

With \( \Delta(G) < m \), by Manvel’s result we can reconstruct the degree list, which determines \( G \) for such \( G \).
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**Thm.** Complete \( r \)-partite is \((r+1)\)-deck reconstructible
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**Pf.** \( P_2 + P_1 \not\in \mathcal{D}_3(G) \) \& \( K_{r+1} \not\in \mathcal{D}_{r+1} \ \Rightarrow \) complete \( r \)-partite.
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With part-sizes \( q_1, \ldots, q_r \), let \( f(x) = \prod_{i=1}^{r}(x - q_i) \).
**$k$-deck Reconstruction for Small $k$**

$k = 2$: only $K_n$, $K_n^\sim$, and their complements. (Manvel [1974]) $\mathcal{D}_{\Delta(G)+2}(G)$ determines the degree list.

**Prop.** $\sum K_{n_i}$ with all $n_i \leq m$ is $(m+1)$-deck reconstructible

**Pf.** $P_3 \notin \mathcal{D}_3(G) \iff G$ has the form $\sum K_{n_i}$.

With $\Delta(G) < m$, by Manvel’s result we can reconstruct the degree list, which determines $G$ for such $G$.

**Thm.** Complete $r$-partite is $(r+1)$-deck reconstructible

**Pf.** $P_2+P_1 \notin \mathcal{D}_3(G) \& K_{r+1} \notin \mathcal{D}_{r+1} \Rightarrow$ complete $r$-partite.

With part-sizes $q_1, \ldots, q_r$, let $f(x) = \prod_{i=1}^{r}(x - q_i)$.

Note $f(x) = \sum_{i=0}^{r}(-1)^i s_i x^{r-i}$, where $s_i$ is the sum of products of $i$ choices from $q_1, \ldots, q_r$. 


**k-deck Reconstruction for Small k**

- **k = 2**: only $K_n, K_n^-$, and their complements. (Manvel [1974]) $D_{\Delta(G)+2}(G)$ determines the degree list.

**Prop.** $\sum K_{n_i}$ with all $n_i \leq m$ is $(m+1)$-deck reconstructible

**Pf.** $P_3 \notin D_3(G) \iff G$ has the form $\sum K_{n_i}$.

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**Pf.** $P_2 + P_1 \notin D_3(G)$ & $K_{r+1} \notin D_{r+1} \Rightarrow$ complete $r$-partite.

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Also $s_i = \#i$-cards that are $K_i$, so $D_i(G)$ determines $s_i$. 
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**Pf.** \( P_3 \notin \mathcal{D}_3(G) \iff G \) has the form \( \sum K_{n_i} \).

With \( \Delta(G) < m \), by Manvel’s result we can reconstruct the degree list, which determines \( G \) for such \( G \).

**Thm.** Complete \( r \)-partite is \( (r+1) \)-deck reconstructible

**Pf.** \( P_2 + P_1 \notin \mathcal{D}_3(G) \& K_{r+1} \notin \mathcal{D}_{r+1} \Rightarrow \) complete \( r \)-partite.

With part-sizes \( q_1, \ldots, q_r \), let \( f(x) = \prod_{i=1}^{r} (x - q_i) \).

Note \( f(x) = \sum_{i=0}^{r} (-1)^i s_i x^{r-i} \), where \( s_i \) is the sum of products of \( i \) choices from \( q_1, \ldots, q_r \).

Also \( s_i = \#i \)-cards that are \( K_i \), so \( \mathcal{D}_i(G) \) determines \( s_i \).

Knowing \( f \), we find \( q_1, \ldots, q_r \).
Almost All Graphs

**Lem.** (Müller [1976]) Fix $\epsilon > 0$. For almost every graph $G$, the induced subgraphs with at least $(1 + \epsilon)\frac{|V(G)|}{2}$ vertices are good, meaning they have no nontrivial automorphisms and are pairwise nonisomorphic.
Almost All Graphs

**Lem.** (Müller [1976]) Fix $\varepsilon > 0$. For almost every graph $G$, the induced subgraphs with at least $(1 + \varepsilon)\frac{|V(G)|}{2}$ vertices are **good**, meaning they have no nontrivial automorphisms and are pairwise nonisomorphic.

**Thm.** If the subgraphs obtained by deleting $\ell + 1$ verts are **good**, then $G$ is reconstructible from some set of $\binom{\ell+2}{2}$ subgraphs obtained by deleting $\ell$ vertices.
**Almost All Graphs**

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**Thm.** If the subgraphs obtained by deleting $\ell+1$ verts are good, then $G$ is reconstructible from some set of $\binom{\ell+2}{2}$ subgraphs obtained by deleting $\ell$ vertices.

**Cor.** Among $n$-vertex graphs, the fraction that are reconstructible from the subgraphs obtained by deleting $(1 - \epsilon)\frac{n}{2}$ vertices tends to 1 as $n \to \infty$. 
Using Some of the Deck

**Thm.** $\mathcal{D}_{n-\ell-1}$ good $\implies \mathcal{D}_{n-\ell}$ determines $G$. 
Using Some of the Deck

**Thm.** $\mathcal{D}_{n-\ell-1}$ good $\Rightarrow$ $\mathcal{D}_{n-\ell}$ determines $G$.

**Pf.** Let $n = |V(G)|$. Fix $S = \{x_1, \ldots, x_{\ell+1}\} \subseteq V(G)$. Let $H = G - S$ and $h = |V(H)| = n - \ell - 1$. 

![Diagram with vertices and edges]
Using Some of the Deck

**Thm.** \( D_{n-\ell-1} \) good \( \Rightarrow \) \( D_{n-\ell} \) determines \( G \).

**Pf.** Let \( n = |V(G)| \). Fix \( S = \{x_1, \ldots, x_{\ell+1}\} \subseteq V(G) \). Let \( H = G - S \) and \( h = |V(H)| = n - \ell - 1 \).

Let \( C_i = G - (S - \{x_i\}) \) (deleting \( \ell \)) and \( C = \{C_i: x_i \in S\} \).

\[ \text{Diagram} \]
Using Some of the Deck

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For \( x_i, x_j \in S \), let \( D_{i,j} = G - (S - \{x_i, x_j\}) - w_{i,j} \),
where \( w_{i,j} \in V(H) \). Let \( D = \{D_{i,j}: x_i, x_j \in S\} \).
Using Some of the Deck

**Thm.** $D_{n-\ell-1}$ good $\Rightarrow$ $D_n$ determines $G$.

**Pf.** Let $n = |V(G)|$. Fix $S = \{x_1, \ldots, x_{\ell+1}\} \subseteq V(G)$. Let $H = G - S$ and $h = |V(H)| = n - \ell - 1$.

Let $C_i = G - (S - \{x_i\})$ (deleting $\ell$) and $C = \{C_i: x_i \in S\}$.

For $x_i, x_j \in S$, let $D_{i,j} = G - (S - \{x_i, x_j\}) - w_{i,j}$, where $w_{i,j} \in V(H)$. Let $D = \{D_{i,j}: x_i, x_j \in S\}$.

**Claim:** $G$ is reconstructible from $C \cup D$. 
Using Some of the Deck

**Thm.** $D_{n-l-1}$ good $\Rightarrow D_{n-l}$ determines $G$.

**Pf.** Let $n = |V(G)|$. Fix $S = \{x_1, \ldots, x_{l+1}\} \subseteq V(G)$. Let $H = G - S$ and $h = |V(H)| = n - l - 1$.

Let $C_i = G - (S - \{x_i\})$ (deleting $l$) and $C = \{C_i : x_i \in S\}$.

For $x_i, x_j \in S$, let $D_{i,j} = G - (S - \{x_i, x_j\}) - w_{i,j}$, where $w_{i,j} \in V(H)$. Let $D = \{D_{i,j} : x_i, x_j \in S\}$.

**Claim:** $G$ is reconstructible from $C \cup D$. many such
Reconstructing $G$ from $C \cup D$.

The cards have $h + 1$ verts; $h$-vertex subgrs are good. Which $h$-vertex subgrs $H'$ appear in the cards in $C \cup D$?
Reconstructing $G$ from $C \cup D$.

The cards have $h + 1$ verts; $h$-vertex subgrs are good. Which $h$-vertex subgrs $H'$ appear in the cards in $C \cup D$?

**Idea:** $H$ is the only $h$-vertex subgraph appearing $\ell + 1$ times in cards in $C \cup D$. This identifies $H$ and all $C_i$, the vertex $x_i$ in $C_i$, and the edges from $x_i$ to $V(H)$. 
Reconstructing $G$ from $C \cup D$.

The cards have $h + 1$ verts; $h$-vertex subgrs are good. Which $h$-vertex subgrs $H'$ appear in the cards in $C \cup D$?

**Idea:** $H$ is the only $h$-vertex subgraph appearing $l + 1$ times in cards in $C \cup D$. This identifies $H$ and all $C_i$, the vertex $x_i$ in $C_i$, and the edges from $x_i$ to $V(H)$.

If $|V(H') \cap S| \geq 3$, then $H'$ appears in no card in $C \cap D$.

If $V(H') \cap S = \{x_i, x_j\}$, then $H'$ appears only in $D_{i,j}$.

If $V(H') \cap S = \{x_i\}$, then $H'$ is in one card in $C$ and can be in cards $D_{i,j}$ as $H' = D_{i,j} - x_j = G[V(H) + x_i - w_{i,j}]$.

If $w_{i,j}$ is not the same for all $j$, then $H'$ is in $\leq l$ cards.

If $V(H') \cap S = \emptyset$, then $H' = H$, in all $l + 1$ cards of $C$. 
Reconstructing $G$ from $C \cup D$.

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If $V(H') \cap S = \{x_i\}$, then $H'$ is in one card in $C$ and can be in cards $D_{i,j}$ as $H' = D_{i,j} - x_j = G[V(H) + x_i - w_{i,j}]$.
If $w_{i,j}$ is not the same for all $j$, then $H'$ is in $\le \ell$ cards.
If $V(H') \cap S = \emptyset$, then $H' = H$, in all $\ell + 1$ cards of $C$.

Note $H = C_i - x_i$. For $w \in V(H)$, a card $D' \in D$ contains both $C_i - w$ and $C_j - w$ only when $D' = D_{i,j}$ and $w = w_{i,j}$. This identifies $D_{i,j}$, used to check whether $x_i x_j \in E(G)$. ■
Connectedness is $l$-Reconstructible for Large $n$

**Def.** Let $c(D) = \#$ of connected cards in a deck $D$. 
Connectedness is $\ell$-Reconstructible for Large $n$

**Def.** Let $c(\mathcal{D}) = \#$ of connected cards in a deck $\mathcal{D}$.

Suppose $G$ connected, $H$ disconn., same $(n - \ell)$-deck $\mathcal{D}$. 
Connectedness is $\ell$-Reconstructible for Large $n$

**Def.** Let $c(D) = \#$ of connected cards in a deck $D$.

Suppose $G$ connected, $H$ disconn., same $(n - \ell)$-deck $D$. $G \Rightarrow c(D) \geq 1$, so $H$ has component $C$ with $|V(C)| \geq n - \ell$. 
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$G \Rightarrow c(\mathcal{D}) \geq 1$, so $H$ has component $C$ with $|V(C)| \geq n - \ell$.

Let $|V(C)| = n - p$, so $H \Rightarrow c(\mathcal{D}) \leq \binom{n-p}{\ell-p} \leq \binom{n-1}{\ell-1}$.

(Keep only vertices from $C$, discarding $\ell - p$ of them.)
Connectedness is $l$-Reconstructible for Large $n$

**Def.** Let $c(D) = \#$ of connected cards in a deck $D$.

Suppose $G$ connected, $H$ disconn., same $(n - l)$-deck $D$. $G \Rightarrow c(D) \geq 1$, so $H$ has component $C$ with $|V(C)| \geq n - l$.

Let $|V(C)| = n - p$, so $H \Rightarrow c(D) \leq \binom{n-p}{l-p} \leq \binom{n-1}{l-1}$.

(Keep only vertices from $C$, discarding $l - p$ of them.)

Also $H \Rightarrow \hat{c}(D) \geq \binom{n-1}{l}$, where $\hat{c}(D) = \#$ cards having a component of order $\leq l$. (Keep a vertex $x$ outside $C$.)
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Suppose $G$ connected, $H$ disconn., same $(n - \ell)$-deck $\mathcal{D}$.
$G \Rightarrow c(\mathcal{D}) \geq 1$, so $H$ has component $C$ with $|V(C)| \geq n - \ell$.

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**Idea:** From $G$ get lower bd on $c(\mathcal{D})$ & upper bd on $\hat{c}(\mathcal{D})$, leading to contradiction when $n$ is large.
**Connectedness is \( l \)-Reconstructible for Large \( n \)**

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**Idea:** From \( G \) get lower bd on \( c(\mathcal{D}) \) & upper bd on \( \hat{c}(\mathcal{D}) \), leading to contradiction when \( n \) is large.

Let \( T \) be a spanning tree of \( G \), and let \( \mathcal{D}' = \mathcal{D}_{n-l}(T) \).
**Connectedness is $l$-Reconstructible for Large $n$**

**Def.** Let $c(D) = \# \text{ of connected cards in a deck } D$.

Suppose $G$ connected, $H$ disconn., same $(n - l)$-deck $D$.  
$G \Rightarrow c(D) \geq 1$, so $H$ has component $C$ with $|V(C)| \geq n - l$.

Let $|V(C)| = n - p$, so $\quad H \Rightarrow c(D) \leq \binom{n-p}{l-p} \leq \binom{n-1}{l-1}$.
(Keep only vertices from $C$, discarding $l - p$ of them.)

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**Idea:** From $G$ get lower bd on $c(D)$ & upper bd on $\hat{c}(D)$, leading to contradiction when $n$ is large.

Let $T$ be a spanning tree of $G$, and let $D' = D_{n-l}(T)$.
• $c(D) \geq c(D')$ and $\hat{c}(D) \leq \hat{c}(D')$ (using same vertices).
Connectedness is \( \ell \)-Reconstructible for Large \( n \)

**Def.** Let \( c(\mathcal{D}) \) = \# of connected cards in a deck \( \mathcal{D} \).

Suppose \( G \) connected, \( H \) disconn., same \((n - \ell)\)-deck \( \mathcal{D} \).
\( G \Rightarrow c(\mathcal{D}) \geq 1 \), so \( H \) has component \( C \) with \( |V(C)| \geq n - \ell \).

Let \( |V(C)| = n - p \), so

\[ H \Rightarrow c(\mathcal{D}) \leq \binom{n-p}{\ell-p} \leq \binom{n-1}{\ell-1}. \]

(Keep only vertices from \( C \), discarding \( \ell - p \) of them.)

Also \( H \Rightarrow \hat{c}(\mathcal{D}) \geq \binom{n-1}{\ell} \), where \( \hat{c}(\mathcal{D}) \) = \# cards having a component of order \( \leq \ell \).

(Keeper a vertex \( x \) outside \( C \).)

**Idea:** From \( G \) get lower bd on \( c(\mathcal{D}) \) & upper bd on \( \hat{c}(\mathcal{D}) \), leading to contradiction when \( n \) is large.

Let \( T \) be a spanning tree of \( G \), and let \( \mathcal{D}' = \mathcal{D}_{n-\ell}(T) \).

- \( c(\mathcal{D}) \geq c(\mathcal{D}') \) and \( \hat{c}(\mathcal{D}) \leq \hat{c}(\mathcal{D}') \) (using same vertices).

Get lower bd on \( c(\mathcal{D}') \) & upper bd on \( \hat{c}(\mathcal{D}') \) instead.
Cards in $D'$

Let $t$ be the number of leaves in $T$. 
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$.
Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \left( \frac{t}{\ell} \right)$. 
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$. Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{\ell}$.

Thus $\frac{t(\ell)}{\ell!} = \binom{t}{\ell} \leq c(\mathcal{D}) \leq \binom{n-1}{\ell-1} \leq \binom{n}{\ell-1} = \frac{n(\ell-1)}{(\ell-1)!}$. 
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Thus $\frac{\binom{t}{\ell}}{\ell!} \leq c(\mathcal{D}) \leq \binom{n-1}{\ell-1} \leq \binom{n}{\ell-1} = \frac{n(n-1)}{(l-1)!}$.  

Hence $(t - \ell)^\ell < \ell n^{\ell-1}$, yielding $t < n(2\ell/n)^{1/\ell}$ for $n > \ell^{\ell^2}$.  

Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$. Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{\ell}$. Thus $\frac{t(\ell)}{\ell!} = \binom{t}{\ell} \leq c(\mathcal{D}) \leq \binom{n-1}{\ell-1} \leq \binom{n}{\ell-1} = \frac{n(\ell-1)}{(\ell-1)!}$. Hence $(t - \ell)^\ell < \ell n^{\ell-1}$, yielding $t < n(2\ell/n)^{1/\ell}$ for $n > \ell^{\ell^2}$.

Every card in $\mathcal{D}'$ counted by $\hat{c}(\mathcal{D}')$ has a tree component $F$ with $|V(F)| \leq \ell$, cut off by at most $\ell$ vertices.
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If $F$ is cut off by $j$ vertices, then $F$ is a component in fewer than $\binom{n}{\ell-j}$ cards.
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Let $t$ be the number of leaves in $T$.
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Hence \((t - \ell)^\ell < \ell n^{\ell-1}\), yielding \(t < n(2\ell/n)^{1/\ell}\) for \(n > \ell^{\ell^2}\).

Every card in $\mathcal{D}'$ counted by $\hat{c}(\mathcal{D}')$ has a tree component $F$ with $|V(F)| \leq \ell$, cut off by at most $\ell$ vertices.

If $F$ is cut off by $j$ vertices, then $F$ is a component in fewer than $\binom{n}{\ell-j}$ cards. Let $b_j = \#$ such subtrees $F$. 
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$. Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{\ell}$.

Thus $\frac{t(\ell)}{\ell!} = \binom{t}{\ell} \leq c(\mathcal{D}) \leq \binom{n-1}{\ell-1} \leq \binom{n}{\ell-1} = \frac{n(n-1)(n-2)\cdots(n-\ell+1)}{(\ell-1)!}$.

Hence $(t - \ell)^\ell < \ell n^{\ell-1}$, yielding $t < n(2\ell/n)^{1/\ell}$ for $n > \ell^2$.

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If $F$ is cut off by $j$ vertices, then $F$ is a component in fewer than $\binom{n}{\ell-j}$ cards. Let $b_j = \#$ such subtrees $F$.

Hence $\hat{c}(\mathcal{D}) \leq \hat{c}(\mathcal{D}') \leq \sum_{j=1}^{\ell} b_j \binom{n}{\ell-j}$.
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Let $t$ be the number of leaves in $T$. Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{\ell}$. Thus $\frac{t}{\ell!} \leq c(\mathcal{D}) \leq \binom{n-1}{\ell-1} \leq \binom{n}{\ell-1} = \frac{n(n-1)}{(l-1)!}$.

Hence $(t - \ell)^l < \ell n^{l-1}$, yielding $t < n\left(\frac{2\ell}{n}\right)^{1/l}$ for $n > \ell^{l^2}$.

Every card in $\mathcal{D}'$ counted by $\hat{c}(\mathcal{D}')$ has a tree component $F$ with $|V(F)| \leq \ell$, cut off by at most $\ell$ vertices. If $F$ is cut off by $j$ vertices, then $F$ is a component in fewer than $\binom{n}{\ell-j}$ cards. Let $b_j = \#$ such subtrees $F$.

Hence $\hat{c}(\mathcal{D}) \leq \hat{c}(\mathcal{D}') \leq \sum_{j=1}^{\ell} b_j \binom{n}{\ell-j}$.

We claim: $b_j \binom{n}{\ell-j} \leq \frac{\ell}{2} n^{l-1} t$. 
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Let $t$ be the number of leaves in $T$.
Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{\ell}$.
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Hence $(t - \ell)^{\ell} < \ell n^{\ell-1}$, yielding $t < n(2\ell/n)^{1/\ell}$ for $n > \ell^2$.

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Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$. Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{\ell}$. Thus $\frac{t(\ell)}{\ell!} = \binom{t}{\ell} \leq c(\mathcal{D}) \leq \binom{n-1}{\ell-1} \leq \binom{n}{\ell-1} = \frac{n(n-1)}{(\ell-1)!}$.

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Requires $n < 2\ell^{(\ell+1)^2}$, roughly $\ell > \left(\frac{2 \log n}{\log \log n}\right)^{1/2}$. \tag*{☐}
Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is $\#$subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside nbrs, then $b_j \leq \binom{t}{j} \binom{\ell+j-1}{j}$.
Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is the number of subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside nbrs, then $b_j \leq \binom{t}{j} \binom{\ell-j+1}{j}$ (except $b_2 \leq nt\ell/2$).
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$t = 8$

$l = 11$

$j = 4$

**Pf. $j \geq 3$:** Let $S$ be the set of outside vertices with neighbors in $F$. 


Counting Small Subtrees

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$t = 8$
$l = 11$
$j = 4$

**Pf.** $j \geq 3$: Let $S$ be the set of outside vertices with nbrs in $F$. $F = \text{component of } T - S \text{ having vertices between those of } S$. 
Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is the number of subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside nbrs, then $b_j \leq \binom{t}{j} \binom{\ell+j-1}{j}$ (except $b_2 \leq nt\ell/2$).

$t = 8$

\[ \ell = 11 \]

\[ j = 4 \]

**Pf.** $j \geq 3$: Let $S$ be the set of outside vertices with nbrs in $F$. $F = \text{component of } T - S \text{ having vertices between those of } S$. Paths from $F$ through $S$ reach leaves $S'$ of $T$. 
Thm. If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq l$, and $b_j$ is #subtrees $F$ with $|V(F)| \leq l$ and exactly $j$ outside nbrs, then $b_j \leq {t \choose j} \left( \frac{l+j-1}{j} \right)$ (except $b_2 \leq ntl/2$).

$t = 8$
$l = 11$
$j = 4$

Pf. $j \geq 3$: Let $S$ be the set of outside vertices with nbrs in $F$. $F = \text{component of } T - S \text{ having vertices between those of } S$. Paths from $F$ through $S$ reach leaves $S'$ of $T$. Bound #subgraphs $F$ generating fixed $S'$ of size $j$. 
**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is #subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside nbrs, then $b_j \leq \binom{t}{j} \binom{\ell+j-1}{j}$ (except $b_2 \leq ntl/2$).

**Pf.** \(j \geq 3\): Let $S$ be the set of outside vertices with nbrs in $F$. $F = \text{component of } T - S$ having vertices between those of $S$. Paths from $F$ through $S$ reach leaves $S'$ of $T$. Given $S'$ (in $\binom{t}{j}$ ways), let $T'$ be the tree generated by $S'$. 

$t = 8$\[\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{tree.png}}
\end{array}\]$
\ell = 11$
\(j = 4\)
Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is the number of subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside nbrs, then $b_j \leq \binom{t}{j} \binom{\ell+j-1}{j}$ (except $b_2 \leq nt\ell/2$).

$t = 8$

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**Pf.** $j \geq 3$: Let $S$ be the set of outside vertices with nbrs in $F$. $F$ = component of $T - S$ having vertices between those of $S$. Paths from $F$ through $S$ reach leaves $S'$ of $T$.

Given $S'$ (in $\binom{t}{j}$ ways), let $T'$ be the tree generated by $S'$.

The vertex $u \in S$ generating $v \in S'$ is on the path from $v$ to the nearest branch vertex $w$ in $T'$. Note $w \in V(F)$. Between $w$ and $u$ are fewer than $\ell$ vertices.
Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is \#subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside nbrs, then $b_j \leq \binom{t}{j} \binom{\ell + j - 1}{j}$ (except $b_2 \leq n t \ell / 2$).

- $t = 8$
- $\ell = 11$
- $j = 4$

**Pf.** $j \geq 3$: Let $S$ be the set of outside vertices with nbrs in $F$. $F = \text{component of } T - S \text{ having vertices between those of } S$. Paths from $F$ through $S$ reach leaves $S'$ of $T$. Given $S'$ (in $\binom{t}{j}$ ways), let $T'$ be the tree generated by $S'$. The vertex $u \in S$ generating $v \in S'$ is on the path from $v$ to the nearest branch vertex $w$ in $T'$. Note $w \in V(F)$. Between $w$ and $u$ are fewer than $\ell$ vertices. \# ways to place the break vertices $u \in S$ is at most \#solutions to $x_1 + \cdots + x_j \leq \ell - 1$, which equals $\binom{\ell + j - 1}{j}$. 
Smaller cases

\[ t = 8 \]
\[ \ell = 5 \]
\[ i = 3 \]

\[ j = 2: \] Since \( T \) has \( t \) leaves, from each vertex \( u \) there are at most \( t \) vertices at distance \( i \), for \( 2 \leq i \leq \ell + 1 \).
Smaller cases

\[ t = 8 \]
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$j = 2$: Since $T$ has $t$ leaves, from each vertex $u$ there are at most $t$ vertices at distance $i$, for $2 \leq i \leq \ell + 1$.

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$j = 1$: From a leaf move toward the centroid at most $\ell$ steps to place the vertex $u$ cutting off $F$;
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\[ j = 1: \text{ From a leaf move toward the centroid at most } \ell \text{ steps to place the vertex } u \text{ cutting off } F; \text{ bound is } t\ell. \]

\[ b_j \left( \begin{array}{c} n \\ \ell-j \end{array} \right) \leq (t_j^{\ell+j-1}) \left( \begin{array}{c} n \\ \ell-j \end{array} \right) \leq \frac{\ell}{2} n^{\ell-1} t \quad \text{(biggest when } j = 1) \]
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\( \ell = 5 \)
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\( j = 1 \): From a leaf move toward the centroid at most \( \ell \) steps to place the vertex \( u \) cutting off \( F \); bound is \( t\ell \).

- \( b_j(\binom{n}{\ell-j}) \leq (\binom{t}{j})(\ell+j-1)(\binom{n}{\ell-j}) \leq \frac{\ell}{2}n^{\ell-1}t \) (biggest when \( j = 1 \))

- For \( \ell = 3 \), these computations imply that connectedness is 3-reconstructible for \( n > 86,000,000 \).
**Smaller cases**

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For \( \ell = 3 \), we reduce this to \( n \geq 25 \).
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Hence each vertex belongs to at most \( t\ell \) sets \( S \) of size 2 that can cut off desired subtrees; the bound is \( ntl/2 \).

\( j = 1 \): From a leaf move toward the centroid at most \( \ell \) steps to place the vertex \( u \) cutting off \( F \); bound is \( t\ell \).

\[ b_j \left( \frac{n}{\ell-j} \right) \leq \binom{t}{j} \binom{\ell+j-1}{j} \binom{n}{\ell-j} \leq \frac{\ell}{2} n^{\ell-1} t \quad \text{(biggest when } j = 1) \]

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For \( \ell = 3 \), we reduce this to \( n \geq 25 \).

For the ideas, see the appendix at the end of the slides.
Prob 11898 (Stanley [2016], Amer. Math. Monthly) If $G$ is an $n$-vertex graph whose components are cycles of length greater than $k$, show that the number of independent sets of size $k$ depends only on $n$ and $k$. 
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Thm. Let $G$ and $G'$ be $n$-vertex graphs with maximum degree 2 and $|E(G)| = |E(G')|$. If every component in each graph is a cycle with more than $k$ vertices or a path with at least $k - 1$ vertices, then $D_k(G) = D_k(G')$. 
Maximum Degree 2

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(1) $D_k(C_{q+r}) = D_k(C_q + C_r)$ if $q, r \geq k + 1$,
(2) $D_k(P_{q+r}) = D_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$, and
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In fact, (1,2,3) suffice to prove the theorem, because:
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In fact, (1,2,3) suffice to prove the theorem, because:

Lem. If $G$, $G'$, and $H$ are graphs, then $D_k(G) = D_k(G')$ if and only if $D_k(G + H) = D_k(G' + H)$. 
The full statement, roughly

(1) $D_k(C_{q+r}) = D_k(C_q + C_r)$ if $q, r \geq k + 1$,
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**Thm.**  Given $G$ with $\Delta(G) = 2$, largest component $F$ with $m = |V(F)|$, and next largest with $m'$ vertices, $G$ is $k$-deck reconstructible iff $k \geq \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}$, where $\epsilon = 1$ if $F = P_m$ (else $\epsilon = 0$), and $\epsilon' \in \{0, 1, 2\}$. 
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Let \(s(G, H) = \#\) induced copies of \(H\) in \(G\).
The full statement, roughly

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Let $s(G, H) = \#$ induced copies of $H$ in $G$.

Let $s'(G', H') = \#$ induced copies of $H'$ having a named vertex $z$ of $G'$ as an isolated vertex in $H'$. ($H' = H + P_1$)
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- $s'(P_n, H')$ is indep of $z$ when far enough from ends.
Independent of the Named Vertex

**Lem.** Let $L^m$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L^m)$ is the same.
Lem. Let $L^m$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L^m)$ is the same.

Pf. Induction on $k$. For $k = 1$, any $w_h$ is in one $P_1$. 
**Lem.** Let $L^m$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L^m)$ is the same.

**Pf.** Induction on $k$. For $k = 1$, any $w_h$ is in one $P_1$. Compare $s'(P_n, L^m)$ to $s'(C_n, L^m)$ with edge $w_n w_1$. 
Independent of the Named Vertex

**Lem.** Let $L^m$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L^m)$ is the same.

**Pf.** Induction on $k$. For $k = 1$, any $w_h$ is in one $P_1$. Compare $s'(P_n, L^m)$ to $s'(C_n, L^m)$ with edge $w_n w_1$. By symmetry, $s'(C_n, L^m)$ is independent of $h$. 
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**Lem.** Let $L^m$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L^m)$ is the same.

**Pf.** Induction on $k$. For $k = 1$, any $w_h$ is in one $P_1$. Compare $s'(P_n, L^m)$ to $s'(C_n, L^m)$ with edge $w_n w_1$. By symmetry, $s'(C_n, L^m)$ is independent of $h$.

$s'(C_n, L^m)$ omits copies of $L^m$ in $P_n$ using $w_1$ and $w_n$. $s'(C_n, L^m)$ counts unwanted subgraphs using $w_n w_1$. 
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**Lem.** Let $L^m$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L^m)$ is the same.

**Pf.** Induction on $k$. For $k = 1$, any $w_h$ is in one $P_1$.

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With $L^m_i = L^m - V(P_{\ell_i})$ and $L^m_{i,j} = L^m - V(P_{\ell_i} + P_{\ell_j})$, we have

$$s'(P_n, L^m) = s'(C_n, L^m) + \sum_{i,j} s'(P_{n-(\ell_i+\ell_j+2)}, L^m_{i,j})$$

$$- \sum_i (\ell_i - 1) s'(P_{n-(\ell_i+2)}, L^m_i)$$
Independent of the Named Vertex

**Lem.** Let $L^m$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L^m)$ is the same.

**Pf.** Induction on $k$. For $k = 1$, any $w_h$ is in one $P_1$. Compare $s'(P_n, L^m)$ to $s'(C_n, L^m)$ with edge $w_n w_1$.

By symmetry, $s'(C_n, L^m)$ is independent of $h$.

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With $L^m_i = L^m - V(P_{\ell_i})$ and $L^m_{i,j} = L^m - V(P_{\ell_i} + P_{\ell_j})$, we have

$$s'(P_n, L^m) = s'(C_n, L^m) + \sum_{i,j} s'(P_n-(\ell_i+\ell_j+2), L^m_{i,j}) - \sum_{i}(\ell_i - 1)s'(P_n-(\ell_i+2), L^m_i)$$

$w_h$ is far enough from the ends to use induction hyp.
Same $k$-deck

(3) $D_k(P_{q-1} + P_r) = D_k(P_q + P_{r-1})$ if $q, r \geq k$.

With $q, r \geq k$, either index $h$ for $z = w_h$ satisfies $k + 1 \leq h \leq (q + r + 3) - (k + 1)$, so $s'(P_{q+r+2}, L^m + P_1)$ is the same for both when $|V(L^m)| = k$. \[\square\]
Same $k$-deck

(3) $\mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1})$ if $q, r \geq k$.

With $q, r \geq k$, either index $h$ for $z = w_h$ satisfies $k + 1 \leq h \leq (q + r + 3) - (k + 1)$, so $s'(P_{q+r+2}, L^m + P_1)$ is the same for both when $|V(L^m)| = k$.

(2) $\mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$. 
Same $k$-deck

(3) $D_k(P_{q-1} + P_r) = D_k(P_q + P_{r-1})$ if $q, r \geq k$.

With $q, r \geq k$, either index $h$ for $z = w_h$ satisfies $k + 1 \leq h \leq (q + r + 3) - (k + 1)$, so $s'(P_{q+r+2}, L^m + P_1)$ is the same for both when $|V(L^m)| = k$.

(2) $D_k(P_{q+r}) = D_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$. Let $P_{q+r} = \langle w_1, \ldots, w_{q+r} \rangle$ and $C_q = [w_1, \ldots, w_q]$. 

\[ D_k(P_{q-1} + P_r) = D_k(P_q + P_{r-1}) \text{ if } q, r \geq k. \]
Same \( k \)-deck

(3) \( \mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1}) \) if \( q, r \geq k \).

With \( q, r \geq k \), either index \( h \) for \( z = w_h \) satisfies \( k + 1 \leq h \leq (q + r + 3) - (k + 1) \), so \( s'(P_{q+r+2}, L^m + P_1) \) is the same for both when \( |V(L^m)| = k \).

(2) \( \mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r) \) if \( q \geq k + 1 \) and \( r \geq k - 1 \).

Let \( P_{q+r} = \langle w_1, \ldots, w_{q+r} \rangle \) and \( C_q = [w_1, \ldots, w_q] \).

If \( w_q \) not in copy of \( L^m \), both cases give \( s(P_{q-1} + P_r, L^m) \).
Same $k$-deck

(3) $\mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1})$ if $q, r \geq k$.

With $q, r \geq k$, either index $h$ for $z = w_h$ satisfies $k + 1 \leq h \leq (q + r + 3) - (k + 1)$, so $s'(P_{q+r+2}, L^m + P_1)$ is the same for both when $|V(L^m)| = k$.

(2) $\mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$.
Let $P_{q+r} = \langle w_1, \ldots, w_{q+r} \rangle$ and $C_q = [w_1, \ldots, w_q]$.
If $w_q$ not in copy of $L^m$, both cases give $s(P_{q-1} + P_r, L^m)$.
If used, sum over position of $w_q$ in which $P_{\ell_i}$ in $L^m$. 

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$G'$

\[ \begin{align*}
q - 1 & \quad z & \quad r \\
q & \quad z & \quad r - 1
\end{align*} \]
Same $k$-deck

(3) $\mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1})$ if $q, r \geq k$.

With $q, r \geq k$, either index $h$ for $z = w_h$ satisfies $k + 1 \leq h \leq (q + r + 3) - (k + 1)$, so $s'(P_{q+r+2}, L^m + P_1)$ is the same for both when $|V(L^m)| = k$.

(2) $\mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$.
Let $P_{q+r} = \langle w_1, \ldots, w_{q+r} \rangle$ and $C_q = [w_1, \ldots, w_q]$.
If $w_q$ not in copy of $L^m$, both cases give $s(P_{q-1} + P_r, L^m)$.
If used, sum over position of $w_q$ in which $P_{\ell_i}$ in $L^m$.
By (3), corresponding terms are equal.
Same $k$-deck

(3) $\mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1})$ if $q, r \geq k$.

With $q, r \geq k$, either index $h$ for $z = w_h$ satisfies $k + 1 \leq h \leq (q + r + 3) - (k + 1)$, so $s'(P_q+r+2, L^m + P_1)$ is the same for both when $|V(L^m)| = k$.

(2) $\mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$.

Let $P_{q+r} = \langle w_1, \ldots, w_{q+r} \rangle$ and $C_q = [w_1, \ldots, w_q]$.

If $w_q$ not in copy of $L^m$, both cases give $s(P_{q-1}+P_r, L^m)$.

If used, sum over position of $w_q$ in which $P_{\ell_i}$ in $L^m$.

By (3), corresponding terms are equal.

(1) $\mathcal{D}_k(C_{q+r}) = \mathcal{D}_k(C_q + C_r)$ if $q, r \geq k + 1$. 
Same $k$-deck

(3) $D_k(P_{q-1} + P_r) = D_k(P_q + P_{r-1})$ if $q, r \geq k$.

With $q, r \geq k$, either index $h$ for $z = w_h$ satisfies $k + 1 \leq h \leq (q + r + 3) - (k + 1)$, so $s'(P_{q+r+2}, L^m + P_1)$ is the same for both when $|V(L^m)| = k$.

(2) $D_k(P_{q+r}) = D_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$.

Let $P_{q+r} = \langle w_1, \ldots, w_{q+r} \rangle$ and $C_q = [w_1, \ldots, w_q]$.

If $w_q$ not in copy of $L^m$, both cases give $s(P_{q-1} + P_r, L^m)$.

If used, sum over position of $w_q$ in which $P_{\ell_i}$ in $L^m$.

By (3), corresponding terms are equal.

(1) $D_k(C_{q+r}) = D_k(C_q + C_r)$ if $q, r \geq k + 1$.

Same idea, reducing to equalities given by (2).
Proving $k$-deck Reconstructibility

**Lem.** If all components with more than $k$ vertices are determined by $\mathcal{D}_k(G)$, then $G$ is determined by $\mathcal{D}_k(G)$. 
Proving $k$-deck Reconstructibility

**Lem.** If all components with more than $k$ vertices are determined by $\mathcal{D}_k(G)$, then $G$ is determined by $\mathcal{D}_k(G)$.

**Pf.** Since $\mathcal{D}_k(G)$ determines $\mathcal{D}_{k-1}(G)$, it suffices to find the $k$-vertex components $F$ and iterate.
Proving $k$-deck Reconstructibility

**Lem.** If all components with more than $k$ vertices are determined by $\mathcal{D}_k(G)$, then $G$ is determined by $\mathcal{D}_k(G)$.

**Pf.** Since $\mathcal{D}_k(G)$ determines $\mathcal{D}_{k-1}(G)$, it suffices to find the $k$-vertex components $F$ and iterate.

$$\#F\text{-components} = \left[ \#F\text{-cards in } \mathcal{D}_k(G) \right] - \sum_{i=1}^{r} s(H_i, F),$$

where $H_1, \ldots, H_r$ are the larger components.  

$\blacksquare$
Proving $k$-deck Reconstructibility

**Lem.** If all components with more than $k$ vertices are determined by $\mathcal{D}_k(G)$, then $G$ is determined by $\mathcal{D}_k(G)$.

**Pf.** Since $\mathcal{D}_k(G)$ determines $\mathcal{D}_{k-1}(G)$, it suffices to find the $k$-vertex components $F$ and iterate.

\[
\text{\#F-components} = \left[ \text{\#F-cards in } \mathcal{D}_k(G) \right] - \sum_{i=1}^{r} s(H_i, F),
\]

where $H_1, \ldots, H_r$ are the larger components.

Let $q$ be \#path components with at least $k-1$ vertices.
**Lem.** If all components with more than $k$ vertices are determined by $D_k(G)$, then $G$ is determined by $D_k(G)$.

**Pf.** Since $D_k(G)$ determines $D_{k-1}(G)$, it suffices to find the $k$-vertex components $F$ and iterate.

$$\#F\text{-components} = \left[ \#F\text{-cards in } D_k(G) \right] - \sum_{i=1}^{r} s(H_i, F),$$
where $H_1, \ldots, H_r$ are the larger components.

Let $q$ be $\# path components with at least $k - 1$ vertices.

**Lem.** If $\Delta(G) = 2$, then $q = s(G,P_{k-1}) - s(G,P_k) - ks(G,C_k)$. 

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**Note:** The $D_k(G)$ notation refers to the $k$-deck of graph $G$, which is a collection of subgraphs of $G$ with $k$ vertices. The $s(H,F)$ notation represents the number of shared cards between components $H$ and $F$. The $\Delta(G)$ notation represents the maximum degree of the graph $G$.
Proving $k$-deck Reconstructibility

**Lem.** If all components with more than $k$ vertices are determined by $\mathcal{D}_k(G)$, then $G$ is determined by $\mathcal{D}_k(G)$.

**Pf.** Since $\mathcal{D}_k(G)$ determines $\mathcal{D}_{k-1}(G)$, it suffices to find the $k$-vertex components $F$ and iterate.

$$\#F\text{-components} = [\#F\text{-cards in } \mathcal{D}_k(G)] - \sum_{i=1}^{r} s(H_i, F),$$

where $H_1, \ldots, H_r$ are the larger components.

Let $q$ be $\#$ path components with at least $k-1$ vertices.

**Lem.** If $\Delta(G) = 2$, then $q = s(G, P_{k-1}) - s(G, P_k) - ks(G, C_k)$.

**Pf.** Each such path contributes 1 to $s(G, P_{k-1}) - s(G, P_k)$. Each $k$-cycle contributes 0 to $s(G, P_{k-1}) - ks(G, C_k)$. Each longer cycle contributes 0 to $s(G, P_{k-1}) - s(G, P_k)$.  


Proving \( k \)-deck Reconstructibility

**Lem.** If all components with more than \( k \) vertices are determined by \( D_k(G) \), then \( G \) is determined by \( D_k(G) \).

**Pf.** Since \( D_k(G) \) determines \( D_{k-1}(G) \), it suffices to find the \( k \)-vertex components \( F \) and iterate.

\[
\text{\#F-components} = \left( \text{\#F-cards in } D_k(G) \right) - \sum_{i=1}^{r} s(H_i, F),
\]
where \( H_1, \ldots, H_r \) are the larger components.

Let \( q \) be \#path components with at least \( k - 1 \) vertices.

**Lem.** If \( \Delta(G) = 2 \), then \( q = s(G, P_{k-1}) - s(G, P_k) - ks(G, C_k) \).

**Pf.** Each such path contributes \( 1 \) to \( s(G, P_{k-1}) - s(G, P_k) \).
Each \( k \)-cycle contributes \( 0 \) to \( s(G, P_{k-1}) - ks(G, C_k) \).
Each longer cycle contributes \( 0 \) to \( s(G, P_{k-1}) - s(G, P_k) \).

**Lem.** If \( \Delta(G) = 2 \), then \( D_k(G) \) determines \( q \).
Proving $k$-deck Reconstructibility

**Lem.** If all components with more than $k$ vertices are determined by $D_k(G)$, then $G$ is determined by $D_k(G)$.

**Pf.** Since $D_k(G)$ determines $D_{k-1}(G)$, it suffices to find the $k$-vertex components $F$ and iterate.

$\#F$-components $= \left[ \#F\text{-cards in } D_k(G) \right] - \sum_{i=1}^{r} s(H_i, F)$, where $H_1, \ldots, H_r$ are the larger components.

Let $q$ be $\#$ path components with at least $k - 1$ vertices.

**Lem.** If $\Delta(G)=2$, then $q = s(G, P_{k-1}) - s(G, P_k) - ks(G, C_k)$.

**Pf.** Each such path contributes 1 to $s(G, P_{k-1}) - s(G, P_k)$. Each $k$-cycle contributes 0 to $s(G, P_{k-1}) - ks(G, C_k)$. Each longer cycle contributes 0 to $s(G, P_{k-1}) - s(G, P_k)$.

**Lem.** If $\Delta(G) = 2$, then $D_k(G)$ determines $q$.

**Pf.** $s(G, P_k)$ and $s(G, C_k)$ just count cards. Each copy of $P_{k-1}$ is in $n-k+1$ cards, so $s(G, P_{k-1}) = \frac{\sum_{Q \in D_k(G)} s(Q, P_{k-1})}{n-k+1}$.
How to Use the Lemmas

\[ k \geq \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\} \implies D_k(G) \text{ determines } G \]
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Manvel [1974] showed that \( D_k(G) \) determines the degree list when \( k \geq \Delta(G) + 2 \). (We need special arguments when \( k = 3 \).) Henceforth \( \Delta(G) = 2 \).
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- If \( q \geq 2 \), then \( k < m' + \epsilon' \), not \( k \)-deck reconstructible.
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\[ k \geq \max \{ \lfloor m/2 \rfloor + \epsilon, m' + \epsilon' \} \implies \mathcal{D}_k(G) \text{ determines } G \]

Manvel [1974] showed that \( \mathcal{D}_k(G) \) determines the degree list when \( k \geq \Delta(G) + 2 \). (We need special arguments when \( k = 3 \).) Henceforth \( \Delta(G) = 2 \).

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- If \( q \geq 2 \), then \( k < m' + \epsilon' \), not \( k \)-deck reconstructible.
- If \( q \in \{0, 1\} \) and \( s(G, P_k) > \{2k+1, k\} \), then \( k < \left\lfloor \frac{m}{2} \right\rfloor + \epsilon \).
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- If \( q = 0 \) and \( 0 < s(G, P_k) \leq 2k + 1 \), then \( G \) has one component with more than \( k \) vertices, \( C_{s(G, P_k)} \).
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\[ k \geq \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\} \Rightarrow D_k(G) \text{ determines } G \]

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- If \( q = 1 \) and \( 0 \leq s(G, P_k) \leq k \), then \( G \) has no cycle with more than \( k \) vertices, and its long path is \( P_{s(G, P_k)+k-1} \).
How to Use the Lemmas

\[ k \geq \max \{ \lfloor m/2 \rfloor + \epsilon, m' + \epsilon' \} \Rightarrow D_k(G) \text{ determines } G \]

Manvel [1974] showed that \( D_k(G) \) determines the degree list when \( k \geq \Delta(G) + 2 \). (We need special arguments when \( k = 3 \).) Henceforth \( \Delta(G) = 2 \).

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This completes the proof except for small \( k \).
Open Questions

**Super-Kelly Conj.** Find $M_\ell$ such that $n \geq M_\ell$ implies $n$-vertex $G$ is $\ell$-reconstructible. Linear? $2\ell + 1$?
Open Questions

**Super-Kelly Conj.** Find $M_\ell$ such that $n \geq M_\ell$ implies $n$-vertex $G$ is $\ell$-reconstructible. Linear? $2\ell + 1$?

**Ques.** What is the least $k$ such that when $G$ has $n$ vertices, $D_k(G)$ determines whether $G$ is connected?
Open Questions

**Super-Kelly Conj.** Find $M_\ell$ such that $n \geq M_\ell$ implies $n$-vertex $G$ is $\ell$-reconstructible. Linear? $2\ell + 1$?

**Ques.** What is the least $k$ such that when $G$ has $n$ vertices, $D_k(G)$ determines whether $G$ is connected?

**Ques.** What values of $k$ suffice for $D_k(G)$ to determine other parameters on $n$-vertex graphs? (connectivity, matching number, chromatic number, planarity)
Open Questions

**Super-Kelly Conj.** Find $M_ℓ$ such that $n \geq M_ℓ$ implies $n$-vertex $G$ is $ℓ$-reconstructible. Linear? $2ℓ + 1$?

**Ques.** What is the least $k$ such that when $G$ has $n$ vertices, $D_k(G)$ determines whether $G$ is connected?

**Ques.** What values of $k$ suffice for $D_k(G)$ to determine other parameters on $n$-vertex graphs? (connectivity, matching number, chromatic number, planarity)

**Ques.** Is reconstructibility monotone for bipartite $G$? ($k = 3$ suffices for $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, but need $k = m$ for $C_{2m}$.)
Open Questions

**Super-Kelly Conj.** Find $M_ℓ$ such that $n \geq M_ℓ$ implies $n$-vertex $G$ is $ℓ$-reconstructible. Linear? $2ℓ + 1$?

**Ques.** What is the least $k$ such that when $G$ has $n$ vertices, $D_k(G)$ determines whether $G$ is connected?

**Ques.** What values of $k$ suffice for $D_k(G)$ to determine other parameters on $n$-vertex graphs? (connectivity, matching number, chromatic number, planarity)

**Ques.** Is reconstructibility monotone for bipartite $G$? ($k = 3$ suffices for $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, but need $k = m$ for $C_{2m}$.)

**Ques.** Compute the least $k$ for other graph classes. ($Δ(G) = 3$, vertex-transitive, etc.)
Open Questions

**Super-Kelly Conj.** Find $M_l$ such that $n \geq M_l$ implies $n$-vertex $G$ is $l$-reconstructible. Linear? $2l + 1$?

**Ques.** What is the least $k$ such that when $G$ has $n$ vertices, $\mathcal{D}_k(G)$ determines whether $G$ is connected?

**Ques.** What values of $k$ suffice for $\mathcal{D}_k(G)$ to determine other parameters on $n$-vertex graphs? (connectivity, matching number, chromatic number, planarity)

**Ques.** Is reconstructibility monotone for bipartite $G$? ($k = 3$ suffices for $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, but need $k = m$ for $C_{2m}$.)

**Ques.** Compute the least $k$ for other graph classes. ($\Delta(G) = 3$, vertex-transitive, etc.)

**Ques.** Do there exist a complete $r$-partite graph and complete $(r + 1)$-partite graph with the same $r$-deck? (Yes for $r \leq 3$: $\mathcal{D}_3(K_{7,4,3}) = \mathcal{D}_3(K_{6,6,1,1})$.)
Appendix - Connectedness from $\mathcal{D}_{n-3}$

Recall $c(\mathcal{D}) = \# \text{ of connected cards in a deck } \mathcal{D}$. 
Appendix - Connectedness from $\mathcal{D}_{n-3}$

Recall $c(\mathcal{D}) = \#$ of connected cards in a deck $\mathcal{D}$.

$G$ is connected if and only if $c(\mathcal{D}_{n-1}(G)) \geq 2$. 
Appendix - Connectedness from $\mathcal{D}_{n-3}$

Recall $c(\mathcal{D}) = \# \; \text{of connected cards in a deck } \mathcal{D}$.

$G$ is connected if and only if $c(\mathcal{D}_{n-1}(G)) \geq 2$. Also, $c(\mathcal{D}_{n-3}(G)) \leq 1 \implies G$ is disconnected.
Appendix - Connectedness from $\mathcal{D}_{n-3}$

Recall $c(\mathcal{D}) = \# \text{ of connected cards in a deck } \mathcal{D}$. $G$ is connected if and only if $c(\mathcal{D}_{n-1}(G)) \geq 2$. Also, $c(\mathcal{D}_{n-3}(G)) \leq 1 \implies G$ is disconnected.

Take $G$ connected, $H$ disconn. with same $(n-3)$-deck $\mathcal{D}$. 
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Take $G$ connected, $H$ disconn. with same $(n-3)$-deck $\mathcal{D}$. This needs $c(\mathcal{D}) \geq 2$, so $H$ has component of order $\leq 2$. 
Appendix - Connectedness from $\mathcal{D}_{n-3}$

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(Taylor [1990]) The degree list is reconstructible from the $k$-deck when $k \geq n(1 - \frac{1}{e})(1 + o(1))$. 
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$\therefore$ we may assume $H = C + C'$ of orders $n - 2$ and 2.
Appendix - Connectedness from $\mathcal{D}_{n-3}$

Recall $c(\mathcal{D}) = \#$ of connected cards in a deck $\mathcal{D}$.

$G$ is connected if and only if $c(\mathcal{D}_{n-1}(G)) \geq 2$.

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Take $G$ connected, $H$ disconn. with same $(n-3)$-deck $\mathcal{D}$.

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(Taylor [1990]) The degree list is reconstructible from the $k$-deck when $k \geq n(1 - \frac{1}{e})(1 + o(1))$.

∴ we may assume $H = C + C'$ of orders $n-2$ and $2$.

Thus $c(\mathcal{D}) \leq n-2$ and $i(\mathcal{D}) \geq \binom{n-2}{3}$, where $i(\mathcal{D}) = \#$ of cards in $\mathcal{D}$ having an isolated edge.
Appendix - Connectedness from $D_{n-3}$

Recall $c(D) = \# \text{ of connected cards in a deck } D$.

$G$ is connected if and only if $c(D_{n-1}(G)) \geq 2$.
Also, $c(D_{n-3}(G)) \leq 1 \implies G$ is disconnected.

Take $G$ connected, $H$ disconn. with same $(n-3)$-deck $D$.
This needs $c(D) \geq 2$, so $H$ has component of order $\leq 2$.

(Taylor [1990]) The degree list is reconstructible from the $k$-deck when $k \geq n(1 - \frac{1}{e})(1 + o(1))$.

∴ we may assume $H = C + C'$ of orders $n - 2$ and 2.

Thus $c(D) \leq n - 2$ and $i(D) \geq \binom{n-2}{3}$,
where $i(D) = \# \text{ of cards in } D \text{ having an isolated edge}$.

Idea: From $G$ get lower bound on $c(D)$ and upper bound on $i(D)$
leading to a contradiction when $n \geq 25$. 

Connected \((n - 3)\)-cards of connected \(G\)

Let \(T\) be a spanning tree of \(G\) having the fewest leaves.
Connected \((n - 3)\)-cards of connected \(G\)

Let \(T\) be a spanning tree of \(G\) having the fewest leaves.

- \(c(D) \geq c(D')\), where \(D' = D_{n-3}(T)\).
Connected \((n - 3)\)-cards of connected \(G\)

Let \(T\) be a spanning tree of \(G\) having the fewest leaves.

- \(c(\mathcal{D}) \geq c(\mathcal{D}')\), where \(\mathcal{D}' = \mathcal{D}_{n-3}(T)\).

Let \(L_1 =\) leaves of \(T\); \(V_2 = \{v \in V(T): d_T(v) = 2\}\)
\(L_2 = N_T(L_1) \cap V_2; \quad L_3 = N_T(L_2) \cap V_2; \quad l_i = |L_i|\).
Connected $(n - 3)$-cards of connected $G$

Let $T$ be a spanning tree of $G$ having the fewest leaves.

- $c(D) \geq c(D')$, where $D' = D_{n-3}(T)$.

Let $L_1 = \text{leaves of } T$; $V_2 = \{v \in V(T): d_T(v) = 2\}$, $L_2 = N_T(L_1) \cap V_2$; $L_3 = N_T(L_2) \cap V_2$; $l_i = |L_i|$.

Connected cards in $D'$: (1) Delete three from $L_1$. (2) Delete from $L_2$, its neighbor in $L_1$, another from $L_1$. (3) Delete an $L_3, L_2, L_1$ path.
Connected \((n-3)\)-cards of connected \(G\)

Let \(T\) be a spanning tree of \(G\) having the fewest leaves.

- \(c(D) \geq c(D')\), where \(D' = D_{n-3}(T)\).

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Hence \(c(D) \geq \binom{l_1}{3} + l_2(l_1 - 1) + l_3\).
Cards with Isolated Edges

Let $T$ be a spanning tree of $G$ having the fewest leaves.
Cards with Isolated Edges

Let $T$ be a spanning tree of $G$ having the fewest leaves.

• $i(D) \leq i(D') + \hat{i}$, where $D' = D_{n-3}(T)$ and $\hat{i}$ counts the cards in $D'$ having two isolated vertices adjacent in $G$. 
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\[ i(D') \leq l_2 \left( \frac{n-3}{2} \right) + (n - 1 - l_2)(n - 4) \]
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Let $T$ be a spanning tree of $G$ having the fewest leaves.

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$x, y \in V_2$ with common nbr: $\leq \binom{n-3}{2}$ (weak bound).
Cards with Isolated Edges

Let \( T \) be a spanning tree of \( G \) having the fewest leaves.

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\( \hat{i} \) counts no \( \{x, y\} \) both in \( L_1 \) (\( T \) has fewest leaves).

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\[
i(D) \leq l_2\left(\frac{n-3}{2}\right) + (n - 1)(n - 4) + (l_1 - 2)(n - 5) + \binom{n-3}{2}
\]
Inequalities

Bounds on $c(D)$: $\binom{l_1}{3} + l_2(l_1 - 1) + l_3 \leq n - 2$
Inequalities

Bounds on $c(\mathcal{D})$: $\binom{l_1}{3} + l_2(l_1 - 1) + l_3 \leq n - 2$

Bounds on $i(\mathcal{D})$:

$\binom{n-2}{3} \leq l_2 \binom{n-3}{2} + (n - 3)(n - 4) + l_1(n - 5) + 2 + \binom{n-3}{2}$
Inequalities

Bounds on $c(D)$: \( \left( \frac{1}{3} \right) + l_2(l_1 - 1) + l_3 \leq n - 2 \)

Bounds on $i(D)$:
\[
{\binom{n-2}{3}} \leq l_2 \left( \frac{n-3}{2} \right) + (n - 3)(n - 4) + l_1(n - 5) + 2 + \left( \frac{n-3}{2} \right)
\]

Divide the second by \( \frac{1}{3} \left( \frac{n-3}{2} \right) \) and combine:
\[
\left( \frac{1}{3} \right) + l_2(l_1 - 1) \leq 3l_2 + 6 + l_1 \frac{n-5}{n-3} + \frac{12}{(n-3)(n-4)} + 3,
\]
Inequalities

Bounds on $c(D)$: $(\binom{l_1}{3}) + l_2(l_1 - 1) + l_3 \leq n - 2$

Bounds on $i(D)$:

$\binom{n-2}{3} \leq l_2 \binom{n-3}{2} + (n - 3)(n - 4) + l_1(n - 5) + 2 + \binom{n-3}{2}$

Divide the second by $\frac{1}{3} \binom{n-3}{2}$ and combine:

$\left(\binom{l_1}{3}\right) + l_2(l_1 - 1) \leq 3l_2 + 6 + l_1 \frac{n-5}{n-3} + \frac{12}{(n-3)(n-4)} + 3,$

leading to $\left(\binom{l_1}{3}\right) + l_2(l_1 - 4) \leq 9 + l_1$. 
Inequalities

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This requires \( l_1 \leq 5 \).
Inequalities

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This requires \( l_1 \leq 5 \). With \( l_1 \leq 5 \), the bound on \# pairs $x, y \in V_2$ with common nbr improves from $\binom{n-3}{2}$ to $n - 1$. 

Inequalities

Bounds on $c(D)$: \( \binom{l_1}{3} + l_2(l_1 - 1) + l_3 \leq n - 2 \)

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This requires $l_1 \leq 5$. With $l_1 \leq 5$, the bound on \# pairs $x, y \in V_2$ with common nbr improves from \( \binom{n-3}{2} \) to $n - 1$.

With $l_2 \leq l_1 \leq 5$, the inequality
\[
\binom{n-2}{3} \leq l_2 \binom{n-3}{2} + (n - 3)(n - 4) + l_1(n - 5) + 2 + n - 1
\]
cannot hold for $n \geq 25$. \(\blacksquare\)