

Reconstruction from the deck of k -vertex induced subgraphs

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Abstract

The k -deck of a graph is its multiset of subgraphs induced by k vertices; we study what can be deduced about a graph from its k -deck. We strengthen a result of Manvel by proving for $\ell \in \mathbb{N}$ that when n is large enough ($n > 2\ell^{\ell+1}$ suffices), the $(n - \ell)$ -deck determines whether an n -vertex graph is connected ($n \geq 25$ suffices when $\ell = 3$, and $n \leq 2\ell$ cannot suffice). The *reconstructibility* $\rho(G)$ of a graph G with n vertices is the largest ℓ such that G is determined by its $(n - \ell)$ -deck. We generalize a result of Bollobás by showing $\rho(G) \geq (1 - o(1))n/2$ for almost all graphs. As an upper bound on $\min \rho(G)$, we have $\rho(C_n) = \lceil n/2 \rceil = \rho(P_n) + 1$. More generally, we compute $\rho(G)$ whenever $\Delta(G) = 2$, which involves extending a result of Stanley. Finally, we show that a complete r -partite graph is reconstructible from its $(r + 1)$ -deck.

MSC Codes: 05C60, 05C07

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1 Introduction

A *card* of a graph G is a subgraph of G obtained by deleting one vertex. Cards are unlabeled, so only the isomorphism class of a card is given. The *deck* of G is the multiset of all cards of G . A graph is *reconstructible* if it is uniquely determined by its deck. The famous Reconstruction Conjecture was first posed in 1942.

Conjecture 1.1 (The Reconstruction Conjecture; Kelly [11, 12], Ulam [33]). *Every graph having more than two vertices is reconstructible.*

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The two graphs with two vertices have the same deck. Graphs in many families are known to be reconstructible; these include disconnected graphs, trees, regular graphs, and perfect graphs. Surveys on graph reconstruction include [3, 4, 13, 14, 17].

Various parameters have been introduced to measure the difficulty of reconstructing a graph. Harary and Plantholt [10] defined the *reconstruction number* of a graph to be the minimum number of cards from its deck that suffice to determine it, meaning that no other graph has the same multiset of cards in its deck (surveyed in [1]). All trees with at least five vertices have reconstruction number 3 (Myrvold [24]), and almost all graphs have reconstruction number 3 (Bollobás [2]). Since $K_{r,r}$ and $K_{r+1,r-1}$ have $r + 1$ common cards, the reconstruction number of an n -vertex graph can be as large as $\frac{n}{2} + 2$ (Myrvold [23]). (Here $K_{r,s}$ is the complete bipartite graph with parts of sizes r and s .)

We can also study the reconstruction number of graph properties. Myrvold [22] and Bowler et al. [5] showed that any $\lfloor n/2 \rfloor + 2$ cards determine whether an n -vertex graph is connected. Woodall [34] proved for $n \geq \max\{34, 3p^2 + 1\}$ that the number of edges in an n -vertex graph is reconstructible within $p - 2$ from $n - p$ cards; this extends [20, 25, 27].

Kelly looked in another direction, considering cards obtained by deleting more vertices. He conjectured a more detailed version of the Reconstruction Conjecture.

Conjecture 1.2 (Kelly [12]). *For $\ell \in \mathbb{N}$, there is an integer M_ℓ such that any graph with at least M_ℓ vertices is reconstructible from its deck of cards obtained by deleting ℓ vertices.*

The original Reconstruction Conjecture is the claim $M_1 = 3$.

A k -card of a graph is an induced subgraph having k vertices. The k -deck of G , denoted $\mathcal{D}_k(G)$, is the multiset of all k -cards.

Definition 1.3. A graph G is k -deck reconstructible if $\mathcal{D}_k(H) = \mathcal{D}_k(G)$ implies $H \cong G$. A graph G (or a graph invariant) is ℓ -reconstructible if it is determined by $\mathcal{D}_{|V(G)|-\ell}(G)$ (agreeing on all graphs having that deck). The *reconstructibility* of G , written $\rho(G)$, is the maximum ℓ such that G is $(|V(G)| - \ell)$ -deck reconstructible.

Kelly's conjecture is that for any $\ell \in \mathbb{N}$, all sufficiently large graphs are ℓ -reconstructible. Having checked by computer that every graph with at least six and at most nine vertices is 2-reconstructible (there are 5-vertex graphs that are not), McMullen and Radziszowski [19] asked whether $M_2 = 6$. With computations up to nine vertices, Rivshin and Radziszowski [29] conjectured $M_\ell \leq 3\ell$. Note that for an n -vertex graph, " k -deck reconstructible" and " ℓ -reconstructible" have the same meaning when $k + \ell = n$.

Since each induced subgraph with $k - 1$ vertices arises exactly $n - k + 1$ times by deleting one vertex from a member of $\mathcal{D}_k(G)$, we have the following.

Observation 1.4. For any graph G , the k -deck $\mathcal{D}_k(G)$ determines the $(k - 1)$ -deck $\mathcal{D}_{k-1}(G)$.

By Observation 1.4, information that is k -deck reconstructible is also j -deck reconstructible when $j > k$. This motivates the definition of reconstructibility; if G is ℓ -reconstructible, then G is also $(\ell - 1)$ -reconstructible, and we seek the largest such ℓ .

The 2-deck of G determines only $|E(G)|$ and $|V(G)|$, but the 3-deck determines the number of edge incidences and whether G is complete multipartite. At the other end, Manvel [18] proved that for $|V(G)| = n \geq 6$, the $(n - 2)$ -deck determines whether G is connected, acyclic, unicyclic, regular, or bipartite. In Section 2 we prove a weak version of Kelly's conjecture for connectedness: for fixed ℓ , when n is sufficiently large the $(n - \ell)$ -deck determines whether an n -vertex graph is connected. Our general argument shows that $n > 2\ell^{(\ell+1)^2}$ suffices, so connectedness is ℓ -reconstructible when $\ell < (1 + o(1))(\frac{2 \log n}{\log \log n})^{1/2}$. With a more detailed argument, we show that $n \geq 25$ suffices when $\ell = 3$ (note that $2\ell^{(\ell+1)^2}$ exceeds 86,000,000 when $\ell = 3$).

We believe (for $n \geq 6$) that connectedness is ℓ -reconstructible whenever $\ell < \lceil n/2 \rceil$ (that is, $n \geq 2\ell + 1$). Sharpness is shown by our result in Section 4 that P_{2k} and $C_{k+1} + P_{k-1}$ have the same k -deck. (Here C_n and P_n are the path and cycle with n vertices, and $G + H$ is the disjoint union of G and H .) This yields $M_\ell \geq 2\ell + 1$ in Kelly's conjecture. Nýdl [26] proved that for any $n_0 \in \mathbb{N}$ and $0 < q < 1$, there are nonisomorphic n -vertex graphs for some n larger than n_0 having the same $\lfloor qn \rfloor$ -deck, but this does not prevent M_ℓ being linear in ℓ .

When studying almost all graphs instead of all graphs, we can say much more. With cards not much larger than those where connectedness is not reconstructible, we can completely reconstruct the graph. The result of Bollobás [2] is that almost all graphs are reconstructible from *any* three cards obtained by deleting one vertex. This is much stronger than having reconstruction number 3, which is reconstructibility from *some* three cards (no graph is reconstructible from two cards, since one cannot determine whether the two vertices deleted to form those cards are adjacent).

In Section 3, we generalize Bollobás' result by proving that if $\ell \leq (1 - o(1))n/2$, then almost all graphs are reconstructible from some set of $\binom{\ell+2}{2}$ cards produced by deleting ℓ vertices (actually from many such sets). This relies on the statement (by Müller [21] and Bollobás [2]) that almost always the induced subgraphs with nearly half the vertices are pairwise nonisomorphic and have no nontrivial automorphisms.

Thus, in term of the number of vertices, n , for almost all graphs the reconstructibility is at least $(1 - o(1))n/2$, and there are graphs where it does not exceed $n/2$. We have mentioned that connectedness is not $\lfloor n/2 \rfloor$ -reconstructible even for graphs with maximum degree 2. In Section 4 we compute $\rho(G)$ whenever $\Delta(G) = 2$; in particular, $\rho(C_n) = \lceil n/2 \rceil = \rho(P_n) + 1$. With m and m' denoting the numbers of vertices in the largest and next largest (possibly equal) components, the least k such that $\mathcal{D}_k(G)$ determines G has the form $\max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}$, where always $\epsilon \in \{0, 1\}$ and $\epsilon' \in \{0, 1, 2\}$. When G is 2-regular, the expression simplifies to $\max\{\lfloor m/2 \rfloor, m'\}$.

Proving that $\mathcal{D}_k(G)$ does not determine G involves showing that another graph has the same k -deck. For this we extend a result of Stanley [30], who showed that for 2-regular n -vertex graphs in which every cycle has more than k vertices, the number of independent k -sets depends only on k and n . We prove that any two graphs of maximum degree 2 with the same numbers of vertices and edges also have the same k -deck if every cycle has more than k vertices and every path component has at least $k - 1$ vertices.

Finally, in Section 5 we study highly reconstructible graphs. We prove that every complete r -partite graph is reconstructible from its $(r + 1)$ -deck.

The terminology here is also used in another model of reconstruction with different definitions. Since the terminology overlaps, we try to reduce confusion by describing the main results in that model. We use “digraph” to mean a general binary relation (no repeated edges).

Two digraphs D and D' on an n -element vertex set V are said to be *k -isomorphic* if for every k -element subset $X \subseteq V$, the subdigraphs of D and D' induced by X are isomorphic. They are *$(\leq k)$ -isomorphic* if they are k' -isomorphic for all k' with $1 \leq k' \leq k$. They are *$(-k)$ -isomorphic* if they are $(n - k)$ -isomorphic. A digraph D is *α -reconstructible*, where $\alpha \in \{k, \leq k, -k\}$, if every digraph α -isomorphic to D is isomorphic to D .

These notions were introduced by Fraïssé [9], who conjectured that for sufficiently large k every digraph is $(\leq k)$ -reconstructible (and analogously for m -ary relations, for each m). The difference between Fraïssé’s model and that of Kelly and Ulam is that in Fraïssé’s problem we are told the identities of the missing vertices, but in the problem of Kelly and Ulam we are given only the multiset of isomorphism types. The notions coincide for the original conjecture: a graph (that is, a symmetric digraph) is reconstructible (in the Kelly–Ulam sense) if and only if it is (-1) -reconstructible (in the Fraïssé sense). Stockmeyer [31] showed that general digraphs (in fact, orientations of complete graphs) are not (-1) -reconstructible.

The difference is clear when $k = 2$. Only graphs with at most one edge (and their complements) are reconstructible from their 2-decks, but every symmetric digraph is 2-reconstructible, since we are told which pairs are adjacent. This does not hold for digraphs; any two orientations of a complete graph are 2-isomorphic.

Fraïssé’s conjecture was proved for digraphs (that is, binary relations) by Lopez [15, 16], who proved that every digraph is (≤ 6) -reconstructible (this is sharp). The theorem was proved independently by Reid and Thomassen [28], and it also follows from the later characterization of the non- $(\leq k)$ -reconstructible digraphs by Boudabbous and Lopez [7]. A history of the topic appears in [6].

Analogously to Observation 1.4, Pouzet showed that if two n -vertex digraphs are p -isomorphic, then they are also q -isomorphic whenever $1 \leq q \leq \min\{p, n - p\}$. With Lopez’s Theorem, this implies that every digraph with at least 11 vertices is 6-reconstructible, and every digraph with at least 12 vertices is (-6) -reconstructible.

2 Reconstructibility of Connectedness

For graphs with at least three vertices, connectedness is 1-reconstructible, because an n -vertex connected graph has at least two connected $(n-1)$ -cards, while a disconnected graph has at most one connected $(n-1)$ -card (when $n \geq 3$). Manvel [18] strengthened this result.

Theorem 2.1 (Manvel [18]). *For $n \geq 6$, the connectedness of an n -vertex graph is 2-reconstructible.*

The condition in Theorem 2.1 is sharp: for $n = 5$, the disconnected graph $C_4 + P_1$ and the tree obtained by subdividing one edge of $K_{1,3}$ have the same 3-deck. These graphs and their complements are the only 5-vertex graphs that are not 2-reconstructible [19]. In Theorem 2.4 we prove that connectedness of n -vertex graphs is ℓ -reconstructible when $n > 2\ell^{(\ell+1)^2}$. Using a more detailed argument, we show in Theorem 2.7 that $n \geq 25$ suffices when $\ell = 3$. Most likely this improved threshold for $\ell = 3$ also is not sharp. In Section 4 we prove $\mathcal{D}_{\lfloor n/2 \rfloor}(P_n) = \mathcal{D}_{\lfloor n/2 \rfloor}(P_{\lfloor (n-1)/2 \rfloor} + C_{\lceil (n+1)/2 \rceil})$. Thus for $\ell \geq 2$, ensuring that connectedness is ℓ -reconstructible for *all* n -vertex graphs requires $n \geq 2\ell + 1$. In light of the 5-vertex example of $C_4 + P_1$ for $\ell = 2$, we pose only this:

Conjecture 2.2. *For $n \geq 2\ell + 2$, the connectedness of an n -vertex graph is ℓ -reconstructible.*

To determine connectedness from the $(n-2)$ -deck, Manvel argued as for the $(n-1)$ -deck, obtaining contradictory bounds on the number of connected $(n-2)$ -cards from a connected graph and a disconnected graph. When more vertices are deleted, this no longer suffices, and we must also consider disconnected cards. We briefly summarize our approach.

Suppose $n > 2\ell$. With ℓ fixed, we call members of an $(n-\ell)$ -deck “cards” rather than $(n-\ell)$ -cards. Let $c(\mathcal{D})$ denote the number of connected cards in a deck \mathcal{D} , and let $\hat{c}(\mathcal{D})$ denote the number of cards having a component with at most ℓ vertices.

When connectedness is not ℓ -reconstructible for n -vertex graphs, there exist a connected graph G and a disconnected graph H such that $\mathcal{D}_{n-\ell}(G) = \mathcal{D}_{n-\ell}(H)$; let \mathcal{D} denote this common $(n-\ell)$ -deck. For $c(\mathcal{D})$, we obtain an upper bound from H and a lower bound from G ; together, these yield an upper bound on the number t of leaves in a spanning tree of G . We obtain an upper bound on $\hat{c}(\mathcal{D})$ from G in terms of t . In order to show that this upper bound is smaller than the lower bound on $\hat{c}(\mathcal{D})$ arising from H (thereby prohibiting such an alternative reconstruction), we need an upper bound on the number of subtrees with at most ℓ vertices in a tree with t leaves. This bound may be of independent interest.

Here a *subtree* of a tree is a connected subgraph with at least one vertex. A *branch vertex* in a tree is a vertex with degree at least 3. The *break* of a subgraph F in a graph G , here denoted $\partial(F)$, is the set of vertices outside F that have at least one neighbor in F .

Theorem 2.3. *Let T be an n -vertex tree with t leaves. Given $1 \leq j \leq \ell < n/2$, let \mathcal{F} be the family of subtrees F of T such that $|V(F)| \leq \ell$ and $|\partial(F)| = j$. If $j = 2$, then $|\mathcal{F}| \leq nt\ell/2$; otherwise, $|\mathcal{F}| \leq \binom{t}{j} \binom{\ell+j-1}{j}$.*

Proof. We consider three cases, depending on j .

Case 1: $j \geq 3$. Note that a set S of vertices in T is the break for some subtree if and only if no vertex of S lies on the path joining two other vertices of S in T . The subtree F such that $S = \partial(F)$ is then the component of $T - S$ containing all vertices on paths joining vertices of S . It may also contain vertices that do not lie on such paths.

For $F \in \mathcal{F}$, a path from F through a vertex $u \in \partial(F)$ eventually reaches a leaf v of T ; there may be many choices for v . Choosing one such leaf for each $u \in \partial(F)$ yields a j -set of leaves in T . We bound the number of subtrees in \mathcal{F} that can produce a particular set S' of j leaves. This times the number $\binom{t}{j}$ of such j -sets gives the desired upper bound on $|\mathcal{F}|$.

Given a set S' of j leaves in T arising from a subtree $F \in \mathcal{F}$ in this way, let T' be the minimal subtree of T containing S' . Let $S' = \{v_1, \dots, v_j\}$. For $v_i \in S'$, the vertex $u_i \in \partial(F)$ through which v_i is chosen is along the path in T' from v_i to the branch vertex of T' that is closest to v_i ; call this branch vertex w_i (the vertices w_1, \dots, w_j need not be distinct; indeed, T' may have only one branch vertex). The vertex u_i must lie on the path from w_i to v_i (and not equal w_i); otherwise, F would be cut off by deleting fewer than j vertices, since u_i would block more than one vertex of S' from F .

When u_i has distance $1 + x_i$ from w_i on the path toward v_i , the $1 + x_i$ vertices on this path from w_i until just before u_i all lie in $V(F)$; hence $x_i \leq \ell - 1$. Thus the number of ways to locate the vertices of $\partial(F)$ along the paths to S' in T' (and hence specify $F \in \mathcal{F}$ that yields the set S' of leaves) is at most the number of nonnegative integer solutions to $\sum_{i=1}^j x_i \leq \ell - 1$, or to $\sum_{i=0}^j x_i = \ell - 1$. This equals the number of orderings of $\ell - 1$ dots and j separators, which is $\binom{\ell+j-1}{j}$.

Case 2: $j = 2$. The distance between the two vertices of $\partial(F)$ is at least 2 and at most $\ell + 1$, since $|V(F)| \leq \ell$ and there is a vertex of F between them. For any vertex v , at most t vertices have distance i from v in T , because extending the paths from v to those vertices reaches distinct leaves. Hence each vertex is paired with at most $t\ell$ others to form a possible break of size 2 when $|V(F)| \leq \ell$, and the total number of such pairs is bounded by $nt\ell/2$.

Case 3: $j = 1$. A *centroid* of T is a vertex z minimizing the maximum number of vertices in a component of $T - z$. It is a standard exercise that each tree has one centroid or two adjacent centroids. A subtree F with $|V(F)| < n/2$ and $|\partial(F)| = 1$ cannot contain a centroid z , because deleting $V(F)$ would leave a subtree with more than $n/2$ vertices inside one component of $T - z$, and then the vertex of $\partial(F)$ would contradict the choice of the centroid.

When $|\partial(F)| = \{u\}$, choose a leaf v reached along a path through u from a centroid (note that $v \in V(F)$). Yielding a given leaf v there are at most ℓ choices for u to create F , since it must be within distance ℓ of v on the path to the centroid. Thus $|\mathcal{F}| \leq t\ell = \binom{t}{j} \binom{\ell+j-1}{j}$. \square

The bounds in Theorem 2.3 seem weak when j is near ℓ , but the most important case in the application is $j = 1$, and that case can be sharp for a subdivided star.

Theorem 2.4. *For fixed ℓ , the $(n - \ell)$ -deck of an n -vertex graph G determines whether G is connected when $n > 2\ell^{(\ell+1)^2}$. That is, connectedness of an n -vertex graph is ℓ -reconstructible when $\ell < (1 + o(1))\left(\frac{2 \log n}{\log \log n}\right)^{1/2}$.*

Proof. We have already noted that the conclusion holds for $n \geq 3$ when $\ell = 1$, so we may assume $\ell \geq 2$. Recall that $c(\mathcal{D})$ is the number of connected cards in a deck \mathcal{D} and that $\hat{c}(\mathcal{D})$ is the number of cards having a component with at most ℓ vertices.

Consider n -vertex graphs G and H such that $\mathcal{D} = \mathcal{D}_{n-\ell}(G) = \mathcal{D}_{n-\ell}(H)$, with G connected and H disconnected. From H we obtain an upper bound on $c(\mathcal{D})$ and a lower bound on $\hat{c}(\mathcal{D})$. From G we obtain a lower bound on $c(\mathcal{D})$ and an upper bound on $\hat{c}(\mathcal{D})$. The resulting constraints will yield an upper bound on n in terms of ℓ .

Since G has connected cards, H must have a component C with at least $n - \ell$ vertices. With $n > 2\ell$, there is only one such component. Let $|V(C)| = n - p$. A connected card in $\mathcal{D}_{n-\ell}(H)$ omits all p vertices outside C and $\ell - p$ others. Hence $c(\mathcal{D}) \leq \binom{n-p}{\ell-p} \leq \binom{n-1}{\ell-1} < \binom{n}{\ell-1}$.

Now consider the cards having components with at most ℓ vertices, counted by $\hat{c}(\mathcal{D})$. Since $p \leq \ell$, we obtain a lower bound from H by counting the cards that contain any fixed vertex outside C , discarding ℓ other vertices. There are $\binom{n-1}{\ell}$ such cards, so $\hat{c}(\mathcal{D}) \geq \binom{n-1}{\ell}$.

To obtain bounds on $c(\mathcal{D})$ and $\hat{c}(\mathcal{D})$ from G , we consider a spanning tree T of G . Let $\mathcal{D}' = \mathcal{D}_{n-\ell}(T)$. Consider a set $S \subseteq V(G)$. If $T - S$ is connected, then $G - S$ is connected. If $G - S$ has a component with at most ℓ vertices, then so does $T - S$. Thus $c(\mathcal{D}) \geq c(\mathcal{D}')$ and $\hat{c}(\mathcal{D}') \geq \hat{c}(\mathcal{D})$. Therefore, we can use a lower bound on $c(\mathcal{D}')$ and upper bound on $\hat{c}(\mathcal{D}')$.

Let t be the number of leaves of the spanning tree T of G . Since the subgraph obtained by deleting any set of leaves in T is connected, $c(\mathcal{D}') \geq \binom{t}{\ell}$. Combined with the upper bound on $c(\mathcal{D})$ from H , we now have

$$\frac{\prod_{i=0}^{\ell-1} (t - i)}{\ell!} = \binom{t}{\ell} \leq c(\mathcal{D}') \leq c(\mathcal{D}) < \binom{n}{\ell-1} = \frac{\prod_{i=0}^{\ell-2} (n - i)}{(\ell-1)!}.$$

Further weakening the inequality by using the smallest factor on the low side and the largest factor on the high side (when $t > \ell$), we have $(t - \ell)^\ell < \ell n^{\ell-1}$, which yields $t < \ell + n(\ell/n)^{1/\ell}$ (true also when $t \leq \ell$). Since $\ell < n(\ell/n)^{1/\ell}$ for $n > \ell > 1$, we may relax the bound to $t < 2n(\ell/n)^{1/\ell}$. This bound on t will be useful in applying Theorem 2.3.

Finally, we need an upper bound on $\hat{c}(\mathcal{D}')$. A component F having at most ℓ vertices in a card counted by $\hat{c}(\mathcal{D}')$ is a subtree of T having at most ℓ vertices. The card is formed by deleting ℓ vertices from T , so $1 \leq |\partial(F)| \leq \ell$. Let $j = |\partial(F)|$. The number of cards in which F appears as a component is at most $\binom{n-j-|V(F)|}{\ell-j}$; we use $\binom{n}{\ell-j}$ as an upper bound.

Let b_j be the total number of such cards resulting from all F such that $|\partial(F)| = j$. By Theorem 2.3, for $j \neq 2$ we have $b_j < f(j)$, where $f(j) = \binom{t}{j} \binom{\ell+j-1}{j} \binom{n}{\ell-j}$. For $1 \leq j < \ell$,

$$\frac{f(j+1)}{f(j)} = \binom{t-j}{j+1} \binom{\ell+j}{j+1} \left(\frac{\ell-j}{n-\ell+j+1} \right) < \frac{(t-1)\ell^2}{4(n-\ell)} < \frac{t\ell^2}{2n} < \left(\frac{\ell}{n} \right)^{1/\ell} \ell^2 < 1$$

when $n > \ell^{2\ell+1}$, since $1 \leq j \leq \ell < n/2$ and $t < 2n(\frac{\ell}{n})^{1/\ell}$. Thus $b_j \leq f(1) = t\ell \binom{n}{\ell-1} < n^{\ell-1}t\ell/2$, since $\ell \geq 2$.

When $j = 2$, Theorem 2.3 yields $b_j < \frac{nt\ell}{2} \binom{n}{\ell-2}$, so again $b_j < n^{\ell-1}t\ell/2$. Summing over j and using the lower bound $\hat{c}(\mathcal{D}) \geq \binom{n-1}{\ell}$ obtained from H and the upper bound $t < 2n(\ell/n)^{1/\ell}$ obtained for spanning trees of G , we now have

$$\binom{n-1}{\ell} \leq \hat{c}(\mathcal{D}) \leq \sum_{j=1}^{\ell} b_j < n^{\ell-1}t\ell^2/2 < n^{\ell-1/\ell}\ell^{2+1/\ell}.$$

Since $\frac{n-i}{\ell-i} > \frac{n}{\ell}$ when $1 \leq i < \ell < n$, we have $\binom{n-1}{\ell} = \frac{n-\ell}{n} \binom{n}{\ell} > (1-\ell/n) \left(\frac{n}{\ell}\right)^\ell$. These upper and lower bounds on $\binom{n-1}{\ell}$ yield $(1-\ell/n) \left(\frac{n}{\ell}\right)^\ell < n^{\ell-1/\ell}\ell^{2+1/\ell}$, which simplifies to $(1-\ell/n)^\ell < \ell^{(\ell+1)^2}/n$. However, $(1-\ell/n)^\ell > (1-\ell^2/n) > 1/2 > \ell^{(\ell+1)^2}/n$ when $n > 2\ell^{(\ell+1)^2} > 2\ell^2$. This contradiction forbids a connected G and a disconnected H with the same $(n-\ell)$ -deck when $n > 2\ell^{(\ell+1)^2}$. \square

Attempts to tighten the relaxed bounds in the argument of Theorem 2.4 generally only affect lower-order terms. For example, by using Theorem 2.5 as we do for the case $\ell = 3$ in Theorem 2.7, we can show that H has no isolated vertex. Hence H has a small component with at least two vertices. This shrinks the upper bound on $c(\mathcal{D})$ from $\binom{n-1}{\ell-1}$ to $\binom{n-2}{\ell-2}$. In the leading behavior of the conclusion about reconstructibility, this only replaces $2 \log n$ with $4 \log n$ under the square root. Similarly, the square in the exponent of $n > 2\ell^{(\ell+1)^2}$ comes from the denominator in the lower bound on $\binom{n-1}{\ell}$; replacing ℓ^ℓ with $\ell!$ would not help much.

When $\ell = 3$, the value of $2\ell^{(\ell+1)^2}$ exceeds 86,000,000. Following the argument of Theorem 2.4 without relaxing the bounds to simplify the general statement yields a threshold around $n > 4800$ for $\ell = 3$. In the spirit of Theorem 2.1 computing the exact threshold on n for $\ell = 2$, we present a more detailed argument to reduce the threshold to 25 when $\ell = 3$. We will need to reconstruct the degree list.

Theorem 2.5 (Manvel [18]). *The degree list of a graph G is reconstructible from $\mathcal{D}_{\Delta(G)+2}(G)$.*

For sharpness, Manvel [18] showed that the maximum degree itself is not always determined by $\mathcal{D}_{\Delta(G)+1}(G)$. He constructed graphs G and H such that $\Delta(G) = k$, $\Delta(H) = k+1$, and $\mathcal{D}_{k+1}(G) = \mathcal{D}_{k+1}(H)$. Both graphs are forests of stars. However, in this construction the number of vertices is exponential in k . It seems to be unknown how large n must be to permit the existence of an n -vertex graph with maximum degree k whose $(k+1)$ -deck has

an alternative reconstruction with maximum degree $k + 1$. Taylor showed that the degree list is reconstructible from the k -deck when the number of vertices is not too much larger than k , regardless of the value of the maximum degree.

Theorem 2.6 (Taylor [32]). *If $n \geq g(\ell)$, then the degree list of any n -vertex graph is determined by its $(n - \ell)$ -deck, where*

$$g(\ell) = (\ell - \log \ell + 1) \left(e + \frac{e \log \ell + e + 1}{(\ell - 1) \log \ell - 1} \right) + 1.$$

Here e denotes the base of the natural logarithm.

In particular, the degree list of an n -vertex graph is reconstructible from the k -deck when $k \geq n(1 - \frac{1}{e})(1 + o(1))$.

For a vertex subset S in a graph G , let $N_G(S)$ denote the neighborhood of S , meaning the set of vertices in G that have a neighbor in S .

Theorem 2.7. *For $n \geq 25$, the connectedness of an n -vertex graph is 3-reconstructible.*

Proof. Set $\ell = 3$. As in Theorem 2.4, G and H are connected and disconnected n -vertex graphs with $\mathcal{D}_{n-3}(G) = \mathcal{D}_{n-3}(H) = \mathcal{D}$. Again $c(\mathcal{D})$ is the number of connected cards in a deck \mathcal{D} . Instead of using $\hat{c}(\mathcal{D})$, let $i(\mathcal{D})$ be the number of cards having an isolated edge.

Since G is connected, it has a spanning tree with at least $\Delta(G)$ leaves, obtained by growing a tree from a largest star centered at a vertex of maximum degree. Hence $c(\mathcal{D}) \geq \binom{\Delta(G)}{3}$. When $n \geq 7$, having connected cards requires H to have a unique component C with at least $n - 3$ and at most $n - 1$ vertices. All vertices outside C must be deleted, so $c(\mathcal{D}) \leq \binom{n-1}{2}$. Since $n \geq 12$, we have $\binom{n-4}{3} > \binom{n-1}{2}$, so we may assume $\Delta(G) < n - 4$. By Theorem 2.5, the common degree list of G and H is reconstructible from \mathcal{D} . Hence H has no isolated vertex.

Consider a spanning tree of G . If it has at least four leaves, then deleting some three of them yields at least four connected cards. When $n \geq 6$, trees with at most three leaves also have at least four connected cards with $n - 3$ vertices. Hence from any spanning tree of G we conclude $c(\mathcal{D}) \geq 4$. Thus the large component C in H must have $n - 2$ vertices, not $n - 3$, since with $n - 3$ vertices there would only be one connected card. Hence $H = C + C'$, where $C' \cong K_2$.

A connected card of H is obtained by deleting $V(C')$ and a non-cut-vertex of C . Hence

$$c(\mathcal{D}) \leq n - 2. \tag{2.1}$$

Also, any card of H obtained by deleting three vertices of C has C' as an isolated edge, so

$$i(\mathcal{D}) \geq \binom{n-2}{3}. \tag{2.2}$$

Let L be the set of leaves in G , and let $l = |L|$. Let T be a spanning tree of G having the fewest leaves, and let $V_2 = \{v \in V(G) : d_T(v) = 2\}$. Let L_1 be the set of leaves in T , let $L_2 = N_T(L_1) \cap V_2$, and let $L_3 = N_T(L_2) \cap V_2$. Let $l_i = |L_i|$ for $i \in \{1, 2, 3\}$.

We consider connected cards of T obtained in two ways: by deleting three vertices of L_1 , or by deleting a vertex of L_2 , its neighbor in L_1 , and another vertex of L_1 . Hence $c(\mathcal{D}') \geq \binom{l_1}{3} + l_2(l_1 - 1)$, where $\mathcal{D}' = \mathcal{D}_{n-3}(T)$. Combining with $c(\mathcal{D}') \leq c(\mathcal{D})$ and (2.1) yields

$$\binom{l_1}{3} + l_2(l_1 - 1) \leq n - 2. \quad (2.3)$$

Now consider a card in \mathcal{D} having an isolated edge. This edge may or may not belong to $E(T)$. Hence $i(\mathcal{D}) \leq i(\mathcal{D}') + \hat{i}$, where \hat{i} is the number of cards in \mathcal{D}' having two isolated vertices that are adjacent in G .

First we bound $i(\mathcal{D}')$. If $e \in E(T)$ can be isolated by deleting one vertex of T , then e has endpoints in L_1 and L_2 . Such an edge is isolated in $\binom{n-3}{2}$ cards of \mathcal{D}' . Other edges of T require at least two vertex deletions to become isolated and hence can be isolated in at most $n - 4$ cards of \mathcal{D}' . Therefore,

$$i(\mathcal{D}') \leq l_2 \binom{n-3}{2} + (n-1-l_2)(n-4). \quad (2.4)$$

To bound \hat{i} , consider cards of T having isolated vertices x and y with $xy \in E(G)$. Every leaf of G is a leaf in T , but a leaf of G cannot be in such a pair $\{x, y\}$. Let $L'_1 = L_1 - L$, $L'_2 = N_T(L'_1) \cap V_2$, and $L'_3 = L'_2 \cap V_2$, with $l'_i = |L'_i|$ for $i \in \{1, 2, 3\}$. Since H has an isolated edge and G has the same degree list, G has at least two leaves, so $l'_1 \leq l_1 - 2$.

If $x, y \in L'_1$, then $xy \notin E(G)$, since otherwise the last edge on the path in T from x or y to the nearest vertex of degree at least 3 in T can be replaced with xy to obtain a spanning tree of G with fewer leaves than T . Hence \hat{i} counts no cards where the two isolated vertices adjacent in G lie in L'_1 .

In order to isolate x and y on a card in \mathcal{D}' , all of $N_T(x) \cup N_T(y)$ must be deleted. We may assume $d_T(x) \leq d_T(y)$, so now $y \notin L'_1$. Let $m(x, y) = |N_T(x) \cup N_T(y)|$. Since $d_T(y) \geq 2$, we have $m(x, y) \in \{2, 3\}$. We consider two types of cards counted by \hat{i} .

Type 1: $x \in L'_1$. Suppose first that x and y have a common neighbor w in T . If $d_T(w) \geq 3$, then $T - wy + xy$ has fewer leaves than T . Thus, $w \in L'_2$. We have $m(x, y) = 2$ with $d_T(y) = 2$ and $y \in L'_3$, or $m(x, y) = 3$ with $d_T(y) = 3$. There are at most l'_2 such cards in the second case and at most $l'_3(n-4)$ such cards in the first case. Since $l'_3 \leq l'_2 \leq l_2$, there are at most $l_2(n-3)$ cards of Type 1 with $N_T(x) \cap N_T(y) \neq \emptyset$.

On the other hand, if $N_T(x) \cap N_T(y) = \emptyset$, then $m(x, y) = 3$ and $d_T(y) = 2$. Such a pair $\{x, y\}$ is isolated in at most one card of $\mathcal{D}_{n-3}(T)$. For each $x \in L'_1$, the vertices excluded from being y include all of L_1 and the vertices within distance 2 of x . Since $x \in L'_1$, we

have $l'_1 \geq 1$, and hence $l_1 \geq l'_1 + 2 \geq 3$, so there are at most $l'_1(n-5)$ cards of Type 1 with $N_T(x) \cap N_T(y) = \emptyset$.

Type 2: $x \notin L'_1$. Since both x and y are non-leaf vertices of T and T contains no cycle, we have $m(x, y) = 3$ with $x, y \in V_2$ and $N_T(x) \cap N_T(y) \neq \emptyset$. Hence \hat{i} counts one card for each such pair $\{x, y\}$. The number of pairs of internal vertices of degree 2 with a common neighbor is at most the number of pairs of incident non-pendant edges in T , which equals $-l_1 + \sum_{v \in V(T)} \binom{d_T(v)}{2}$.

Since $\sum_{v \in V(T)} d_T(v)$ is the fixed value $2n - 2$, convexity implies that $\sum_{v \in V(T)} \binom{d_T(v)}{2}$ is maximized when the $n - l_1$ non-leaf vertices consist of one with degree l_1 and $n - l_1 - 1$ with degree 2. Hence the number of cards of this type is at most $n - 1 - 2l_1 + \binom{l_1}{2}$. When $l_1 \leq 5$, we have $\binom{l_1}{2} \leq 2l_1$ and can simplify this bound to $n - 1$. When $l_1 \geq 6$, (2.3) yields $\binom{l_1}{2} \leq \binom{l_1}{3} - 5 \leq n - 7$, and then $n - 1 - 2l_1 + \binom{l_1}{2} \leq 2n - 20 < 2n - 10$. Note that $2n - 10 \geq n - 1$ when $n \geq 9$.

Summing the bounds from Type 1 and Type 2 cards yields $\hat{i} \leq l_2(n-3) + l'_1(n-5) + 2n - 10$. Applying $i(\mathcal{D}) \leq i(\mathcal{D}') + \hat{i}$ and (2.4) and simplifying, we have

$$i(\mathcal{D}) \leq l_2 \binom{n-3}{2} + (n-1)(n-4) + l_2 + l'_1(n-5) + 2n - 10. \quad (2.5)$$

Using the upper and lower bounds on $i(\mathcal{D})$ from (2.2) and (2.5),

$$\binom{n-2}{3} \leq l_2 \binom{n-3}{2} + (n-1)(n-4) + l_2 + l'_1(n-5) + 2n - 10. \quad (2.6)$$

Using $l_2 \leq l_1$ and $l'_1 \leq l_1 - 2$ yields $l_2 + l'_1(n-5) + 2n - 10 \leq l_1(n-4)$. After substituting this into (2.6), we divide the result by $\frac{1}{3} \binom{n-3}{2}$ to obtain $n - 2 < 3l_2 + 6\frac{n-1}{n-3} + l_1\frac{6}{n-3}$. We then combine this with (2.3) to obtain

$$\binom{l_1}{3} + l_2(l_1 - 4) < 6\frac{n-1}{n-3} + l_1\frac{6}{n-3} \leq 6 + l_1$$

for $n \geq 9$. The inequality $\binom{l_1}{3} + l_2(l_1 - 4) < 6 + l_1$ fails when $l_1 \geq 6$, even if $l_2 = 0$.

Hence we may assume $l_1 \leq 5$. Using $l_2 \leq l_1 \leq 5$ and $l'_1 \leq l_1 - 2$, (2.6) becomes

$$\binom{n-2}{3} \leq 5 \binom{n-3}{2} + (n-1)(n-4) + 5 + 3(n-5) + 2n - 10 = \frac{7(n-1)(n-4)}{2}.$$

Canceling $(n-4)/6$ yields $(n-2)(n-3) \leq 21(n-1)$. This inequality fails when $n \geq 25$, so there cannot then be both a connected and a disconnected reconstruction. \square

3 Almost All Graphs

We say that a property holds for almost every graph if the fraction of graphs with vertex set $\{1, \dots, n\}$ for which it holds tends to 1 as n tends to ∞ . In order to prove that almost every graph is reconstructible, Chinn proved the following (in a stronger form):

Theorem 3.1 (Chinn [8]). *If the subgraphs of a graph G obtained by deleting two vertices are pairwise nonisomorphic, then G is reconstructible.*

Under this hypothesis, for any $u, v, w \in V(G)$, vertex u is identifiable in $G - w$ because it is the only vertex in $G - v$ whose deletion yields a subgraph obtainable from $G - u$ by deleting one vertex. After similarly identifying v in $G - w$, one can check whether u and v are adjacent there. Bollobás [2] strengthened conclusion from Chinn’s hypothesis.

Theorem 3.2 (Bollobás [2]). *For almost every graph, any three cards determine G .*

This conclusion is stronger than saying that *some* three cards determine G , which is the meaning of reconstruction number 3 (note that two cards cannot determine whether the two deleted vertices are adjacent). Bollobás [2] used a lemma also proved earlier by Müller [21].

Lemma 3.3 (Müller [21]). *Let ϵ be a small positive real number. For almost every graph G , the induced subgraphs with at least $(1 + \epsilon)\frac{|V(G)|}{2}$ vertices have no nontrivial automorphisms and are pairwise nonisomorphic.*

In fact, this was proved in greater generality for the random graph model $\mathcal{G}(n, p)$ where the graphs with vertex set n are generated by letting each pair be an edge with probability p , where p can depend on n , but $p = 1/2$ (used here) suffices for our purposes. Our result is valid for any $p(n)$ where the conclusion of Lemma 3.3 holds, since we use only that conclusion.

Lemma 3.3 suggests considering subgraphs obtained by deleting more vertices. Letting $n = |V(G)|$, the ℓ -reconstruction number of G is the minimum size of a multiset from $\mathcal{D}_{n-\ell}(G)$ that occurs in the $(n - \ell)$ -deck of no other n -vertex graph. We generalize the consequence of Theorem 3.2 for reconstruction number by bounding the ℓ -reconstruction number for almost every graph. The special case $\ell = 1$ is the statement that almost every graph has reconstruction number 3.

Theorem 3.4. *For $\epsilon > 0$, if $\ell \leq (1 - \epsilon)\frac{|V(G)|}{2}$ as a function of $|V(G)|$, then almost every graph has ℓ -reconstruction number at most $\binom{\ell+2}{2}$.*

Given Lemma 3.3, Theorem 3.4 follows from the next result.

Theorem 3.5. *Given $n \geq \ell + 4 \geq 6$, if the subgraphs obtained by deleting $\ell + 1$ vertices from an n -vertex graph G have no nontrivial automorphisms and are pairwise nonisomorphic, then G is reconstructible from some subset of $\mathcal{D}_{n-\ell}(G)$ having size $\binom{\ell+2}{2}$.*

Proof. Fix $S = \{x_1, \dots, x_{\ell+1}\} \subseteq V(G)$. Let $H = G - S$, and let $h = |V(H)| = n - \ell - 1$. Let C_i be the card $G - (S - \{x_i\})$ in $\mathcal{D}_{n-\ell}(G)$; note that $H = C_i - x_i$.

For $x_i, x_j \in S$, let $D_{i,j} = G - (S - \{x_i, x_j\}) - w_{i,j}$, where $w_{i,j}$ is a vertex in $V(H)$ (note that $V(H) \cap S = \emptyset$). The subscripts can be read in either order, so $D_{i,j} = D_{j,i}$ and $w_{i,j} = w_{j,i}$.

The deleted vertices are chosen so that for fixed i , the vertex $w_{i,j}$ is not the same for all j , which is possible since $\ell \geq 2$ and $h \geq 3$. Let $D = \{D_{i,j}: x_i, x_j \in S\}$ and $C = \{C_i: x_i \in S\}$.

Each graph in $C \cup D$ has $n - \ell$ vertices, and $|C \cup D| = \binom{\ell+2}{2}$. We prove that G is reconstructible from $C \cup D$. In this discussion, “subgraph” means “induced subgraph”

The h -vertex subgraphs of G are nonisomorphic. Thus G has exactly one subgraph isomorphic to H . We first retrieve H from $C \cup D$ by showing that H is the only h -vertex subgraph appearing in $l + 1$ cards in $C \cup D$; it appears in all $l + 1$ cards of C . Let H' be an h -vertex subgraph of G . If $|V(H') \cap S| \geq 3$, then H' appears in no card in $C \cup D$. If $V(H') \cap S = \{x_i, x_j\}$, then H' appears only in the card $D_{i,j}$. If $V(H') \cap S = \emptyset$, then $H' = H$.

If $V(H') \cap S = \{x_i\}$, then H' appears in C_i and possibly in cards of the form $D_{i,j}$ as $D_{i,j} - x_j$. Since $D_{i,j} - x_j$ is the subgraph of G induced by $V(H) \cup \{x_i\} - w_{i,j}$ and the subgraphs with $h - 1$ vertices are non-isomorphic, the ℓ graphs of the form $D_{i,j} - x_j$ cannot all equal H' when the vertices of the form $w_{i,j}$ are not the same for fixed i and all j . Hence in this case H' appears in at most ℓ cards in $C \cup D$.

Thus H is the unique h -vertex subgraph that appears in $\ell + 1$ cards in $C \cup D$. This identifies H and C from $C \cup D$.

Having identified H and the $\ell + 1$ cards forming C , we also know x_i in C_i , and we know the names of all the vertices of H in C_i since H has no nontrivial automorphisms. Hence C_i also gives us the neighbors of x_i in $G - S$. It remains only to determine the edges within S .

We use $D_{i,j}$ to check whether x_i and x_j are adjacent. For $w \in V(H)$, a card $D' \in D$ contains both $C_i - w$ and $C_j - w$ (which have $h - 1$ vertices) if and only if $D' = D_{i,j}$ and $w = w_{i,j}$. Since the subgraphs with $h - 1$ vertices are distinct, this identifies $D_{i,j}$, and we can then use $C_i - w$ and $C_j - w$ to determine x_i and x_j within $D_{i,j}$ and check whether they are adjacent. \square

As in the result by Bollobás [2], many sets of $\binom{\ell+2}{2}$ cards in $\mathcal{D}_{n-\ell}(G)$ suffice when the hypothesis of Theorem 3.5 holds. The set S can be any $\ell + 1$ vertices of G , and $D_{i,j} \in D$ can be obtained by choosing any $w_{i,j} \in V(G) - S$, as long as we do not choose the same $w_{i,j}$ for all j . We allow all h choices of $w_{i,j}$ for $1 \leq i < j \leq \ell$. We choose $w_{\ell,\ell+1}$ to ensure that the vertices of the form $w_{i,j}$ are not the same for all j ; hence there may only be $h - 1$ choices when $i + 1 < j = \ell + 1$, and perhaps only $h - 2$ for $w_{\ell,\ell+1}$. For $\ell > 1$, the resulting lower bound on the number of sets of $\binom{\ell+2}{2}$ cards that determine G is asymptotic to $\binom{n}{\ell+1} h^{\binom{\ell+1}{2}}$, where $h = n - \ell - 1$.

4 Graphs with Maximum Degree 2

From Manvel’s theorem (Theorem 2.5), when $k \geq 4$ we can recognize from the k -deck whether a graph has maximum degree 2. However, we will show that much larger cards are needed to guarantee determining whether a graph with maximum degree 2 is connected. In Problem

11898 of the American Mathematical Monthly, Richard Stanley posed a question related to reconstructing 2-regular graphs from their k -decks.

Problem 4.1 (Stanley [30]). *Let n and k be integers, with $n \geq k \geq 2$. Let G be a graph with n vertices whose components are cycles of length greater than k . Let $i_k(G)$ be the number of k -element independent sets of vertices of G . Show that $i_k(G)$ depends only on k and n .*

Let $s(G, H)$ denote the number of induced subgraphs of G isomorphic to H . Stanley's problem asserts $s(G, \overline{K}_k) = s(G', \overline{K}_k)$ for n -vertex 2-regular graphs G and G' whose components have length greater than k (here \overline{H} denotes the complement of H). Stanley's proposed solution of Problem 4.1 used generating functions. Our proof and its generalization for Theorem 4.2 are bijective.

Problem 4.1 considers only subgraphs with no edges. We prove the same conclusion for all subgraphs with k vertices. That is, n -vertex 2-regular graphs whose components have more than k vertices all have the same k -deck. Our technique of proof further generalizes to graphs with maximum degree 2.

Theorem 4.2. *Let G and G' be graphs with maximum degree 2 having the same number of vertices and the same number of edges. If every component in each graph is a cycle with at least $k + 1$ vertices or a path with at least $k - 1$ vertices, then $\mathcal{D}_k(G) = \mathcal{D}_k(G')$.*

Before we prove Theorem 4.2, some discussion is in order. First, the most important aspect of the theorem and the essence of the proof is what the theorem says for graphs with one or two components. In particular,

- (4.1) $\mathcal{D}_k(C_{q+r}) = \mathcal{D}_k(C_q + C_r)$ if $q, r \geq k + 1$,
- (4.2) $\mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$, and
- (4.3) $\mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1})$ if $q, r \geq k$.

In this section, the focus of discussion is on the number of vertices in the cards rather than on the number of vertices deleted. Hence it is natural to let $\rho'(G)$ be the least k such that $\mathcal{D}_k(G)$ determines G . Note that $\rho'(G) = |V(G)| - \rho(G)$. Statements (4.1) and (4.2) above now yield the following result.

Corollary 4.3. *For $n \geq 3$, the least k such that connectedness of an n -vertex graph G can always be determined from its k -deck is at least $\lfloor n/2 \rfloor + 1$ (even when restricted to $\Delta(G) = 2$). Furthermore, $\rho'(P_n) = \lfloor n/2 \rfloor + 1$ and $\rho'(C_n) = \lfloor n/2 \rfloor$ when $n \geq 6$.*

Proof. By (4.2), $\mathcal{D}_k(P_n) = \mathcal{D}_k(C_{\lfloor n/2 \rfloor + 1} + P_{\lfloor n/2 \rfloor - 1})$ when $k \leq \lfloor n/2 \rfloor$. This proves the claim about connectedness and also $\rho'(P_n) \geq \lfloor n/2 \rfloor + 1$.

Consider $\mathcal{D}_k(P_n)$ with $k = \lfloor n/2 \rfloor + 1$. If $n \geq 6$, then $k \geq 4$, and by Theorem 2.5 we can reconstruct the degree list. The components of any reconstruction G are cycles except for one path. Since the k -deck has no cycle, the length of any cycle must exceed k . Since $n \leq 2k - 1$,

there can only be one cycle, and the path component then has at most $k - 2$ vertices. In $\mathcal{D}_k(P_n)$, there are $n - k + 1$ copies of P_k . However, when $l < k - 1$, in $\mathcal{D}_k(P_l + C_{n-l})$ there are $n - l$ copies of P_k , which is larger than in P_n . Hence the deck differs from $\mathcal{D}_k(P_n)$ unless $G = P_n$. This proves $\rho'(P_n) = \lfloor n/2 \rfloor + 1$ for $n \geq 6$.

By (4.1), $\mathcal{D}_k(C_n) = \mathcal{D}_k(C_{\lfloor n/2 \rfloor} + C_{\lfloor n/2 \rfloor})$ when $k < \lfloor n/2 \rfloor$, so $\rho'(C_n) \geq \lfloor n/2 \rfloor$. Suppose $k = \lfloor n/2 \rfloor$. If $n \geq 8$, then $\lfloor n/2 \rfloor \geq 4$, and by Theorem 2.5 we can reconstruct the degree list from $\mathcal{D}_k(C_n)$. Any 2-regular graph other than C_n has a cycle of length at most $\lfloor n/2 \rfloor$, and this can be seen in the $\lfloor n/2 \rfloor$ -deck.

For $n \in \{6, 7\}$, let G be a reconstruction from $\mathcal{D}_3(C_n)$. We know $|E(G)|$ from the 2-deck, and we know the number of incidences (edges in the line graph) from the 3-deck. This yields $\sum_{v \in V(G)} \binom{d(v)}{2} = n = \sum_{v \in V(G)} \frac{d(v)}{2}$. Now it is a standard exercise by convexity of $\binom{x}{2}$ that G is 2-regular. With $n \leq 7$, a cycle appears in the 3-deck if $G \neq C_n$. Thus $\rho'(C_n) = \lfloor n/2 \rfloor$. \square

As remarked after Theorem 2.1, the unique 5-vertex tree that is not a path or a star has the same 3-deck as the disconnected graph $C_4 + P_1$. There are also three pairs of 7-vertex graphs that have the same 4-deck, but all six graphs are connected. This suggests the question that motivates Conjecture 2.2.

Question 4.4. For $n \in \mathbb{N}$, what is the least k such that for every n -vertex graph G , it can be determined from $\mathcal{D}_k(G)$ whether G is connected? Does $\lfloor n/2 \rfloor + 1$ suffice when $n \geq 6$?

Sections 4.1 and 4.2 are devoted to the proof of Theorem 4.2, whose facts (4.1)–(4.3) yield lower bounds on $\rho'(G)$ whenever $\Delta(G) = 2$. In Section 4.3 we prove that these lower bounds are always sharp, showing how to reconstruct G from its $\rho'(G)$ -deck in all cases.

A useful technical lemma implies that when two graphs have the same k -deck, taking the disjoint union of either with a third graph again yields two graphs with the same k -deck. This will allow us to change one or two components of a graph while keeping the rest of the graph unchanged. Note that $G[X]$ denotes the subgraph of G induced by a vertex subset X .

Lemma 4.5. *If G , G' , and H are graphs, then $\mathcal{D}_k(G) = \mathcal{D}_k(G')$ if and only if $\mathcal{D}_k(G + H) = \mathcal{D}_k(G' + H)$.*

Proof. Necessity. For a graph F , let $\mathcal{S}_k(F)$ denote the set of labeled induced subgraphs with at most k vertices. Given $\mathcal{D}_k(G) = \mathcal{D}_k(G')$, there is a bijection $g: \mathcal{S}_k(G) \rightarrow \mathcal{S}_k(G')$ that pairs isomorphic subgraphs of G and G' . From g we define $h: \mathcal{S}_k(G + H) \rightarrow \mathcal{S}_k(G' + H)$. For a set $U \subseteq V(G) \cup V(H)$ with $|U| \leq k$, let $X = U \cap V(G)$ and $Y = U \cap V(H)$. Since $|X|, |Y| \leq k$, we can set $h(U) = g(G[X]) + H[Y]$. Since g is a bijection, h is a bijection, and $G[X] + H[Y] \cong g(G[X]) + H[Y]$. Thus $\mathcal{D}_k(G + H) = \mathcal{D}_k(G' + H)$.

Sufficiency. Assume $\mathcal{D}_k(G + H) = \mathcal{D}_k(G' + H)$. By Observation 1.4, we also have $\mathcal{D}_j(G + H) = \mathcal{D}_j(G' + H)$ for $j \leq k$. Let X be a graph with k vertices and r components. We claim $s(G, X) = s(G', X)$, by induction on $k + r$. If $r = 1$, then $s(G, X) =$

$s(G + H, X) - s(H, X) = s(G' + H, X) - s(H, X) = s(G', X)$. Let $[r] = \{1, \dots, r\}$. For $r > 1$, let X_1, \dots, X_r be the components of X . For $T \subseteq [r]$, let X_T denote the disjoint union of $\{X_i : i \in T\}$, and let $\bar{T} = [r] - T$. Using the induction hypothesis, we compute

$$\begin{aligned} s(G, X) &= s(G + H, X) - \sum_{\emptyset \neq T \subseteq [r]} s(H, X_T) s(G, X_{\bar{T}}) \\ &= s(G' + H, X) - \sum_{\emptyset \neq T \subseteq [r]} s(H, X_T) s(G', X_{\bar{T}}) = s(G', X). \end{aligned}$$

Thus $\mathcal{D}_k(G) = \mathcal{D}_k(G')$. □

We will use this lemma in both directions. In one direction, it tells us that any lower bound on $\rho'(G)$ is a lower bound on $\rho'(G + H)$ and that (4.1)–(4.3) imply Theorem 4.2. In the other, it tells us that when two graphs with the same k -deck have a common component, deleting the shared component leaves two smaller graphs with the same k -deck.

4.1 Common k -Decks for Linear Forests

For graphs with maximum degree 2 in which every cycle has length larger than k , every k -card is a linear forest, meaning a disjoint union of paths. We will first prove the equal-deck result (4.3) for pairs of paths (Theorem 4.8); we then use this to prove (4.2) (Lemma 4.10) and (4.1) (Theorem 4.12).

Definition 4.6. For distinct positive integers ℓ_1, \dots, ℓ_p , let m denote m_1, \dots, m_p , and let L^m denote the linear forest $\sum m_i P_{\ell_i}$ having m_i components isomorphic to P_{ℓ_i} for $1 \leq i \leq p$. Let L_i^m denote the linear forest obtained from L^m by deleting a component isomorphic to P_{ℓ_i} , and let $L_{i,j}^m$ denote the result of deleting components isomorphic to P_{ℓ_i} and P_{ℓ_j} (we allow $i = j$ when $m_i \geq 2$). Again $s(G, H)$ is the number of induced subgraphs of G isomorphic to H . For $w \in V(G)$, let $s'(G, H, w)$ be the number of induced subgraphs of G isomorphic to H in which w is used as an isolated vertex in H .

Lemma 4.7. *Let L^m be a linear forest with k vertices, and let the vertices of P_n be w_1, \dots, w_n in order. For all h such that $k \leq h \leq n - k + 1$, the quantity $s'(P_n, L^m, w_h)$ has the same value, written $s'(P_n, L^m)$.*

Proof. We use induction on k . When $k = 1$, there is exactly one copy of P_1 containing any specified vertex. Note that $s'(P_n, L^m, w_h) = 0$ unless L^m has an isolated vertex. Hence we may assume $k \geq 2$ and that L^m has at least two components.

We compare $s'(P_n, L^m, w_h)$ with $s'(C_n, L^m, w_h)$, where C_n is obtained by adding the edge $w_n w_1$. The quantity $s'(C_n, L^m, w_h)$ omits copies of L^m in P_n in which some path starts with w_1 and another ends with w_n ; let there be A such copies. On the other hand, $s'(C_n, L^m, w_h)$

counts unwanted subgraphs using the edge $w_n w_1$; let there be B of these. Before computing A and B , note that symmetry yields $s'(C_n, L^m, w_h) = s'(C_n, L^m)$, independent of h , so

$$s'(P_n, L^m, w_h) = s'(C_n, L^m) + A - B,$$

Note that w_h is far enough from the ends of P_n that w_h will not be touched by P_{ℓ_i} containing w_1 or P_{ℓ_j} containing w_n plus one more vertex at each end to separate components of L^m . Thus $A = \sum_{i,j} s'(P_{n-(\ell_i+\ell_j+2)}, L_{i,j}^m, w_h)$. The sum allows $i = j$ when $m_i \geq 2$, and the set $\{i, j\}$ yields two terms when $i \neq j$.

To compute B , note that in C_n the edge $w_n w_1$ may occupy any of $\ell_i - 1$ positions within a copy of P_{ℓ_i} , and w_h belongs to no such copy, even when augmented by an extra vertex at each end. Thus $B = \sum_i (\ell_i - 1) s'(P_{n-(\ell_i+2)}, L_i^m, w_h)$. Implicitly this is a double-sum, where the inner sum is over the $\ell_i - 1$ positions of the path containing $w_n w_1$; it will follow inductively that the summand in the inner sum is constant.

To obtain independence of h for the terms in A and B , we check the conditions in the statement of the induction hypothesis. For terms in A , the values n and k are replaced by n' and k' , where $n' = n - (\ell_i + \ell_j + 2)$ and $k' = k - (\ell_i + \ell_j)$. Since P_{ℓ_i} contains w_1 , deleting P_{ℓ_i} and the neighboring vertex from the beginning of P_n (and deleting P_{ℓ_j} and the neighboring vertex from the end) leaves the vertex w_h with a new index h' in the shorter path $P_{n'}$; we obtain $h' = h - \ell_i - 1$. Since $h \geq k$ and $\ell_j \geq 1$, we have $h' = h - \ell_i - 1 \geq k - (\ell_i + \ell_j) = k'$. Similarly, since $h \leq n - k + 1$ and $\ell_i \geq 1$,

$$h' = h - \ell_i - 1 \leq n - k - 1 = n - (\ell_i + \ell_j + 2) - (k - (\ell_i + \ell_j)) + 1 = n' - k' + 1.$$

For terms involving ℓ_i in B , the values n and k are replaced by n' and k' , where $n' = n - (\ell_i + 2)$ and $k' = k - \ell_i$. For the new index h' of w_h , we lose at least two and at most ℓ_i vertices from each end of the path. Thus $h' \geq h - \ell_i \geq k - \ell_i = k'$ and

$$h' \leq h - 2 \leq n - k - 1 = (n - \ell_i - 2) - (k - \ell_i) + 1 = n' - k' + 1.$$

By the induction hypothesis, in all cases the new index is within the range needed to make the contributions independent of the index. Hence $s'(P_n, L^m, w_h)$ is independent of h in the specified range. \square

Lemma 4.7 enables us to prove the special case of Theorem 4.2 for pairs of paths.

Theorem 4.8. *If L^m is any linear forest with k vertices, and $q, r \geq k$, then $s(P_{q-1} + P_r, L^m) = s(P_q + P_{r-1}, L^m)$. Hence also $\mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1})$.*

Proof. Consider P_{q+r+2} with $V(P_{q+r+2}) = \{w_1, \dots, w_{q+r+2}\}$. Deleting $\{w_q, w_{q+1}, w_{q+2}\}$ yields $P_{q-1} + P_r$, while deleting $\{w_{q+1}, w_{q+2}, w_{q+3}\}$ yields $P_q + P_{r-1}$. Thus $s(P_{q-1} + P_r, L^m) =$

$s'(P_{q+r+2}, L^m + P_1, w_{q+1})$, while $s(P_q + P_{r-1}, L^m) = s'(P_{q+r+2}, L^m + P_1, w_{q+2})$. Letting $n = |V(P_{q+r+2})| = q + r + 2$, from $q, r \geq k$ we conclude

$$|V(L^m + P_1)| = k + 1 \leq q + 1 < q + 2 = n - r \leq n - k = n - |V(L^m + P_1)| + 1.$$

Now $s'(P_{q+r+2}, L^m + P_1, w_{q+1}) = s'(P_{q+r+2}, L^m + P_1, w_{q+2})$ by Lemma 4.7, which suffices. \square

Corollary 4.9. *If G and G' are linear forests with the same number of vertices and the same number of edges whose components have at least $k - 1$ vertices, then $\mathcal{D}_k(G) = \mathcal{D}_k(G')$.*

Proof. Fixing the numbers of vertices and edges fixes the number of components. By keeping all but two components fixed and applying Lemma 4.5, Theorem 4.8 allows iteratively shifting a vertex from one component to another to turn G into G' without changing the k -deck. \square

4.2 Common k -Decks for Maximum Degree 2

In order to extend the results to allow cycles, note that deleting any vertex of a cycle leaves the same path. Again the problem reduces to working with just two components.

Lemma 4.10. *Let L^m be a linear forest with k vertices. If $q \geq k + 1$ and $r \geq k - 1$, then $s(P_{q+r}, L^m) = s(C_q + P_r, L^m)$. Hence also $\mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r)$.*

Proof. Let u_1, \dots, u_{q+r} be the vertices of P_{q+r} in order. Consider an induced copy of L^m . Either u_q is not used, or it appears in a path of some length ℓ_i . In the latter case let t be the number of vertices starting with u_q that lie in the copy of P_{ℓ_i} . Since $\ell_i \leq k$, the hypotheses on q and r allow t to run from 1 to ℓ_i . In the notation of Definition 4.6, these possibilities yield

$$s(P_{q+r}, L^m) = s(P_{q-1} + P_r, L^m) + \sum_{i=1}^p \sum_{t=1}^{\ell_i} s(P_{q-(\ell_i-t)-2} + P_{r-t}, L_i^m).$$

Now consider a vertex x on C_q in $C_q + P_r$. By symmetry, the choice of x does not matter. As above, in a copy of L^m the vertex x may be omitted or appear in a copy of P_{ℓ_i} for some i . The position of x in its copy of P_{ℓ_i} also does not matter, since deleting $V(P_{\ell_i})$ and two additional unused vertices always leaves $P_{q-\ell_i-2}$. Thus

$$s(C_q + P_r, L^m) = s(P_{q-1} + P_r, L^m) + \sum_{i=1}^p \ell_i s(P_{q-\ell_i-2} + P_r, L_i^m).$$

It suffices to prove that the right sides of these two equations are equal. The first term is identical. It remains to show $s(P_{q-\ell_i-2} + P_r, L_i^m) = s(P_{q-(\ell_i-t)-2} + P_{r-t}, L_i^m)$ for $1 \leq i \leq p$ and $1 \leq t \leq \ell_i$. In each case, adding three vertices connects the two host paths to form P_n with $n = q + r + 1 - \ell_i$, and the specified values equal $s'(P_n, L_i^m + P_1, w_h)$ for appropriate

h , namely $h = q - \ell_i$ or $h = q - \ell_i + t$. Let $k' = |V(L_i^m + P_1)| = k - \ell_i + 1$. By Lemma 4.7, $s'(P_n, L_i^m + P_1, w_h)$ does not depend on h when $k' \leq h \leq n - k' + 1$, which is equivalent to $k - \ell_i + 1 \leq h \leq q + r - k + 1$. In the cases of interest, the lowest value taken by h is $q - \ell_i$, and the highest is q (when $t = \ell_i$). Since $q \geq k + 1$ and $r \geq k - 1$, the desired inequalities hold (and we cannot weaken the hypotheses). \square

Corollary 4.11. *Let G and G' be non-regular graphs with maximum degree 2, equal numbers of vertices, and equal numbers of edges. If all cycles in G and G' have at least $k + 1$ vertices and all path components have at least $k - 1$ vertices, then $\mathcal{D}_k(G) = \mathcal{D}_k(G')$.*

Proof. Since G and G' are not regular, each has at least one path component. Using Lemma 4.10 to absorb cycles into paths, each has the same k -deck as some linear forest with at least $k - 1$ vertices in each component. By Corollary 4.9, the resulting linear forests H and H' have the same k -deck. \square

Our original motivation for Theorem 4.2 was 2-regular graphs. The earlier cases now simplify the proof of this case.

Theorem 4.12. *If L^m is a linear forest with k vertices, and $q, r \geq k + 1$, then $s(C_{q+r}, L^m) = s(C_q + C_r, L^m)$, and hence $\mathcal{D}_k(C_{q+r}) = \mathcal{D}_k(C_q + C_r)$.*

Proof. Choose $x \in V(C_{q+r})$ and $y \in V(C_r)$. We consider the usage of x and y in induced copies of L^m . The specified vertex may be omitted or may occur in a copy of some path P_{ℓ_i} . In the latter case, it may occur with any position in P_{ℓ_i} , but the resulting number of subgraphs is the same for each position, since deleting any ℓ_i -vertex path from a cycle leaves a path of the same length. We thus have the following two expansions.

$$s(C_{q+r}, L^m) = s(P_{q+r-1}, L^m) + \sum_{i=1}^p \ell_i s(P_{q+r-\ell_i-2}, L_i^m)$$

$$s(C_q + C_r, L^m) = s(C_q + P_{r-1}, L^m) + \sum_{i=1}^p \ell_i s(C_q + P_{r-\ell_i-2}, L_i^m)$$

It suffices to use Lemma 4.10 to show that corresponding terms on the right are equal. Equality of the first terms follows from $q \geq k + 1$ and $r - 1 \geq k - 1$, which hold by assumption. For the other case it suffices to have $q \geq k - \ell_i + 1$ and $r - \ell_i - 2 \geq k - \ell_i - 1$. The first inequality holds since $q \geq k + 1$. The second simplifies to $r \geq k + 1$, which holds by assumption. \square

Corollary 4.13. *Any two n -vertex graphs whose components all are cycles with at least $k + 1$ vertices have identical k -decks.*

Proof. Repeated application of Lemma 4.5 and Theorem 4.12 shows that any two such graphs have the same k -deck as C_n . \square

Corollaries 4.11 and 4.13 together complete the proof of Theorem 4.2.

4.3 $\rho'(G)$ for Graphs with Maximum Degree 2

We first reduce k -deck reconstruction to the problem of finding all components with more than k vertices.

Lemma 4.14. *If all the components with more than k vertices in a graph G can be determined from $\mathcal{D}_k(G)$, then G is k -deck reconstructible.*

Proof. It suffices to show that all components of order k can be determined, since we already observed that $\mathcal{D}_k(G)$ determines $\mathcal{D}_{k-1}(G)$. We then iterate to find all smaller components.

Let H_1, \dots, H_r be the components of G with more than k vertices. Let F be a component with exactly k vertices. The number of components of G isomorphic to F is obtained by subtracting $\sum_{i=1}^r s(H_i, F)$ from the number of cards in $\mathcal{D}_k(G)$ isomorphic to F . \square

Definition 4.15. Given a graph G , let $t_k(G)$ denote the number of components of G that are paths with at least $k - 1$ vertices.

We show first that $t_k(G)$ is reconstructible from $\mathcal{D}_k(G)$ when $\Delta(G) = 2$. We then show that when $t_k(G)$ is small and $\Delta(G) = 2$ we can reconstruct the components of G with more than k vertices; hence G is k -deck reconstructible.

Lemma 4.16. *If $\Delta(G) = 2$, then the quantities $s(G, P_k)$, $s(G, C_k)$, $s(G, P_{k-1})$, and $t_k(G)$ are determined by $\mathcal{D}_k(G)$.*

Proof. The quantities $s(G, P_k)$ and $s(G, C_k)$ just count cards that are paths or cycles, respectively. With $n = |V(G)|$, each induced $(k - 1)$ -vertex path in G occurs in exactly $n - k + 1$ cards, by adding one vertex. Thus $s(G, P_{k-1}) = \frac{\sum_{Q \in \mathcal{D}_k(G)} s(Q, P_{k-1})}{n - k + 1}$.

To determine $t_k(G)$, we show $t_k(G) = s(G, P_{k-1}) - s(G, P_k) - ks(G, C_k)$. Each path component with at least $k - 1$ vertices contributes exactly 1 to $s(G, P_{k-1}) - s(G, P_k)$. Each m -cycle with $m > k$ contributes m to both $s(G, P_{k-1})$ and $s(G, P_k)$. Each k -cycle contributes k to both $s(G, P_{k-1})$ and $ks(G, C_k)$. No smaller component contributes to either side. Hence each component contributes the same amount to each side. \square

We will see that $t_k(G) \geq 2$ requires k being smaller than the threshold we prove suffices for k -deck reconstructibility. When $t_k(G) < 2$, we will apply the following lemma.

Lemma 4.17. *Let G be a graph with maximum degree 2. If $t_k(G) = 0$ and $0 < s(G, P_k) \leq 2k + 1$, then G has exactly one component with more than k vertices, and it is a cycle with $s(G, P_k)$ vertices. If $t_k(G) = 1$ and $0 \leq s(G, P_k) \leq k$, then G has no cycle with more than k vertices, and its one path component with at least $k - 1$ vertices has $s(G, P_k) + k - 1$ vertices.*

Proof. If $t_k(G) = 0$, then every component with more than k vertices is a cycle, each contributing at least $k + 1$ toward $s(G, P_k)$. With $s(G, P_k) \leq 2k + 1$, there is at most one such cycle. With $s(G, P_k) > 0$, there is at least one.

If $t_k(G) = 1$ and $s(G, P_k) \leq k$, then no component with more than k vertices is a cycle. Hence all copies of P_k come from the unique component that is a path with at least $k - 1$ vertices. The result now follows from $s(P_m, P_k) = m - k + 1$ for $m \geq k - 1$. \square

In order to use the lemmas above to prove the upper bounds, we need to determine from $\mathcal{D}_k(G)$ that G has maximum degree 2. When $k \geq 4$, this follows from Manvel's result (Theorem 2.5), but we will need it also sometimes when $k = 3$. We will consider the graphs in a particular family.

Definition 4.18. Let \mathcal{F} denote the family of graphs with maximum degree 2 whose components include at least one of the following: two components forming $H + P_1$ where $|V(H)| \geq 5$, or two forming $C_4 + P_1$, or three forming $P_4 + C_3 + P_1$, or eight forming $4C_3 + 4P_1$.

Lemma 4.19. *Let G be a graph with maximum degree 2. If G lies in the family \mathcal{F} of Definition 4.18, then there is an alternative reconstruction from $\mathcal{D}_3(G)$ that has maximum degree 3. However, every reconstruction from $\mathcal{D}_3(G)$ has maximum degree 2 in the following cases: G has no isolated vertices, $G = P_4 + aP_1$ with $a > 0$, or $G = aP_3 + bC_3 + cP_2 + dP_1$ with $\min\{b, d\} \leq 3$ and $a \leq 1$.*

Proof. We first exhibit the alternative reconstructions for $G \in \mathcal{F}$. Let Y_r be any tree with r vertices and three leaves. Note that $\Delta(Y_r) = 3$.

For $m \geq 4$, the graph $C_m + P_1$ has the same 3-deck as Y_{m+1} . The 3-deck has no triangles, m copies of P_3 , and $m(m - 3)$ copies of $P_2 + P_1$, with the other cards being $3P_1$.

For $m \geq 5$, the graph $P_m + P_1$ has the same 3-deck as $Y_{m-1} + P_2$. The 3-deck has no triangles, $m - 2$ copies of P_3 , and $(m - 2)^2 + 1$ copies of $P_2 + P_1$; the other cards are $3P_1$.

In addition, $\mathcal{D}_3(P_4 + C_3 + P_1) = \mathcal{D}_3(K_{1,3}^+ + 2P_2)$, where $K_{1,3}^+$ arises from $K_{1,3}$ by adding one edge (the 3-deck has one triangle, two copies of P_3 , and 29 of $P_2 + P_1$). Also, $\mathcal{D}_3(4C_3 + 4P_1) = \mathcal{D}_3(K_4 + 6P_2)$ (the 3-deck has four triangles, no P_3 , and 156 copies of $P_2 + P_1$).

For graphs in \mathcal{F} with additional components, Lemma 4.5 applies.

Now suppose $G \notin \mathcal{F}$, with G having n vertices and m edges and $\Delta(G) = 2$. Given $\mathcal{D}_3(G)$, we know n from $\mathcal{D}_1(G)$ and m from $\mathcal{D}_2(G)$. Also, $\mathcal{D}_3(G)$ tells us the number I of incidences between edges, which equals $\sum_{v \in V(H)} \binom{d_H(v)}{2}$ for any reconstruction H . With fixed degree-sum $2m$ satisfying $n \leq 2m \leq 2n$, the value I is minimized and equals $2m - n$ if and only if all degrees lie in $\{1, 2\}$. Thus if $I = 2m - n$, then we know the maximum degree (and degree list). Hence when G has no isolated vertices, we reconstruct $\Delta(G)$.

When $G = P_4 + aP_1$, every reconstruction H from $\mathcal{D}_3(G)$ has three edges. Thus the non-isolated vertices of H induce P_4 , $K_{1,3}$, C_3 , $P_3 + P_2$, or $3P_2$. Among these, only G has exactly two copies of P_3 in its 3-deck.

It remains to consider $G = aP_3 + bC_3 + cP_2 + dP_1$ with $a \leq 1$. If $a = 1$, then $\mathcal{D}_3(G)$ has exactly one copy of P_3 . This copy lies in one component of a reconstruction H , and the only connected graph with exactly one copy of P_3 in its 3-deck is P_3 . Hence H has P_3 as one component. By Lemma 4.5, we therefore need only consider $G = bC_3 + cP_2 + dP_1$. Now let H be an alternative reconstruction from the 3-deck of a minimal such graph G having an alternative reconstruction. By Lemma 4.5, G and H have no common components.

Since P_3 is not a 3-card, H is a disjoint union of complete graphs. When $b > 0$, the graph C_3 is not a component of H . Hence b counts the triangles in the components of H with more than three vertices. In G , we have three edges per triangle. In H the components generating triangles have fewer than three edges per triangle. Hence H has isolated edges, and G does not. A copy of K_m in H with $m > 3$ uses $\binom{m}{2}$ edges to generate $\binom{m}{3}$ triangles, which account for $3\binom{m}{3}$ edges in G . Since G and H have the same number of edges, H has $3\binom{m}{3} - \binom{m}{2}$ isolated edges associated with component isomorphic to K_m . Associated with each such component in H , we thus have $m + 6\binom{m}{3} - 2\binom{m}{2}$ vertices in H and $3\binom{m}{3}$ vertices in G . Since G and H have the same number of vertices, this requires at least $3\binom{m}{3} - m(m-2)$ isolated vertices in G . If $\Delta(H) \neq 2$, then H has a component with $m \geq 4$, which requires that G has at least four isolated vertices and at least four components that are triangles, so $\min\{b, d\} \geq 4$.

Finally, if $G = cP_2 + dP_1$, then G is reconstructible from $\mathcal{D}_3(G)$ because $\mathcal{D}_1(G)$ determines $|V(G)|$, $\mathcal{D}_2(G)$ determines $|E(G)|$, and $\mathcal{D}_3(G)$ tells us that no two edges are incident. \square

These exceptions in Lemma 4.19 yield exceptions to the general formula we now define.

Definition 4.20. By “largest component”, we mean a component with the largest number of vertices. Given a graph G with n vertices and maximum degree at most 2, let m and m' be the numbers of vertices in two largest components of G , with $m \geq m'$ (possibly $m' = 0$). Let $\epsilon = 1$ if P_m is a component; otherwise $\epsilon = 0$. Let $\epsilon' = 2$ if $m' < m - 1$ and $P_{m'}$ is a component, including the degenerate case where $m' = 0$ and G is connected. Let $\epsilon' = 1$ if $m' = m - 1$ and $P_{m'}$ is a component, if $m' < m$ and $P_{m'-1}$ but not $P_{m'}$ is a component, or if $m' = m$ and at least two components equal P_m . Otherwise, let $\epsilon' = 0$. Define

$$k_G = \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}. \quad (*)$$

Theorem 4.21. For $n \geq 2$, let G be a graph with n vertices and maximum degree at most 2. Using notation $m, m', \epsilon, \epsilon', k_G, \mathcal{F}$ as in Definitions 4.20–4.18, always $\rho'(G) = k_G$, except that $\rho'(G) = 4$ when $k_G = 3$ and $G \in \mathcal{F}$, and $\rho'(C_4) = \rho'(C_5) = 3$.

Proof. When $n \geq 2$, the graphs with maximum degree at most 2 that are determined by their numbers of vertices and edges are nK_1 , $K_2 + (n-2)K_1$, P_3 , and C_3 . For these graphs, k_G and $\rho'(G)$ both equal 2. Hence we may assume that $n \geq 4$, that G has at least two edges, and that $\rho'(G) \geq 3$. Note that $\rho'(C_4) = \rho'(C_5) = 3$ although $k_{C_4} = k_{C_5} = 2$.

Lower bounds. For $k_G = 3$ and $G \in \mathcal{F}$, Lemma 4.19 shows $\rho'(G) \geq 4$. Thus it suffices to prove $\rho'(G) \geq k_G$. We use facts (4.1),(4.2),(4.3) listed after Theorem 4.2. When we provide another graph having the same k -deck, we obtain $\rho'(G) > k$. By Lemma 4.5, it suffices to do this for a graph consisting of one or two components of G .

We first show $\rho'(G) \geq \lfloor m/2 \rfloor + \epsilon$; this holds when $m \leq 5$ since $\rho'(G) \geq 3$, so we may assume $m \geq 6$. Since $C_{\lfloor m/2 \rfloor}$ exists when $m \geq 6$,

(4.1) yields $\mathcal{D}_k(C_m) = \mathcal{D}_k(C_{\lfloor m/2 \rfloor} + C_{\lfloor m/2 \rfloor})$ when $k < \lfloor m/2 \rfloor$, and

(4.2) yields $\mathcal{D}_k(P_m) = \mathcal{D}_k(C_{\lfloor m/2 \rfloor + 1} + P_{\lfloor m/2 \rfloor - 1})$ when $k \leq \lfloor m/2 \rfloor$.

If $\epsilon = 0$, then C_m is a component of G ; if $\epsilon = 1$, then P_m is a component. Hence we obtain $\rho'(G) \geq \lfloor m/2 \rfloor + \epsilon$ in both cases.

Next we show $\rho'(G) \geq m' + \epsilon'$. When $m' = 0$ we have $\epsilon' \leq 2$, so the inequality holds. Hence we may assume $m' > 0$, with $P_{m'}$ or $C_{m'}$ being a component. We consider two cases to complete the proof of $\rho'(G) \geq m' + \epsilon'$ and the lower bound.

Case 1: $m' > 0$ and $P_{m'}$ is a component. Recall that

(4.2) yields $\mathcal{D}_k(C_m + P_{m'}) = \mathcal{D}_k(P_{m+m'})$ when $k \leq \min\{m-1, m'+1\}$, and

(4.3) yields $\mathcal{D}_k(P_m + P_{m'}) = \mathcal{D}_k(P_{m-1} + P_{m'+1})$ when $k \leq \min\{m, m'+1\}$.

Both observations are valid when $k \leq \min\{m'+1, m-1\}$. When $m' < m-1$ we therefore obtain $\rho'(G) \geq m'+2 \geq m'+\epsilon'$. Next consider $m' = m$. If two components equal P_m , then $\epsilon' = 1$, and the second observation with $k = m$ gives $\rho(G) \geq m+1 = m'+\epsilon'$. When $m' = m$ and only one component equals P_m , then C_m is a component, we have $\epsilon' = 0$, and the first observation with $k = m-1$ gives $\rho'(G) \geq m = m'+\epsilon'$.

This leaves the case $m' = m-1$, where $\epsilon' = 1$. We may assume $m' \geq 3$, since $\rho'(G) \geq 3$. If C_m is a component, then the first observation with $k = m-1$ gives $\rho'(G) \geq m = m'+\epsilon'$. If P_m is a component, then $P_m + P_{m'} = P_{m-1} + P_{m'+1}$. In this case, we use $\mathcal{D}_k(P_m + P_{m'}) = \mathcal{D}_k(P_{m+1} + P_{m'-1})$, valid when $k \leq m'$. Thus $\rho'(G) \geq m'+1 = m'+\epsilon'$.

Case 2: $m' > 0$ and $P_{m'}$ is not a component. In this case $C_{m'}$ is a component. Since $P_{m'}$ is not a component, either $\epsilon' = 1$ with $m' < m$ and $P_{m'-1}$ being a component, or $\epsilon' = 0$. In either case, C_m or P_m is a component. Now

(4.1) yields $\mathcal{D}_k(C_m + C_{m'}) = \mathcal{D}_k(C_{m+m'})$ when $k < m' \leq m$,

(4.2) yields $\mathcal{D}_k(P_m + C_{m'}) = \mathcal{D}_k(P_{m+m'})$ when $k < m' \leq m$,

(4.2) yields $\mathcal{D}_k(C_m + P_{m'-1}) = \mathcal{D}_k(P_{m'+m-1})$ when $k \leq m' < m$, and

(4.3) yields $\mathcal{D}_k(P_m + P_{m'-1}) = \mathcal{D}_k(P_{m-1} + P_{m'})$ when $k \leq m' < m$.

In each situation listed, we conclude $\rho'(G) \geq m'$, which suffices whenever $\epsilon' = 0$. Since $P_{m'}$ is not a component, the situation with $\epsilon' = 1$ occurs only when $m' < m$ and $P_{m'-1}$ is a component, in which case the last two observations yield $\rho'(G) \geq m'+1 = m'+\epsilon'$ whether the largest component is C_m or P_m .

This completes the proof of the lower bound.

Upper bounds. We have discussed the cases with $|E(G)| \leq 1$ or $n \leq 3$ or $G \in \{C_4, C_5\}$ at the beginning of the proof. If $|E(G)| \geq 2$ and $\Delta(G) = 1$, then at least two components of G equal P_m , and $k_G = m' + \epsilon' = 3$. In this case Manvel's result (Theorem 2.5) determines the degree list from $\mathcal{D}_3(G)$, which in turn determines G .

Otherwise $\Delta(G) = 2$, which yields $m \geq 3$. If $k_G \leq 2$, then a component with m vertices must lie in $\{C_5, C_4, C_3, P_3\}$. We have discussed the outcome when G equals one of these graphs. If G contains one of them and another component, then we claim $k_G \geq 3$. In particular, if $C_{m'}$ is a component, then $k_G \geq m' \geq 3$. Otherwise, $P_{m'}$ is a component, and $k_G \geq m' + \epsilon'$, where $m' + \epsilon'$ is at least $3 + 0$ if $m' = m$, at least $2 + 1$ if $m' = m - 1$, and at least $1 + 2$ if $m' < m - 1$.

Therefore, we may assume $k_G \geq 3$. If $k_G = 3$ and $G \in \mathcal{F}$, then set $k = 4$. Otherwise, set $k = k_G$.

If $G \notin \mathcal{F}$ and G does not have one of the three special forms in the last sentence of Lemma 4.19, then G must have an isolated vertex, and G cannot have both P_4 and C_3 as components (it would be in \mathcal{F}). Further avoiding other graphs in G , either $G = aP_4 + bP_3 + cP_2 + dP_1$ with $a \geq 1$ and $a + b + c \geq 2$, or $G = aP_3 + bC_3 + cP_2 + dP_1$ with $a \geq 2$. In either case, $k_G \geq m' + \epsilon' \geq 4$. Therefore, if $k_G = 3$ and $G \notin \mathcal{F}$, then Lemma 4.19 implies that every reconstruction from $\mathcal{D}_3(G)$ has maximum degree 2. In all other cases $k \geq 4$, and Manvel's result (Theorem 2.5) implies that every reconstruction has maximum degree 2. Hence from $\mathcal{D}_k(G)$ we can reconstruct $\Delta(G) = 2$.

By Lemma 4.14, it suffices to show that $\mathcal{D}_k(G)$ determines the components of G with more than k vertices; we now know that they can only be paths and cycles. Since $k \geq k_G$, we have $k \geq \lfloor m/2 \rfloor + \epsilon$ and $k \geq m' + \epsilon'$.

Case 1: $m' = m$ and at least two components of G equal P_m . Here $\epsilon' = 1$ and $k = m + 1$. Since no component of G has at least k vertices, no card is connected; hence every reconstruction has no component with at least k vertices.

Case 2: $m' = m$ and G does not have two components equal to P_m , or $m' = m - 1$ and $P_{m'}$ is a component of G . In each of these possibilities, $k = m' + \epsilon' = m$. Since no component has more than k vertices and P_m occurs at most once, at most one card is P_k . Since $s(G, P_k) \leq 1$ and $\Delta(G) = 2$, again the deck determines that every reconstruction has no component with more than k vertices.

Case 3: $m' < m - 1$, or $m' = m - 1$ and $P_{m'}$ is not a component. This case occupies the remainder of the proof. Recall that $t_k(G)$ is the number of components of G that are paths with at least $k - 1$ vertices. In order to apply Lemma 4.17, we first prove $t_k(G) \leq 1$.

Since $k \geq m' + \epsilon'$, we have $m' \leq k - \epsilon'$. If $\epsilon' = 2$, then G has $P_{m'}$ as a component and has at most one component with more vertices, which suffices since $m' < k - 1$. If $\epsilon' = 1$, then G has $P_{m'-1}$ but not $P_{m'}$ as a component: at most one path component has at least m' vertices, which suffices since $m' \leq k - 1$. Otherwise $\epsilon' = 0$ and neither $P_{m'}$ nor $P_{m'-1}$ is a component. Now at most one path component has at least $m' - 1$ vertices (it can only be

P_m), which suffices since $m' \leq k$.

Hence $t_k(G) \leq 1$ and $\Delta(G) = 2$. By Lemma 4.16, $\mathcal{D}_k(G)$ determines $t_k(G)$ and $s(G, P_k)$.

When $t_k(G) = 0$, cards that are paths arise only from cycles with more than k vertices. Since $k \geq m'$, no such cards arise from m' -cycles, and by the definition of m' only one cycle can be longer. Since $k \geq \lfloor m/2 \rfloor + \epsilon$ and here $\epsilon = 0$, we have $s(G, P_k) = m \leq 2k + 1$. Now Lemma 4.17 implies that a reconstruction from $\mathcal{D}_k(G)$ has one component with more than k vertices, and it is C_m .

When $t_k(G) = 1$, let Q be the path component with at least $k - 1$ vertices. If $Q = P_m$, then since $k \geq m'$ no copies of P_k arise from m' -cycles, so $s(G, P_k) = m - k + 1$. Since $k \geq \lfloor m/2 \rfloor + \epsilon$ and $\epsilon = 1$, we have $m \leq 2k - 1$, and hence $s(G, P_k) \leq k$. Now Lemma 4.17 implies that any reconstruction from $\mathcal{D}_k(G)$ has no cycle with more than k vertices, and its one path component with at least $k - 1$ vertices is P_m .

If $t_k(G) = 1$ and $Q = P_r$ with $r < m$, then C_m is a component and $r \leq m'$. The conditions of Case 3 and definition of ϵ' now leave several possibilities. Either $r = m'$ with $m' < m - 1$ and $\epsilon' = 2$, or $r = m' - 1$ with $m' \leq m - 1$ and $\epsilon' = 1$, or $r < m' - 1$ with $m' \leq m - 1$ and $\epsilon = 0$. Since $k \geq m' + \epsilon'$, in each case $r \leq k - 2$, which contradicts $t_k(G) = 1$. \square

5 Complete Multipartite Graphs

In this section we study reconstructibility of complete multipartite graphs. Since $\rho(G) = \rho(\overline{G})$ for every graph G , we are also studying their complements, which are disjoint unions of complete graphs. Note first that membership in this family is 3-deck reconstructible, since a graph is complete multipartite if and only if it does not have $P_2 + P_1$ as an induced subgraph (similarly, it is a disjoint union of complete graphs if and only if it does not have P_3 as an induced subgraph).

Only graphs with at most one edge (and their complements) are 2-deck reconstructible, since these are the only graphs determined by their numbers of vertices and edges. Graphs with maximum degree 1 (and their complements) are 3-deck reconstructible, since by Manvel's result the 3-deck determines the degree list. This statement generalizes.

Proposition 5.1. *If G is a disjoint union of complete graphs of order at most m (or a complete multipartite graph with parts of size at most m), then G is $(m + 1)$ -deck reconstructible.*

Proof. For $m = 1$, the edgeless graph is 2-deck reconstructible. For $m \geq 2$, the absence of P_3 in the 3-deck determines that G is a disjoint union of complete graphs (since the 3-deck is available from the $(m + 1)$ -deck). Since $\Delta(G) < m$, by Theorem 2.5 we can reconstruct the degree list. Every disjoint union of complete graphs is determined by its degree list. \square

When the parts are large, we still have an upper bound on the size of the cards needed.

Theorem 5.2. *Every complete r -partite graph is $(r + 1)$ -deck reconstructible (as are disjoint unions of r complete graphs).*

Proof. Let G be a complete r -partite graph. Since \overline{K}_n is 2-deck reconstructible, we may assume $r \geq 2$. By Observation 1.4, $\mathcal{D}_{r+1}(G)$ also determines $\mathcal{D}_k(G)$ when $k \leq r$. The absence of $P_2 + P_1$ in $\mathcal{D}_3(G)$ requires G to be a complete multipartite graph. The absence of K_{r+1} in $\mathcal{D}_{r+1}(G)$ implies that G is a complete r -partite graph (parts may be empty).

Let the sizes of the parts be q_1, \dots, q_r (not necessarily distinct). Let $f(x) = \prod_{i=1}^r (x - q_i)$. The r roots of f are the integers q_1, \dots, q_r . Since a polynomial determines its roots, it suffices to reconstruct f from $\mathcal{D}_{r+1}(G)$.

Let R_i be the family of i -element subsets of the index set $\{1, \dots, r\}$. Let $s_i = \sum_{J \in R_i} \prod_{j \in J} q_j$. Note that $f(x) = \sum_{i=0}^r (-1)^i s_i x^{r-i}$.

In fact, $s_i = s(G, K_i)$ (recall that $s(G, H)$ is the number of induced subgraphs of G isomorphic to H), and $\mathcal{D}_i(G)$ determines $s(G, K_i)$. Hence $\mathcal{D}_{r+1}(G)$ determines f . \square

We know that complete bipartite graphs are not 2-deck reconstructible, since they are not determined by their numbers of edges and vertices. Similarly, even though the 3-deck determines that a complete multipartite graph is complete multipartite, not all complete tripartite graphs are 3-deck reconstructible.

Example 5.3. The complete multipartite graphs $K_{7,4,3}$ and $K_{6,6,1,1}$ have the same 3-deck. It consists of 84 copies of K_3 , 240 copies of P_3 , and 40 copies of \overline{K}_3 .

By the argument in Theorem 5.2, the only thing we need in order to exclude complete $(r + 1)$ -partite graphs as alternative reconstructions from the r -deck of a complete r -partite graph is the absence of K_{r+1} . Thus, among K_{r+1} -free graphs, a complete r -partite graph is determined by its r -deck. We have not generalized the construction in Example 5.3 to obtain a complete r -partite graph that is not r -deck reconstructible. Hence we ask:

Question 5.4. For $r > 3$, do there exist a complete r -partite graph and a complete $(r + 1)$ -partite graph having the same r -deck?

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References

- [1] K. J. Asciak, M. A. Francalanza, J. Lauri, and W. Myrvold, A survey of some open questions in reconstruction numbers, *Ars Combin.* 97 (2010), 443–456.

- [2] B. Bollobás, Almost every graph has reconstruction number three, *J. Graph Theory* 14 (1990), 1–4.
- [3] J. A. Bondy, A graph reconstructor’s manual, in *Surveys in Combinatorics (Guildford, 1991)*, Lond. Math. Soc. Lec. Notes 166 (Cambridge U. Press, 1991), 221–252.
- [4] J. A. Bondy and R. L. Hemminger, Graph reconstruction—a survey, *J. Graph Theory* 1 (1977), 227–268.
- [5] A. Bowler, P. Brown, T. Fenner, and W. Myrvold, Recognizing connectedness from vertex-deleted subgraphs, *J. Graph Theory* 67 (2011), 285–299.
- [6] Y. Boudabbous, L. Haddad, and N. Zaguia, The mathematics of Gérard Lopez and Maurice Pouzet, *J. Mult.-Valued Logic & Soft Computing* 27 (2016), 117–132.
- [7] Y. Boudabbous and G. Lopez, The minimal non- $(\leq k)$ -reconstructible relations, *Discrete Math.* 291 (2005), 19–40.
- [8] P. Chinn, A graph with p points and enough distinct $(p-2)$ -order subgraphs is reconstructible, *Recent Trends in Graph Theory Lecture Notes in Mathematics* 186 (Springer, 1971).
- [9] R. Fraïssé, L’intervalle en théorie des relations, ses généralisations, filtre intervallaire et clôture d’une relation, in: M. Pouzet and D. Richard eds., *Order, Description and Roles*, (North-Holland, Amsterdam, 1984), 313–342.
- [10] F. Harary and M. Plantholt, The graph reconstruction number, *J. Graph Theory* 9 (1985), 451–454.
- [11] P. J. Kelly, On isometric transformations, PhD Thesis, University of Wisconsin-Madison, 1942.
- [12] P. J. Kelly, A congruence theorem for trees, *Pacific J. Math.* 7 (1957), 961–968.
- [13] J. Lauri, Pseudosimilarity in graphs—a survey, *Ars Combin.* 46 (1997), 77–95.
- [14] J. Lauri and R. Scapellato, *Topics in Graph Automorphism and Reconstruction*, *London Math. Soc. Student Texts* 54, (Cambridge Univ. Press, Cambridge, 2003). Second edition: *London Math. Soc. Lect. Note Series* 432, (Cambridge Univ. Press, Cambridge, 2016).
- [15] G. Lopez, Deux résultats concernant la détermination d’une relation par les types d’isomorphie de ses restrictions, *C. R. Acad. Sci. Paris Sér. A-B* 274 (1972), 1525–1528.
- [16] G. Lopez, L’indéformabilité des relations et multirelations binaires, *Z. Math. Logik Grundlag. Math.* 24 (1978), 303–317.
- [17] A. Maccari, O. Rueda, V. Viazzi, A survey on edge reconstruction of graphs, *J. Discr. Math. Sci. Cryptog.* 5 (2002), 1–11.
- [18] B. Manvel, Some basic observations on Kelly’s conjecture for graphs, *Discrete Math.* 8 (1974), 181–185.

- [19] B. McMullen and S. Radziszowski, Graph reconstruction numbers, *J. Combin. Math. Combin. Comput.* 62 (2007), 85–96.
- [20] S. Monikandan and J. Balakumar, On pairs of graphs having $n - 2$ cards in common, *Ars Combin.* 124 (2016), 289–302.
- [21] V. Müller, Probabilistic reconstruction from subgraphs, *Comment. Math. Univ. Carolinae* 17 (1976), 709–719.
- [22] W. J. Myrvold, Ally and adversary reconstruction problems, Ph.D. thesis, Univ. Waterloo (1988).
- [23] W. Myrvold, The ally-reconstruction number of a disconnected graph, *Ars Combin.* 28 (1989), 123–127.
- [24] W. Myrvold, The ally-reconstruction number of a tree with five or more vertices is three, *J. Graph Theory* 14 (1990), 149–166.
- [25] W. Myrvold, The degree sequence is reconstructible from $n - 1$ cards, *Discrete Math.* 102 (1992), 187–196.
- [26] V. Nýdl, Finite undirected graphs which are not reconstructible from their large cardinality subgraphs, *Discrete Math.* 108 (1992), 373–377.
- [27] S. Ramachandran and S. Monikandan, Pairs of graphs having $n - 2$ cards in common, *Ars Combin.* 112 (2013), 213–223.
- [28] K. B. Reid and C. Thomassen, Strongly self-complementary and hereditarily isomorphic tournaments, *Monatshefte Math.* 81 (1976), 291–304.
- [29] D. Rivshin and S. Radziszowski, Multi-vertex deletion graph reconstruction numbers. *J. Combin. Math. Combin. Comput.* 78 (2011), 303–321.
- [30] R. P. Stanley, in Problems and Solutions, *American Math. Monthly*, 123(3) (2016), 296–303.
- [31] P. K. Stockmeyer, The falsity of the reconstruction conjecture for tournaments, *J. Graph Theory* 1 (1977), 19–25; erratum *J. Graph Theory* 62 (2009), 199–200.
- [32] R. Taylor, Reconstructing degree sequences from k -vertex-deleted subgraphs. *Discrete Math.* 79 (1990), 207–213.
- [33] S. M. Ulam, A collection of mathematical problems, *Interscience Tracts in Pure and Applied Mathematics* 8 (Interscience Publishers, 1960).
- [34] D. R. Woodall, Towards size reconstruction from fewer cards. *Discrete Math.* 338 (2015), 2514–2522.