Reconstruction from the Subgraphs Obtained by Deleting $\ell$ Vertices

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Joint work with
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(and earlier with) Hannah Spinoza
Def. A card of a graph $G$ is an induced subgraph $G - v$. The deck of a graph is the multiset of its cards.
Extending The Classical Problem

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\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{card1}\quad \Leftrightarrow \quad \includegraphics[width=0.4\textwidth]{card2}
\end{array}
\]

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![Graphs and cards](image)

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**Obs.** $\mathcal{D}_k(G)$ determines $\mathcal{D}_{k-1}(G)$. 
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Obs. $\mathcal{D}_k(G)$ determines $\mathcal{D}_{k-1}(G)$.

Pf. Each member of $\mathcal{D}_{k-1}$ arises $n - k + 1$ times by deleting one vertex from a graph in $\mathcal{D}_k(G)$. \[\blacksquare\]
A More General Conjecture

If the RC holds, then for every graph $G$ there is a least $k$ such that $G$ is determined by $\mathcal{D}_k(G)$, meaning that every graph with $k$-deck $\mathcal{D}_k(G)$ is isomorphic to $G$. 
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**Conj.** “Kelly’s Conjecture” (Manvel [1964, 1969]): For $\ell \in \mathbb{N}$, $\exists M_\ell \in \mathbb{N}$ such that $|V(G)| \geq M_\ell \implies G$ is determined by its subgraphs deleting $\ell$ vertices.
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For $\ell \in \mathbb{N}$, $\exists M_\ell \in \mathbb{N}$ such that $|V(G)| \geq M_\ell \Rightarrow G$ is determined by its subgraphs deleting $\ell$ vertices.

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Sharp: $C_4 + K_1$ and the tree $K_{1,3}'$ are not 2-reconstructible.
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**Sharp:** $C_4 + K_1$ and the tree $K_{1,3}'$ are not 2-reconstructible.

3-deck of each: four of $P_3$, four of $P_2 + P_1$, two of $3K_1$. 

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![Diagram](null)
Warmup: Complete Multipartite Graphs

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**Sharpness?** $\mathcal{D}_3(K_{7,4,3}) = \mathcal{D}_3(K_{6,6,1,1})$, so $\mathcal{D}_r$ does not suffices when $r = 3$, but what about larger $r$?
Results on Manvel’s Conjecture

**Thm.** Nýdl [1992]: For $\epsilon > 0$ and sufficiently large $n_0$, $\exists n$-vertex $G$ with $n \geq n_0$ that is not $\epsilon n$-reconstructible.
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**Theme:** For special classes of graphs, find a threshold $c$ such that $n \geq cl$ implies $l$-reconstructibility.
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**Thm.** Müller [1976]: When $n$ and $\ell$ are restricted by $n \geq (2 + \epsilon)\ell$, almost all graphs are $\ell$-reconstructible.
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**Thm.** Spinoza–West [2019]: If $D_{n-l-1}$ is good, then $G$ is reconstructible from some $\binom{l+2}{2}$ subgraphs in $D_{n-l}$.
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**Thm.** Spinoza–West [2019]: If $D_{n-\ell-1}$ is good, then $G$ is reconstructible from some $\binom{\ell+2}{2}$ subgraphs in $D_{n-\ell}$.

Generalizes Chinn ’71, Müller ’76, Bollobás ’90 for $\ell = 1$: Almost every $G$ is 1-reconstructible from any three cards.
Almost All Graphs

**Thm.** \( \mathcal{D}_{n-\ell-1} \text{ good } \Rightarrow \mathcal{D}_{n-\ell} \text{ determines } G. \)
Almost All Graphs

**Thm.** \( D_{n-\ell-1} \) good \( \Rightarrow \) \( D_{n-\ell} \) determines \( G \).

**Pf.** Let \( n = |V(G)| \). Fix \( S = \{x_1, \ldots, x_{\ell+1}\} \subseteq V(G) \). Let \( H = G - S \) and \( h = |V(H)| = n - \ell - 1 \).
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Let \( C_i = G - (S - \{x_i\}) \) (deleting \( \ell \)) and \( C = \{C_i : x_i \in S\} \).
Almost All Graphs

**Thm.** $D_{n-\ell-1}$ good $\Rightarrow$ $D_{n-\ell}$ determines $G$.

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Let $C_i = G - (S - \{x_i\})$ (deleting $\ell$) and $C = \{C_i : x_i \in S\}$.

For $x_i, x_j \in S$, let $D_{i,j} = G - (S - \{x_i, x_j\}) - w_{i,j}$, where $w_{i,j} \in V(H)$. Let $D = \{D_{i,j} : x_i, x_j \in S\}$.
Almost All Graphs

**Thm.** \( D_{n-1} \) good \( \Rightarrow D_{n-\ell} \) determines \( G \).

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For \( x_i, x_j \in S \), let \( D_{i,j} = G - (S - \{x_i, x_j\}) - w_{i,j} \), where \( w_{i,j} \in V(H) \). Let \( D = \{D_{i,j}: x_i, x_j \in S\} \).

**Claim:** \( G \) is reconstructible from \( C \cup D \).
Reconstructing $G$ from $C \cup D$.

**Idea:** $H$ is the only $h$-vertex subgraph $H'$ appearing in $\ell + 1$ cards in $C \cup D$. This identifies $H$ and all $C_i$, the vertex $x_i$ in $C_i$, and the edges from $x_i$ to $V(H)$. 
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If $|V(H') \cap S| \geq 3$, then $H'$ appears in no card in $C \cap D$.

If $V(H') \cap S = \{x_i, x_j\}$, then $H'$ appears only in $D_{i,j}$.

If $V(H') \cap S = \{x_i\}$, then $H'$ is in one card in $C$ and can be in cards $D_{i,j}$ as $H' = D_{i,j} - x_j = G[V(H) + x_i - w_{i,j}]$.

If $w_{i,j}$ is not the same for all $j$, then $H'$ is in $\leq \ell$ cards.

If $V(H') \cap S = \emptyset$, then $H' = H$, in all $\ell + 1$ cards of $C$. 
Reconstructing $G$ from $C \cup D$.

Idea: $H$ is the only $h$-vertex subgraph $H'$ appearing in $l + 1$ cards in $C \cup D$. This identifies $H$ and all $C_i$, the vertex $x_i$ in $C_i$, and the edges from $x_i$ to $V(H)$.

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If $w_{i,j}$ is not the same for all $j$, then $H'$ is in $\leq l$ cards.

If $V(H') \cap S = \emptyset$, then $H' = H$, in all $l + 1$ cards of $C$.

Idea: $H$ can also be used to identify the card $D_{i,j}$, which is used to check whether $x_ix_j \in E(G)$. 
Reconstructing $G$ from $C \cup D$.

**Idea:** $H$ is the only $h$-vertex subgraph $H'$ appearing in $\ell + 1$ cards in $C \cup D$. This identifies $H$ and all $C_i$, the vertex $x_i$ in $C_i$, and the edges from $x_i$ to $V(H)$.

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If $V(H') \cap S = \{x_i, x_j\}$, then $H'$ appears only in $D_{i,j}$.
If $V(H') \cap S = \{x_i\}$, then $H'$ is in one card in $C$ and can be in cards $D_{i,j}$ as $H' = D_{i,j} - x_j = G[V(H)+x_i-w_{i,j}]$.
If $w_{i,j}$ is not the same for all $j$, then $H'$ is in $\leq \ell$ cards.
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**Idea:** $H$ can also be used to identify the card $D_{i,j}$, which is used to check whether $x_ix_j \in E(G)$.

Note $H = C_i - x_i$. For $w \in V(H)$, a card $D' \in D$ contains both $C_i - w$ and $C_j - w$ only when $D' = D_{i,j}$ and $w = w_{i,j}$.
Thm. Connectedness (Manvel [1974]) and the degree list (Chernyak [1982]) are 2-reconstructible for $n \geq 6$. (Sharp by $C_4 + K_1$ and $K'_{1,3}$ having same 3-deck.)
Thm. Connectedness (Manvel [1974]) and the degree list (Chernyak [1982]) are 2-reconstructible for $n \geq 6$.
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Thm. Kostochka-Nahvi-West-Zirlin [2020]: Connectedness & the degree list are 3-reconstructible for $n \geq 7$.
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**Thm.** Taylor [1990]: The degree list is $l$-reconstructible for $n \geq \ell(1 + o(1))$. 
Degree Lists and Connectedness

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**Thm.** Groenland–Johnston–Scott–Tan [2021+]: The degree list is reconstructible from $D_k(G)$ when $k \geq \sqrt{2n \log 2n}$. 
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**Thm.** Groenland–Johnston–Scott–Tan [2021+]: The degree list is reconstructible from $D_k(G)$ when $k \geq \sqrt{2n \log 2n}$.

**Cor.** Degree list is $l$-reconstr’ble when $n \geq l + O(\sqrt{l})$. 
Degree Lists and Connectedness

**Thm.** Connectedness (Manvel [1974]) and the degree list (Chernyak [1982]) are 2-reconstructible for $n \geq 6$. (Sharp by $C_4 + K_1$ and $K'_{1,3}$ having same 3-deck.)

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Theorem. Connectedness (Manvel [1974]) and the degree list (Chernyak [1982]) are 2-reconstructible for \( n \geq 6 \).
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Theorem. Kostochka-Nahvi-West-Zirlin [2020]: Connectedness & the degree list are 3-reconstructible for \( n \geq 7 \).
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Theorem. Taylor [1990]: The degree list is \( l \)-reconstructible for \( n \geq e\ell(1 + o(1)) \).

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Theorem. GJST’21: Connectedness is \( l \)-reconstr. for \( n \geq 10l \).
Degree Lists — via an algebraic result
Lem. Borwein–Ingalls [1999], GJST [2021+]: If $\alpha$ and $\beta$ are distinct nonincreasing lists in $\{0, \ldots, n\}^m$ such that

$\binom{\alpha_1}{j} + \cdots + \binom{\alpha_m}{j} = \binom{\beta_1}{j} + \cdots + \binom{\beta_m}{j}$

for $0 \leq j \leq k$, then $k + 1 \leq \sqrt{2n \log(2m)}$. 

\[\]
Lem. Borwein–Ingalls [1999], GJST [2021+]: If $\alpha$ and $\beta$ are distinct nonincreasing lists in $\{0, \ldots, n\}^m$ such that

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\[\blacksquare\]
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**Sharpness?** GJST observe that when $k \in \Omega(\sqrt{\log n})$ there are more degree lists than possible $k$-decks.
Graphs with Max Degree 2 (Spinoza–West [2019])

**Ques.** When is the \( n \)-vertex cycle \( ℓ \)-reconstructible?
Graphs with Max Degree 2 (Spinoza–West [2019])

Ques. When is the $n$-vertex cycle $\ell$-reconstructible?

Prob 11898 (Stanley [2016], Amer. Math. Monthly)
In an $n$-vertex graph whose components are cycles of length greater than $k$, the number of independent sets of size $k$ depends only on $n$ and $k$. 
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**Thm.** Let $G$ and $G'$ be $n$-vertex graphs with maximum degree 2 and $|E(G)| = |E(G')|$. If every component in each graph is a cycle with more than $k$ vertices or a path with at least $k - 1$ vertices, then $D_k(G) = D_k(G')$. 

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1. \( \mathcal{D}_k(C_{q+r}) = \mathcal{D}_k(C_q + C_r) \) if \( q, r \geq k + 1 \),
2. \( \mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r) \) if \( q \geq k + 1 \) and \( r \geq k - 1 \), and
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3. $D_k(P_{q-1} + P_r) = D_k(P_q + P_{r-1})$ if $q, r \geq k$.

**Ex.** $D_l(P_{2l}) = D_l(C_{l+1} + P_{l-1})$, so the threshold on $n$ for $l$-reconstructibility of connectedness is at least $2l + 1$. 
Key Ideas

(1) $\mathcal{D}_k(C_{q+r}) = \mathcal{D}_k(C_q + C_r)$ if $q, r \geq k + 1$,
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In fact, (1,2,3) suffice to prove the theorem, because:

Lem. If $G, G'$, and $H$ are graphs, then $\mathcal{D}_k(G) = \mathcal{D}_k(G')$ if and only if $\mathcal{D}_k(G + H) = \mathcal{D}_k(G' + H)$. 
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• Show $s'(P_n, H)$ is indep of $z$ when $z$ is far from ends.
Independent of the Named Vertex

**Lem.** Let $L$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k + 1 \leq h \leq n - k$, the value $s'(P_n, L)$ is the same.
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**Pf.** Induction on $k$. If $k = 1$, then $s'(P_n, L) = n - 3$ when $w_h$ is not an endpoint of $P_n$. (Avoid nbrs of $w_h$.)
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**Pf.** Induction on $k$. If $k = 1$, then $s'(P_n, L) = n - 3$ when $w_h$ is not an endpoint of $P_n$. (Avoid nbrs of $w_h$.) Compare $s'(P_n, L)$ to $s'(C_n, L)$ by adding edge $w_nw_1$. 
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$s'(C_n, L)$ omits copies of $L + K_1$ in $P_n$ using $w_1$ and $w_n$.

$s'(C_n, L)$ counts unwanted subgraphs using $w_n w_1$. 
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**Lem.** Let $L$ be the linear forest $\sum_{i=1}^{p} m_i P_{l_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k + 1 \leq h \leq n - k$, the value $s'(P_n, L)$ is the same.

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$s'(C_n, L)$ omits copies of $L + K_1$ in $P_n$ using $w_1$ and $w_n$. $s'(C_n, L)$ counts unwanted subgraphs using $w_nw_1$.

$$s'(P_n, L) = s'(C_n, L) + \sum_{i,j} s'(P_{n-l_i-l_j-2}, L - P_{l_i} - P_{l_j})$$
$$- \sum_{i} (l_i - 1) s'(P_{n-l_i-2}, L - P_{l_i})$$

$w_h$ is far enough from the ends to use induction hyp. ■
Let $L$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k + 1 \leq h \leq n - k$, the value $s'(P_n, L)$ is the same.
Same $k$-deck, and Sharpness Claim

**Lem.** Let $L$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k + 1 \leq h \leq n - k$, the value $s'(P_n, L)$ is the same.

Implies (3) $D_k(P_{q-1} + P_r) = D_k(P_q + P_{r-1})$ if $q, r \geq k$. 

\[
\begin{array}{cccccccc}
q-1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & r \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
q & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & r-1
\end{array}
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**Lem.** Let $L$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k + 1 \leq h \leq n - k$, the value $s'(P_n, L)$ is the same.

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By summing over cases, also (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1).

(2) $D_k(P_{q+r}) = D_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$.

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**Lem.** Let $L$ be the linear forest $\sum_{i=1}^{\ell} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k + 1 \leq h \leq n - k$, the value $s'(P_n, L)$ is the same.

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![Diagram]

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(1) $D_k(C_{q+r}) = D_k(C_q + C_r)$ if $q, r \geq k + 1$.

Full solution for maxdegree 2:

**Thm.** If $\Delta(G) = 2$, and two largest components have $m$ and $m'$ vertices, then $G$ is reconstructible from $D_k(G)$ iff $k \geq \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}$, where $\epsilon \in \{0, 1\}$ and $\epsilon' \in \{0, 1, 2\}$. ($\epsilon = 1$ if largest component is $P_m$.)
Regular Graphs and Disconnected Graphs

**Thm.** Spinoza–West [2019]: For 2-regular graphs we know all $l$ such that the graph is $l$-reconstructible.
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**Thm.** KW’21: If \( n \geq 2l + 1 \) and every component of \( G \) has at most \( n - l \) vertices, then \( G \) is \( l \)-reconstructible.
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**Prop.** All regular graphs are 1-reconstructible.

**Ques.** Mohar: Are regular graphs 2-reconstructible?

**Thm.** KNWZ’21: 3-regular graphs are 2-reconstructible.

**Thm.** Kelly’57: Disconnected graphs are 1-reconstructible.

**Thm.** KW’21: If \( n \geq 2l + 1 \) and every component of \( G \) has at most \( n - l \) vertices, then \( G \) is \( l \)-reconstructible.

**Thm.** KW’21: If \( \forall l \), graphs with \( n \geq l + 2 \) having one component with \( n - l + 1 \) vertices and the rest isolated are \( l \)-reconstructible, then the original RC holds.
The Counting Lemma

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**Def.** \( \mathcal{F} \)-subgraph = induced subgraph of \( G \) in family \( \mathcal{F} \).

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s(F, G) = \# \text{ induced copies of } F \text{ in } G.
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**Pf.** By induction on $n − |V(F)|$; given when $|V(F)| \geq n − \ell$. 
The Counting Lemma


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For smaller $F$, group the copies of $F$ by the unique maximal $\mathcal{F}$-subgraph $H$ containing them.
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Now \(s(F, G) = \sum_{H \in \mathcal{F}} s(F, H)m(H, G)\), solve for \(m(F, G)\). \[\blacksquare\]
Easy Applications of the Counting Lemma

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Cor. For $n > 2\ell$, every $n$-vertex graph having no component w. more than $n - \ell$ vertices is $\ell$-reconstr’bl.

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Sharpness by \( P_{\ell} + P_{\ell} \) vs. \( P_{\ell+1} + P_{\ell-1} \) when \( n = 2\ell \).
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Trees - I

**Thm.** Kelly [1957]: Trees with at least 3 vertices are $1$-reconstructible.
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Two steps to reconstruction of graphs in a family \( \mathcal{F} \).

1. **Recognition**: Every graph with deck \( D \) lies in \( \mathcal{F} \).
2. **Weak reconstruction**: Given that \( D \) is the deck of a graph in \( \mathcal{F} \), the deck determines which \( F \in \mathcal{F} \).
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**Conj.** Nýdl [1981]: Trees with \( n \geq 2\ell + 1 \) are weakly \( \ell \)-reconstructible.
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**Thm.** Kostochka-Nahvi-West-Zirlin [2021+]: For $n \geq 2\ell + 1$, $n$-vertex acyclic graphs are $\ell$-recognizable.
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![Diagram of trees with labels](image)

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Ours, only for \( l = 3 \), takes 48 pages (uses rooted trees).
Recognizing Acyclic Graphs

**Thm.** For $n \geq 2l + 2$, the $(n - l)$-deck $D$ of an $n$-vertex graph $G$ determines whether $G$ has a cycle.
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**Def.** \( j \)-vine = a tree with diameter \( 2j \).
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**Idea:** for suitable $j$, show $D$ determines $\#j$-centers in any reconstruction.

Acyclic reconstruction has $\#j$-centers $\leq a$.

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Use $b \leq a$ to show $n \leq 2\ell + 1$ if both types occur.
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(Thus \( j \)-centers correspond to maximal \( j \)-vines.)

**Pf.** Girth \( \geq 2j+2 \) \implies a \( j \)-vine \( B \) is an induced subgraph.
Diameter \( 2j \) \implies \( B \) has unique center \( v \).
No \( j \)-vine with another center contains \( B \).
\( \therefore \) unique maximal \( j \)-vine containing \( B \) is the \( j \)-ball at \( v \).
Counting the $j$-Centers

Lem. (restated) If $G$ has girth $\geq 2j + 2$, then \{$j$-vines\} is absorbing for $G$ and $j$-centers $\iff$ maximal $j$-vines.
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**Pf.** all cards acyclic with radius $> j$
- connected cards contain $P_{2j+2}$
- $n − \ell \geq 2j + 2$
- girth $\geq 2j + 3$ (shorter cycles would be in cards).
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Previous Lemma $\Rightarrow$ Family of $j$-vines is absorbing.

$j$-vine with $\geq n - \ell$ vertices
\[ \Rightarrow \text{ connected card with radius } \leq j, \text{ so no such } j$-vine. \]
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$j$-vine with $\geq n - \ell$ vertices

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$\therefore$ Counting Lemma yields $\#$ maximal $j$-vines.
Counting the $j$-Centers

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These correspond bijectively to $j$-centers.
Ambiguous Decks

**Def.** ambiguous deck $\mathcal{D} = \text{the } (n - \ell)\text{-deck of both an acyclic } F \text{ and non-acyclic } H \text{ with } n \text{ vertices.}$

**Def.** Let $k$ be one less than minimum radius of cards.
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All cards from $F$ are acyclic, so $H$ has girth $\geq n - \ell + 1$. 
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Already all reconstructions have same $\#k$-centers. All cards from $F$ are acyclic, so $H$ has girth $\geq n - \ell + 1$.

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \Rightarrow k \geq 1$. 
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Already all reconstructions have same $\#k$-centers. All cards from $F$ are acyclic, so $H$ has girth $\geq n - \ell + 1$.

**Lem.** ambiguous $\mathcal{D}$ and $n \geq 2\ell + 2 \Rightarrow k \geq 1$.

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Girth \( \geq n - l + 1 \geq 4 \) \( \Rightarrow \) star & shortest cycle share \( \leq 3 \) verts.
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Now $(n - l + 1) + (n - l) - 3 \leq n - 1 \Rightarrow n \leq 2l + 1$. ■
**The Marking Lemma**

**Def.** Marking process: From forest $F$, let $C$ be a card with radius $j + 1$ and central vertex $z$. Each $j$-center $x$ other than $z$ marks a vertex $x'$ at distance $j$ from $x$ (in the direction away from $z$).

![Diagram of the marking process]

- $j = 2$
- $d = 3 = \#\text{paths}$
The Marking Lemma

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$Lem.$ With $C$ as above, $F$ has $\leq 1 + d + \ell$ $j$-centers, with $d = \#edge-disjoint paths of length $j + 1$ leaving $z$ in $C$. 
The Marking Lemma

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**Lem.** With $C$ as above, $F$ has $\leq 1 + d + \ell j$-centers, with $d = \#$ edge-disjoint paths of length $j + 1$ leaving $z$ in $C$.

**Pf.** $j$-centers not adj. to $z$ mark vertices outside $C$, and each outside vert. (at most $\ell$) is marked at most once. ■
Cards of Diameter $2k + 1$

Trees with radius $k + 1$ may have diam $2k+1$ or $2k+2$. 
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**Pf.** $F$ and $H$ have same number of $k$-centers.

For card $C$ with diameter $2k+1$, we have $d = 1$. ( $\exists$ One path of length $k+1$ from $z$ in $C$.)

![Diagram](image)
Cards of Diameter $2k + 1$

Trees with radius $k + 1$ may have diam $2k+1$ or $2k+2$.

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**Lem.** ambiguous deck $D$ and $n \geq 2\ell + 2 \implies$ no card has diameter $2k + 1$.

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For card $C$ with diameter $2k + 1$, we have $d = 1$.  
( $\exists$ One path of length $k + 1$ from $z$ in $C$.)

$\therefore$ Marking Lemma $\implies \#k$-centers in $F$ is at most $2 + \ell$. 

![Diagram of a graph with nodes z, z', and an edge labeled k+1]
Cards of Diameter \(2k + 1\)

Trees with radius \(k + 1\) may have diam \(2k + 1\) or \(2k + 2\).

**Cases:** (1) some card has diam \(2k + 1\); (2) none do.

**Lem.** ambiguous deck \(D\) and \(n \geq 2\ell + 2 \Rightarrow\) no card has diameter \(2k + 1\).

**Pf.** \(F\) and \(H\) have same number of \(k\)-centers.
For card \(C\) with diameter \(2k + 1\), we have \(d = 1\).

(∃ One path of length \(k + 1\) from \(z\) in \(C\).)

\[\therefore \text{Marking Lemma} \Rightarrow \#k\text{-centers in } F \text{ is at most } 2 + \ell.\]

Since girth \(\geq 2k + 3\), all vertices on a cycle are \(k\)-centers,
\[\therefore \#k\text{-centers in } H \text{ is at least } n - \ell + 1.\]
Cards of Diameter $2k + 1$

Trees with radius $k + 1$ may have diam $2k + 1$ or $2k + 2$.

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Now $n - \ell + 1 \leq 2 + \ell \implies n \leq 2\ell + 1$. \[ \blacksquare \]
**k-Central Edges and Cards of Diameter** \(2k + 2\)

**Def.** \(k\)-evine = a tree with diameter \(2k + 1\).

\(k\)-central edge = the central edge of a \(k\)-evine.
Def. $k$-evine = a tree with diameter $2k + 1$. $k$-central edge = the central edge of a $k$-evine.

Lem. If all cards acyclic w. radius $> k$ and none have diameter $2k + 1$, then $D$ determines $\#k$-central edges.
**Def.** $k$-evine = a tree with diameter $2k + 1$. 
$k$-central edge = the central edge of a $k$-evine.

**Lem.** If all cards acyclic w. radius $> k$ and none have diameter $2k + 1$, then $\mathcal{D}$ determines $\#k$-central edges.

**Pf.** No card w. diam $2k + 1 \Rightarrow$ no $k$-evine has $\geq n - ℓ$ vrts. 
Girth $\geq 2k + 3 \Rightarrow \{k$-evines$\}$ absorbing; Counting Lem. ■
**k-Central Edges and Cards of Diameter 2k + 2**

**Def.**  *k-evine* = a tree with diameter $2k + 1$.

*k-central edge* = the central edge of a *k-evine*.

**Lem.**  If all cards acyclic w. radius $> k$ and none have diameter $2k + 1$, then $\mathcal{D}$ determines $\#k$-central edges.

**Pf.**  No card w. diam $2k+1$ ⇒ no *k-evine* has $\geq n - \ell$ vrts.

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**Thm.** For $n \geq 2\ell + 2$, acyclicity is $\ell$-recognizable.

**Pf.** No cards of diam $2k + 1 \Rightarrow \#k$-central edges known.
**k-Central Edges and Cards of Diameter 2k + 2**

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$k$-evine = a tree with diameter $2k + 1$.  
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Short card shows $d$ $k$-central edges w. common endpt.
**k-Central Edges and Cards of Diameter $2k + 2$**

**Def.** $k$-evine = a tree with diameter $2k + 1$. 
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**Lem.** If all cards acyclic w. radius $> k$ and none have diameter $2k + 1$, then $D$ determines $\#k$-central edges.

**Pf.** No card w. diam $2k + 1 \Rightarrow$ no $k$-evine has $\geq n - l$ vrts. 
Girth $\geq 2k + 3 \Rightarrow \{k$-evines$\}$ absorbing; Counting Lem. ■

**Thm.** For $n \geq 2l + 2$, acyclicity is $l$-recognizable.

**Pf.** No cards of diam $2k + 1 \Rightarrow \#k$-central edges known. 
Short card shows $d \ k$-central edges w. common endpt. 

In $F$, edge is $k$-central $\iff$ end away from $z$ is $k$-cntr. 

$\therefore \#k$-centers $\leq 1 + d + l \Rightarrow \#k$-central edges $\leq d + l$. 
**Def.** $k$-evine = a tree with diameter $2k + 1$.
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**Lem.** If all cards acyclic w. radius $> k$ and none have diameter $2k+1$, then $\mathcal{D}$ determines $\#k$-central edges.

**Pf.** No card w. diam $2k+1$ $\Rightarrow$ no $k$-evine has $\geq n-\ell$ vrts.
Girth $\geq 2k+3$ $\Rightarrow$ \{$k$-evines\} absorbing; Counting Lem. ■

**Thm.** For $n \geq 2\ell + 2$, acyclicity is $\ell$-recognizable.

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Short card shows $d$ $k$-central edges w. common endpt.
In $F$, edge is $k$-central $\iff$ end away from $z$ is $k$-cntr.
$\therefore$ $\#k$-centers $\leq 1+d+\ell$ $\Rightarrow$ $\#k$-central edges $\leq d+\ell$.
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plus $\geq d - 2$ with common endpoint.
Now $(n - \ell + 1) + (d - 2) \leq d + \ell \Rightarrow n \leq 2\ell + 1$. □
Open Questions

**Conj.** Trees with $n \geq 2\ell + 1$ are $\ell$-reconstructible. (Not for $(5, 2)$ or $(13, 6)$; OK for $n \geq 9\ell + O(\sqrt{\ell})$ [GJST].)
Conj. Trees with \( n \geq 2l + 1 \) are \( l \)-reconstructible. (Not for \((5, 2)\) or \((13, 6)\); OK for \( n \geq 9l + O(\sqrt{l}) \) [GJST].)

Conj. Connectedness is \( l \)-recognizable for \( n \geq 2l + 1 \). (True for \( l = 3 \) [KNWZ’20]; OK for \( n \geq 10l \) [GJST].)
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**Ques.** Least \( k \) so degree list determined by \( \mathcal{D}_k(G) \)? (Between \( \Omega(\log n) \) and \( \sqrt{2n\log(2n)} \) [GJST].)
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(Between $\Omega(\log n)$ and $\sqrt{2n \log(2n)}$ [GJST].)

**Ques.** What is the max $n$ such that every $n$-vertex complete multipartite $G$ is determined by its $k$-deck?
(Nýdl [1985]: it is between $k \ln(k/2)$ and $(k + 1)2^{k-1}$.)

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**Prob.** Find thresholds on \( n \) for \( \ell \)-reconstructibility of connectivity, matching number, \( \chi(G) \), planarity, etc.