Reconstruction from the Subgraphs Obtained by Deleting ℓ Vertices

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slides and papers on preprint page from
https://faculty.math.illinois.edu/

Joint work with
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(and earlier with) Hannah Spinoza

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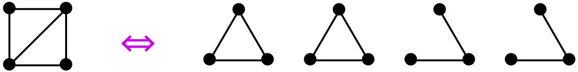
Def. A card of a graph G is an induced subgraph $G - v$. The deck of a graph is the multiset of its cards.



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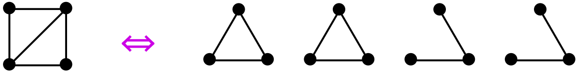

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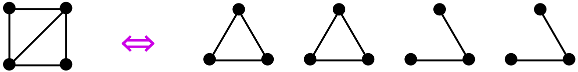

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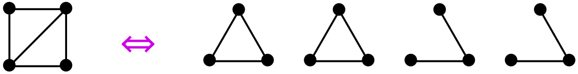

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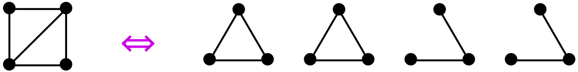

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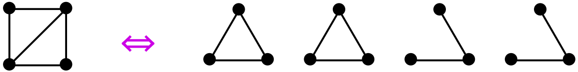

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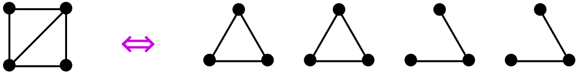

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Pf. Each member of $\mathcal{D}_{k-1}$ arises $n - k + 1$ times by deleting one vertex from a graph in $\mathcal{D}_k(G)$. ∎

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If the **RC** holds, then for every graph G there is a least k such that G is determined by $\mathcal{D}_k(G)$, meaning that every graph with k-deck $\mathcal{D}_k(G)$ is isomorphic to G.

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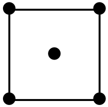

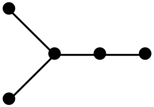

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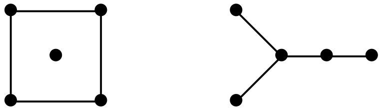

3-deck of each: four of P_3, four of $P_2 + P_1$, two of $3K_1$.

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Sharpness? $\mathcal{D}_3(K_{7,4,3}) = \mathcal{D}_3(K_{6,6,1,1})$, so $\mathcal{D}_r$ does not suffices when $r = 3$, but what about larger r?

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Generalizes Chinn '71, Müller '76, Bollobás '90 for $\ell = 1$:
Almost every G is 1-reconstr'bl from any three cards.

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Pf. Let $n = |V(G)|$. Fix $S = \{x_1, \ldots, x_{\ell+1}\} \subseteq V(G)$.
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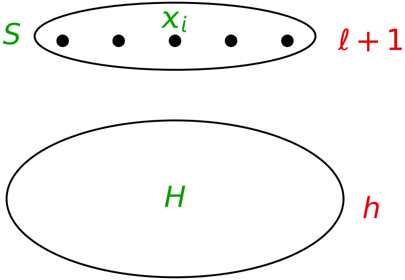

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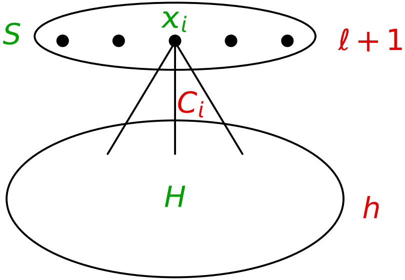

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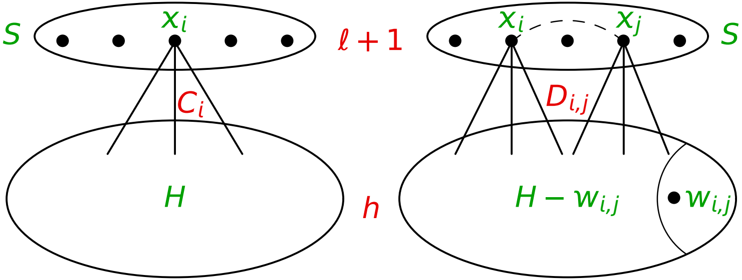

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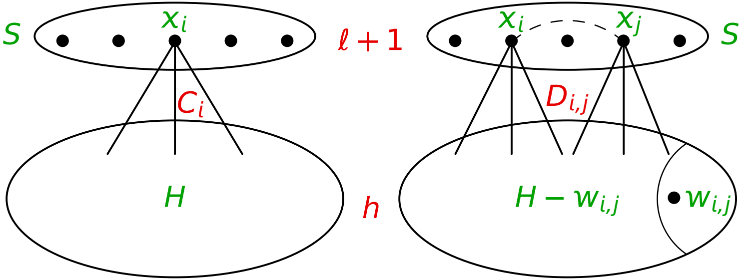

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Claim: G is reconstructible from $C \cup D$.

Reconstructing G from $C \cup D$.

Idea: H is the only h-vertex subgraph H' appearing in $\ell + 1$ cards in $C \cup D$. This identifies H and all C_i, the vertex x_i in C_i, and the edges from x_i to $V(H)$.

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If $|V(H') \cap S| \geq 3$, then H' appears in no card in $C \cap D$.

If $V(H') \cap S = \{x_i, x_j\}$, then H' appears only in $D_{i,j}$.

If $V(H') \cap S = \{x_i\}$, then H' is in one card in C and can be in cards $D_{i,j}$ as $H' = D_{i,j} - x_j = G[V(H) + x_i - w_{i,j}]$.
If $w_{i,j}$ is not the same for all j, then H' is in $\leq \ell$ cards.

If $V(H') \cap S = \varnothing$, then $H' = H$, in all $\ell + 1$ cards of C.

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Idea: H can also be used to identify the card $D_{i,j}$, which is used to check whether $x_i x_j \in E(G)$.

Reconstructing G from $C \cup D$.

Idea: H is the only h-vertex subgraph H' appearing in $\ell+1$ cards in $C \cup D$. This identifies H and all C_i, the vertex x_i in C_i, and the edges from x_i to $V(H)$.

If $|V(H') \cap S| \geq 3$, then H' appears in no card in $C \cap D$.

If $V(H') \cap S = \{x_i, x_j\}$, then H' appears only in $D_{i,j}$.

If $V(H') \cap S = \{x_i\}$, then H' is in one card in C and can be in cards $D_{i,j}$ as $H' = D_{i,j} - x_j = G[V(H) + x_i - w_{i,j}]$.
If $w_{i,j}$ is not the same for all j, then H' is in $\leq \ell$ cards.

If $V(H') \cap S = \varnothing$, then $H' = H$, in all $\ell+1$ cards of C.

Idea: H can also be used to identify the card $D_{i,j}$, which is used to check whether $x_i x_j \in E(G)$.

Note $H = C_i - x_i$. For $w \in V(H)$, a card $D' \in D$ contains both $C_i - w$ and $C_j - w$ only when $D' = D_{i,j}$ and $w = w_{i,j}$. ■

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Sharpness? GJST observe that when $k\in\Omega(\sqrt{\log n})$ there are more degree lists than possible k-decks.

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Ques. When is the n-vertex cycle ℓ-reconstructible?

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Ex. $\mathcal{D}_\ell(P_{2\ell}) = \mathcal{D}_\ell(C_{\ell+1} + P_{\ell-1})$, so the threshold on n for ℓ-reconstructibility of connectedness is at least $2\ell + 1$.

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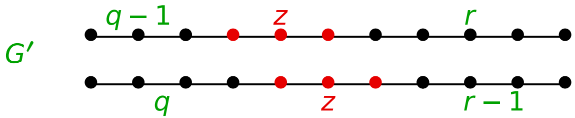

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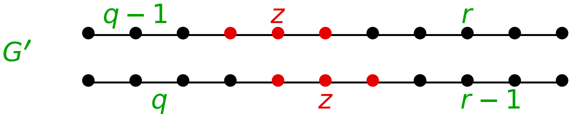

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- Show $s'(P_n, H)$ is indep of z when z is far from ends.

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Lem. Let L be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with k vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k+1 \leq h \leq n-k$, the value $s'(P_n, L)$ is the same.

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$$s'(P_n, L) = s'(C_n, L) + \sum_{i,j} s'(P_{n-\ell_i-\ell_j-2}, L - P_{\ell_i} - P_{\ell_j}) - \sum_i (\ell_i - 1) s'(P_{n-\ell_i-2}, L - P_{\ell_i})$$

w_h is far enough from the ends to use induction hyp. ■

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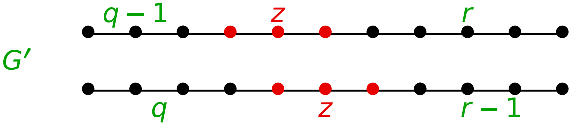

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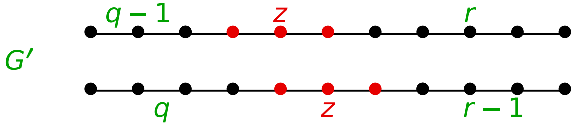

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Same k-deck, and Sharpness Claim

Lem. Let L be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with k vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k+1 \le h \le n-k$, the value $s'(P_n, L)$ is the same.

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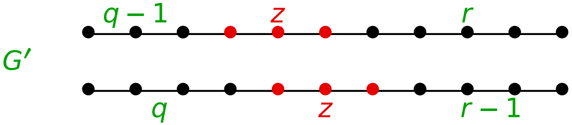

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Full solution for maxdegree 2:

Thm. If $\Delta(G) = 2$, and two largest components have m and m' vertices, then G is reconstructible from $\mathcal{D}_k(G)$ iff $k \ge \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}$, where $\epsilon \in \{0,1\}$ and $\epsilon' \in \{0,1,2\}$. ($\epsilon = 1$ if largest component is P_m.)

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Thm. KW'21: If $\forall \ell$, graphs with $n \geq \ell + 2$ having one component with $n - \ell + 1$ vertices and the rest isolated are ℓ-reconstructible, then the original **RC** holds.

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Following Manvel [1974] and Greenwell–Hemminger [1973].

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Now $s(F, G) = \sum_{H \in \mathcal{F}} s(F, H) m(H, G)$, solve for $m(F, G)$. ■

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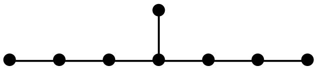

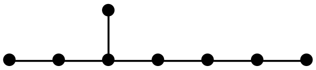

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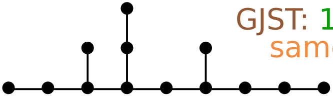

GJST: 13 vertices,
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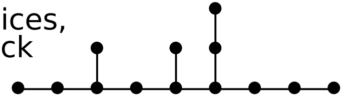

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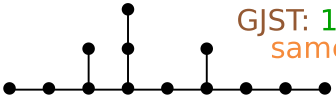

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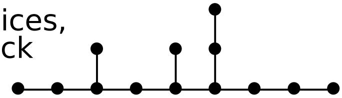

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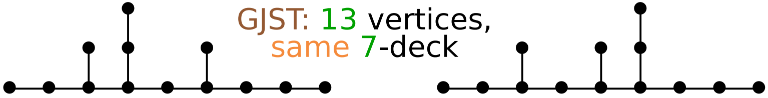

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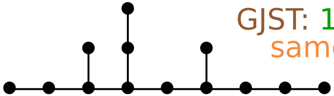

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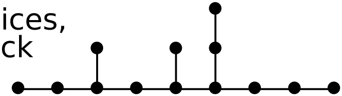

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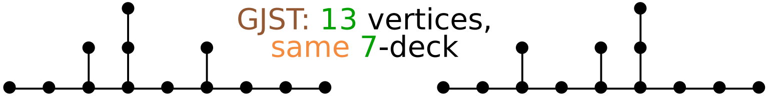

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Ours, only for $\ell = 3$, takes 48 pages (uses rooted trees).

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Lem. In a graph G with girth at least $2j + 2$, every j-vine is contained in a unique maximal j-vine.
(Thus j-centers correspond to maximal j-vines.)

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Non-acyclic reconstruction has j-centers $\ge b$.
Use $b \le a$ to show $n \le 2\ell + 1$ if both types occur.

Lem. In a graph G with girth at least $2j + 2$, every j-vine is contained in a unique maximal j-vine.
(Thus j-centers correspond to maximal j-vines.)

Pf. Girth $\ge 2j+2 \Rightarrow$ a j-vine B is an induced subgraph.
Diameter $2j \Rightarrow B$ has unique center v.
No j-vine with another center contains B.
$\therefore$ unique maximal j-vine containing B is the j-ball at v. ■

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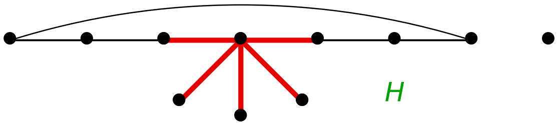

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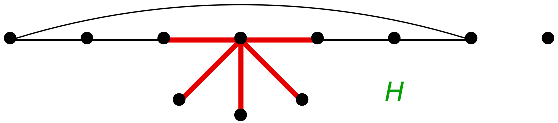

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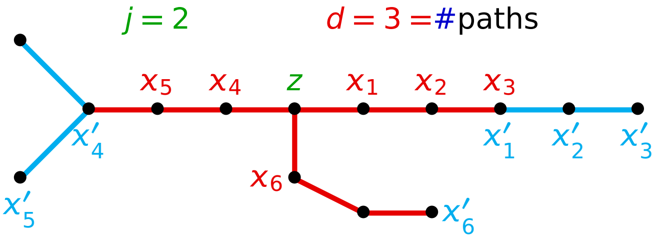

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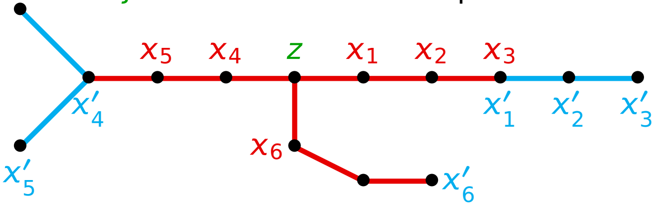

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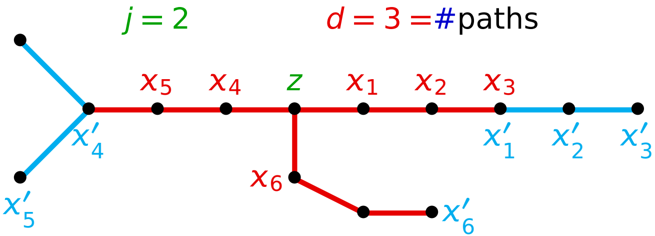

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Pf. j-centers not adj. to z mark vertices outside C, and each outside vert. (at most ℓ) is marked at most once. ■

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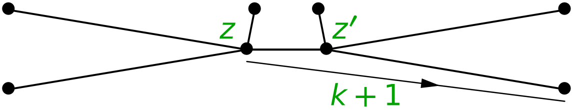

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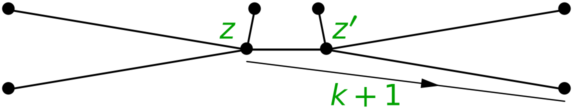

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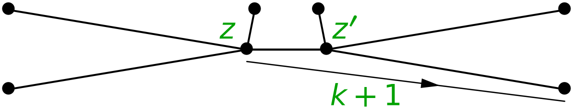

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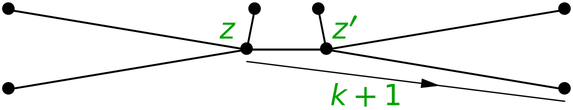

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Ques. What is the max n such that every n-vertex complete multipartite G is determined by its k-deck?
(Nýdl [1985]: it is between $k \ln(k/2)$ and $(k+1)2^{k-1}$.)

Open Questions

Conj. Trees with $n \geq 2\ell + 1$ are ℓ-reconstructible.
(Not for $(5,2)$ or $(13,6)$; OK for $n \geq 9\ell + O(\sqrt{\ell})$ [GJST].)

Conj. Connectedness is ℓ-recognizable for $n \geq 2\ell+1$.
(True for $\ell = 3$ [KNWZ'20]; OK for $n \geq 10\ell$ [GJST].)

Ques. Are d-regular graphs ℓ-reconstructible?
(3-regular graphs are 2-reconstructible [KNWZ'21].)

Ques. Are bipartite graphs 2-reconstructible ($n \geq 6$)?
(For 1-reconstructibility, see [Monikandan–Balakumar'12].)

Ques. Least k so degree list determined by $\mathcal{D}_k(G)$?
(Between $\Omega(\log n)$ and $\sqrt{2n\log(2n)}$ [GJST].)

Ques. What is the max n such that every n-vertex complete multipartite G is determined by its k-deck?
(Nýdl [1985]: it is between $k\ln(k/2)$ and $(k+1)2^{k-1}$.)

Prob. Find thresholds on n for ℓ-reconstructibility of connectivity, matching number, $\chi(G)$, planarity, etc.