Reconstruction from the Subgraphs Obtained by Deleting ℓ Vertices

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slides and papers on preprint page from
https://faculty.math.illinois.edu/

Joint work with
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(and earlier with) Hannah Spinoza

The Classical Problem

Def. A card of a graph G is an induced subgraph $G - v$.
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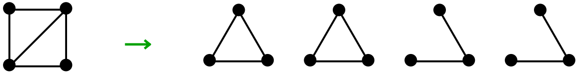

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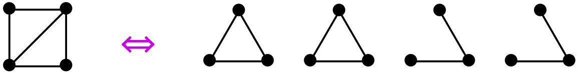

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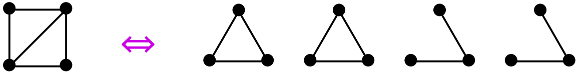

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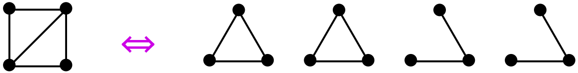

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- Surveys: Bondy-Hemminger ['77], Lauri ['87], Ellingham ['88], Manvel ['88], Bondy ['91], Lauri ['97], Nýdl ['01], Maccari-Rueda-Viazzi ['02], Asciak-Francalanza-Lauri-Myrvold ['10]

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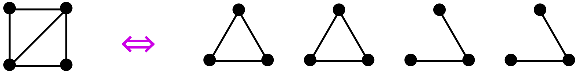

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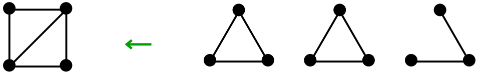

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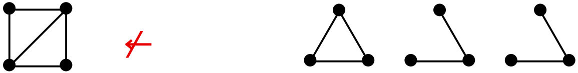

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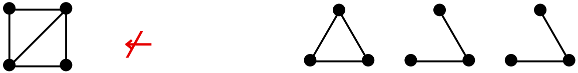

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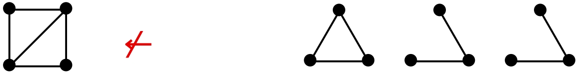

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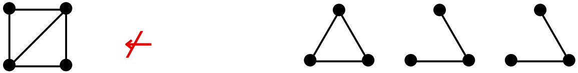

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Conj. $n/2+2$ is the max of $\mathrm{rn}(G)$ over n-vertex graphs.

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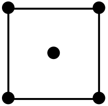

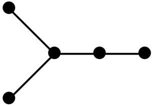

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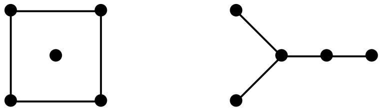

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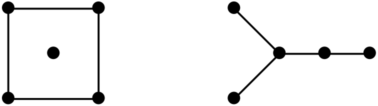

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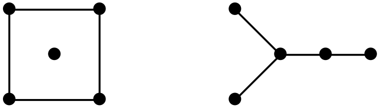

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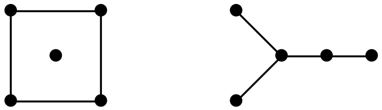

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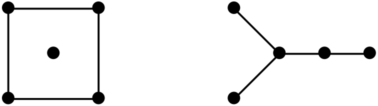

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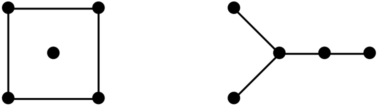

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- Another way to ask how hard it is to reconstruct G.

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Knowing f, we find $q_1, \dots, q_r$. ■

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Theme: For special families or parameters, find thresholds c such that $n \ge c\ell$ implies ℓ-reconstructibility.

Almost All Graphs

Lem. (Müller [1976]) Fix $\epsilon > 0$. For almost every graph G, the induced subgraphs with at least $(1+\epsilon)\frac{|V(G)|}{2}$ vertices are good, meaning they have no nontrival automorphisms and are pairwise nonisomorphic.

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Cor. Among n-vertex graphs, the fraction that are reconstructible from the subgraphs obtained by deleting $(1-\epsilon)\frac{n}{2}$ vertices tends to 1 as $n \to \infty$.

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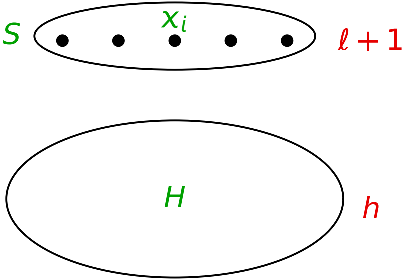

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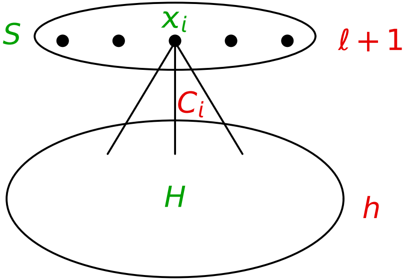

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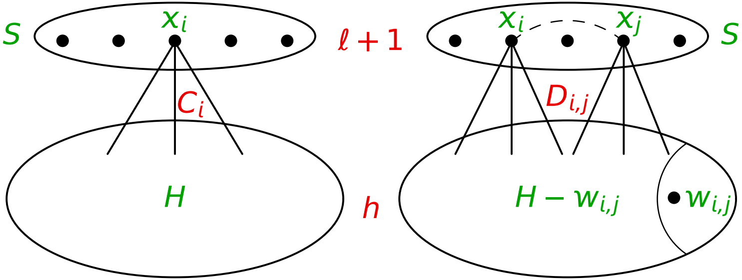

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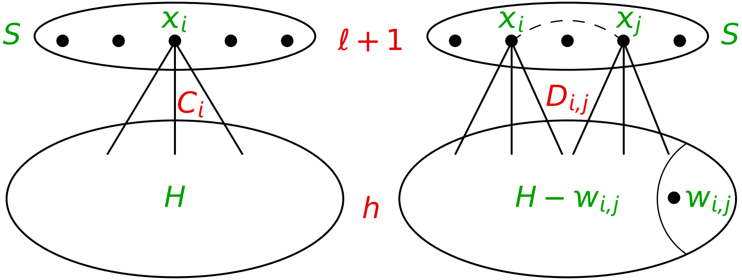

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Claim: G is reconstructible from $C \cup D$.

Reconstructing G from $C \cup D$.

Idea: H is the only h-vertex subgraph H' appearing in $\ell + 1$ cards in $C \cup D$. This identifies H and all C_i, the vertex x_i in C_i, and the edges from x_i to $V(H)$.

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If $V(H') \cap S = \{x_i\}$, then H' is in one card in C and can be in cards $D_{i,j}$ as $H' = D_{i,j} - x_j = G[V(H) + x_i - w_{i,j}]$.
If $w_{i,j}$ is not the same for all j, then H' is in $\leq \ell$ cards.

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Note $H = C_i - x_i$. For $w \in V(H)$, a card $D' \in D$ contains both $C_i - w$ and $C_j - w$ only when $D' = D_{i,j}$ and $w = w_{i,j}$. ■

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Ex. $\mathcal{D}_\ell(P_{2\ell}) = \mathcal{D}_\ell(C_{\ell+1} + P_{\ell-1})$, so the threshold on n for ℓ-reconstructibility of connectedness is at least $2\ell + 1$.

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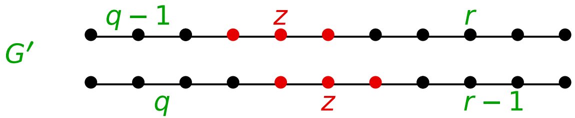

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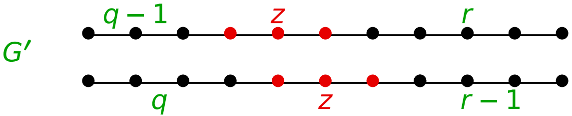

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- Show $s'(P_n, H)$ is indep of z when z is far from ends.

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Lem. Let L be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with k vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k+1 \leq h \leq n-k$, the value $s'(P_n, L)$ is the same.

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$s'(C_n, L)$ omits copies of L in P_n using w_1 and w_n.
$s'(C_n, L)$ counts unwanted subgraphs using $w_n w_1$.

With $L_i = L - V(P_{\ell_i})$ and $L_{i,j} = L - V(P_{\ell_i} + P_{\ell_j})$, we have

$$s'(P_n, L) = s'(C_n, L) + \sum_{i,j} s'(P_{n-(\ell_i+\ell_j+2)}, L_{i,j}) - \sum_i (\ell_i - 1) s'(P_{n-(\ell_i+2)}, L_i)$$

w_h is far enough from the ends to use induction hyp. ■

Same k-deck, and Sharpness Claim

Lem. Let L be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with k vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k+1 \le h \le n-k$, the value $s'(P_n, L)$ is the same.

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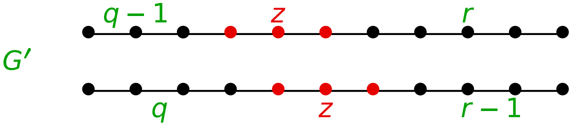

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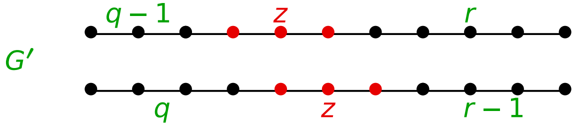

By summing over cases, also (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1).
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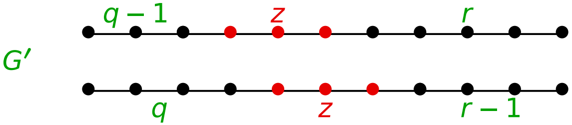

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Full solution for maxdegree 2:

Thm. If $\Delta(G) = 2$, and two largest components have m and m' vertices, then G is reconstructible from $\mathcal{D}_k(G)$ iff $k \ge \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}$, where $\epsilon \in \{0, 1\}$ and $\epsilon' \in \{0, 1, 2\}$. ($\epsilon = 1$ if largest component is P_m.)

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Now $s(F, G) = \sum_{H \in \mathcal{F}} s(F, H) m(H, G)$, solve for $m(F, G)$. ∎

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(1) Recognition: Every graph with deck $\mathcal{D}$ lies in $\mathcal{F}$.
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Conj. Nýdl'81: Trees with $n \geq 2\ell + 1$ are weakly ℓ-reconstructible.

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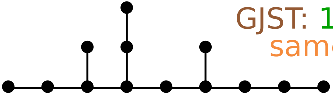

GJST: 13 vertices,
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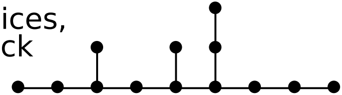

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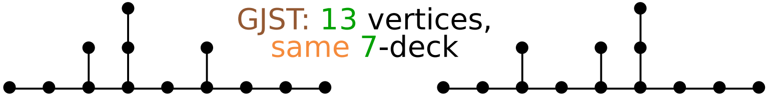

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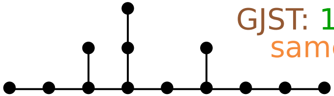

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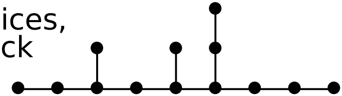

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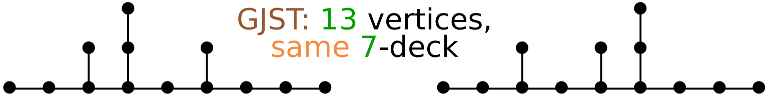

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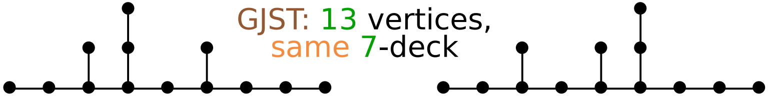

Thm. GJST'21: n-vertex trees are reconstructible from their k-decks when $k \geq \frac{8}{9}n + \frac{4}{9}\sqrt{8n+5} + 1$.

Cor. Trees ℓ-reconstr'ble for $n \geq 9\ell + 24\sqrt{2\ell} + o(\sqrt{\ell})$.

Thm. KNWZ: Trees are 3-reconstr'ble for $n \geq 22$.

The case $\ell = 3$ of GJST covers $n \geq 194$.
Ours, only for $\ell = 3$, takes 48 pages (uses rooted trees).

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Cor. To count k-centers, count maximal k-vines.

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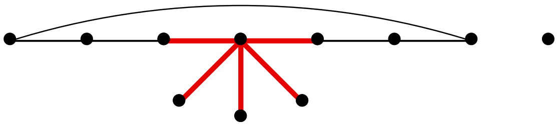

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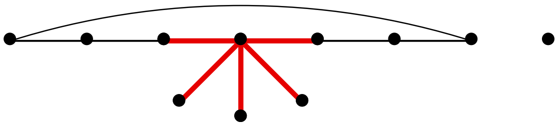

Now $(n-\ell+1)+(n-\ell)-3 \leq n-1 \Rightarrow n \leq 2\ell+1$. ■

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Def. Let d = #edge-disjoint paths of length $\hat{k}$ emanating from the center z in a short card C.

Def. Marking argument: For short card C from forest F, each k-center x other than the center z marks a vertex x' at distance k from x (direction away from z).

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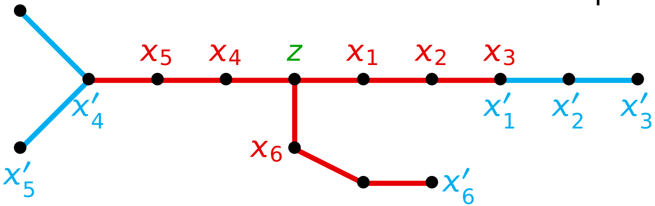

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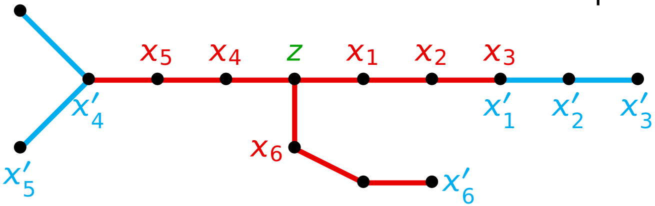

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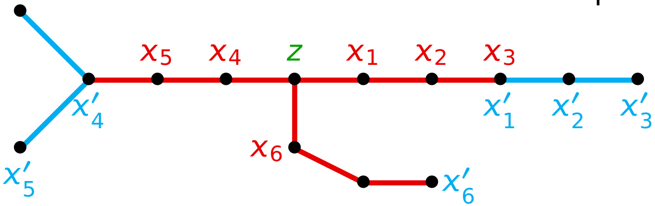

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Pf. k-centers not adj. to z must mark verts. outside C, and each outside vertex is marked at most once. ∎

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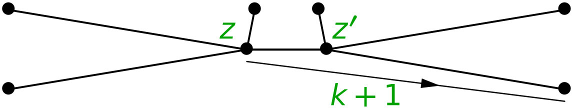

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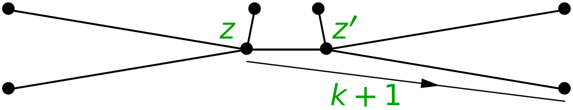

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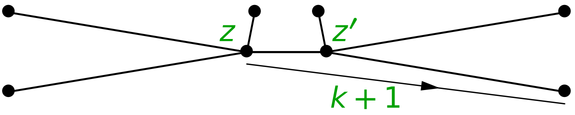

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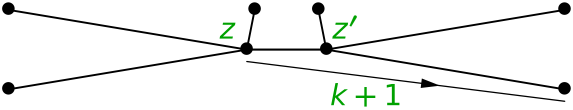

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Now $n-\ell+1 \leq 2+\ell \Rightarrow n \leq 2\ell+1$. ∎

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$\therefore$ $n-\ell+1+d-2 \leq d+\ell$. Hence $n \leq 2\ell+1$. ■

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Prob. Find thresholds on n for ℓ-reconstructibility of connectivity, matching number, $\chi(G)$, planarity, etc.