Reconstruction of graphs from their $k$-vertex Induced Subgraphs

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Joint work with
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A Matrix Question

Is a symmetric matrix determined by the isomorphism types of its principal submatrices (simultaneous permutation of rows and columns allowed)?
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Such matrices, with the two values written as 0 and 1 and the diagonal constant as 0, are just the adjacency matrices of graphs, and the principal submatrices correspond to the induced subgraphs.
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Hence the matrix question reduces to the classical Reconstruction Problem in graph theory.
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The deck of a graph is the multiset of its cards.
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![Diagram of cards and decks]

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- **Surveys:** Bondy-Hemminger ['77], Lauri ['87], Ellingham ['88], Manvel ['88], Bondy ['91], Lauri ['97], Nýdl ['01], Maccari-Rueda-Viazzi ['02], Asciak-Francalanza-Lauri-Myrvold['10]
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**Ex.** $K^-_4$ is determined by three of its cards.
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Ex. $K_4$ is determined by three cards. Which three?
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**Ex.** $K_4^-$ is determined by three cards. **Which** three?

**Def.** Harary-Plantholt [1985]: The **reconstruction number** $\text{rn}(G)$ is the least number of cards that determine $G$. 
Reconstruction numbers

- (Myrvold [1989]) \( \text{rn}(G) = 3 \) for disconnected graphs with (at least) two nonisomorphic components.

- (Myrvold [1990]) \( \text{rn}(G) = 3 \) for trees with \( \geq 5 \) vertices.
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\( K_{n/2,n/2} \) shares \( \frac{n}{2} + 1 \) cards with \( K_{n/2+1,n/2-1} \).
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$K_{n/2, n/2}$ shares $\frac{n}{2} + 1$ cards with $K_{n/2+1, n/2-1}$.

**Conj.** (Harary–Plantholt [1985]) $\text{rn}(G) \leq \frac{n}{2} + 2$, with equality only for $K_{n/2, n/2}$ and $2K_{n/2}$ when $n > 4$. 
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• Much work has studied the number of cards needed to reconstruct $|E(G)|$: Myrvold [1992], Woodall [2015], Monikandan–Balakumar [2016], Brown–Fenner [2018].
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- Much work has studied the number of cards needed to reconstruct \( |E(G)| \): Myrvold [1992], Woodall [2015], Monikandan–Balakumar [2016], Brown–Fenner [2018].

Groenland–Guggiari–Scott [2018arXiv] proved that \( |E(G)| \) is determined by any \( n - \sqrt{n}/20 \) cards when \( n \) is large.
Another Direction

**Conj.** Kelly [1957], Manvel [1969]: For \( \ell \in \mathbb{N} \), \( \exists M_\ell \in \mathbb{N} \) s.t. \( |\mathcal{V}(G)| \geq M_\ell \) \( \implies \) \( G \) is reconstructible from the deck obtained by deleting \( \ell \) vertices.
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![Diagram](image)
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**Obs.** $\mathcal{D}_k(G)$ determines $\mathcal{D}_{k-1}(G)$.

**Pf.** Each graph in $\mathcal{D}_{k-1}$ arises $n - k + 1$ times by deleting one vertex from a graph in $\mathcal{D}_k(G)$. □
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- Another way to ask how hard it is to reconstruct \( G \).
What is known?

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**Thm.** Nýdl [1992]: For $\epsilon > 0$, $\exists$ arb. large graphs not $\epsilon n$-reconstructible. $\therefore M_\ell$ grows superlinearly.
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More Results

Connectedness is not $\frac{n}{2}$-reconstructible, but . . .
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Connectedness is not \( n/2 \)-reconstructible, but . . .

**Thm.** Müller [1976], S-W [2019]: For \( \ell \leq (1 - o(1))\frac{n}{2} \), almost every graph is \( \ell \)-reconstr’bl.
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Generalizes Chinn ’71, Müller ’76, Bollobás ’90 for $\ell = 1$. 
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**Thm.** Spinoza-West’19: When \( \text{maxdeg}(G) \leq 2 \), we know \( \max\{\ell : G \text{ is } \ell\text{-reconstructible}\} \).
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**Thm.** Kostochka-Nahvi-West-Zirlin [2021+]: For \( n \geq 2\ell + 3 \), \( n \)-vertex acyclic graphs are \( \ell \)-recognizable.
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Generalizes Chinn ’71, Müller ’76, Bollobás ’90 for $\ell = 1$.

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**Thm.** KNWZ’21: 3-regular graphs are 2-reconstr’ble.
Warmup: Complete Multipartite Graphs

(Manvel [1974]) $\mathcal{D}_{\Delta(G)+2}(G)$ determines the degree list.
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Prop. $G=K_{n_1,\ldots,n_r}$ with all $n_i \leq m$ is determined by $\mathcal{D}_{m+1}(G)$
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**Prop.** $G = K_{n_1, \ldots, n_r}$ with all $n_i \leq m$ is determined by $\mathcal{D}_{m+1}(G)$

**Pf.** $P_3 \notin \mathcal{D}_3(\overline{G})$ $\iff$ $G$ is complete multipartite.
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With $\Delta(\bar{G}) < m$, by Manvel’s result we can reconstruct the degree list, which determines $G$ for such $G$.  ■
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**Thm.** Complete \( r \)-partite \( G \) is determined by \( \mathcal{D}_{r+1}(G) \)
Warmup: Complete Multipartite Graphs

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**Thm.** Complete $r$-partite $G$ is determined by $\mathcal{D}_{r+1}(G)$

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\[ f(x) = \prod_{j=1}^{r}(x - q_j). \]

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Note $f(x) = \sum_{i=0}^{r} (-1)^i s_i x^{r-i}$, where $s_i$ is the sum of products of $i$ choices from $q_1, \ldots, q_r$. 

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Also $s_i = \# i$-cards that are $K_i$, so $\mathcal{D}_i(G)$ determines $s_i$. 
Warmup: Complete Multipartite Graphs

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With part-sizes \( q_1, \ldots, q_r \), let \( f(x) = \prod_{j=1}^{r} (x - q_j) \).
Note \( f(x) = \sum_{i=0}^{r} (-1)^i s_i x^{r-i} \), where \( s_i \) is the sum of products of \( i \) choices from \( q_1, \ldots, q_r \).
Also \( s_i = \# i \)-cards that are \( K_i \), so \( \mathcal{D}_i(G) \) determines \( s_i \).
Knowing \( f \), we find \( q_1, \ldots, q_r \).
Almost All Graphs

**Lem.** (Müller [1976]) Fix $\epsilon > 0$. For almost every graph $G$, the induced subgraphs with at least $(1 + \epsilon)\frac{|V(G)|}{2}$ vertices are good, meaning they have no nontrivial automorphisms and are pairwise nonisomorphic.
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**Thm.** If the subgraphs obtained by deleting $\ell + 1$ verts are **good**, then $G$ is reconstructible from some set of $\binom{\ell+2}{2}$ subgraphs obtained by deleting $\ell$ vertices.
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**Thm.** If the subgraphs obtained by deleting $\ell + 1$ verts are good, then $G$ is reconstructible from some set of $\binom{\ell + 2}{2}$ subgraphs obtained by deleting $\ell$ vertices.

**Cor.** Among $n$-vertex graphs, the fraction that are reconstructible from the subgraphs obtained by deleting $(1 - \varepsilon)\frac{n}{2}$ vertices tends to 1 as $n \to \infty$. 
Using Some of the Deck

**Thm.** $\mathcal{D}_{n-\ell-1}$ good $\Rightarrow$ $\mathcal{D}_{n-\ell}$ determines $G$. 
Using Some of the Deck

**Thm.** \( D_{n-\ell-1} \) good \( \implies \) \( D_{n-\ell} \) determines \( G \).

**Pf.** Let \( n = |V(G)| \). Fix \( S = \{x_1, \ldots, x_{\ell+1}\} \subseteq V(G) \).
Let \( H = G - S \) and \( h = |V(H)| = n - \ell - 1 \).
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Let $C_i = G - (S - \{x_i\})$ (deleting $\ell$) and $C = \{C_i : x_i \in S\}$.
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For \( x_i, x_j \in S \), let \( D_{i,j} = G - (S - \{ x_i, x_j \}) - w_{i,j} \), where \( w_{i,j} \in V(H) \). Let \( D = \{ D_{i,j} : x_i, x_j \in S \} \).
Using Some of the Deck

**Thm.** \( D_{n-l-1} \) good \( \Rightarrow \) \( D_{n-l} \) determines \( G \).

**Pf.** Let \( n = |V(G)| \). Fix \( S = \{x_1, \ldots, x_{l+1}\} \subseteq V(G) \).
Let \( H = G - S \) and \( h = |V(H)| = n - l - 1 \).
Let \( C = G - (S - \{x_i\}) \) (deleting \( l \)) and \( C = \{C_i: x_i \in S\} \).

For \( x_i, x_j \in S \), let \( D_{i,j} = G - (S - \{x_i, x_j\}) - w_{i,j} \), where \( w_{i,j} \in V(H) \). Let \( D = \{D_{i,j}: x_i, x_j \in S\} \).

**Claim:** \( G \) is reconstructible from \( C \cup D \).
Reconstructing $G$ from $C \cup D$. 

**Idea:** $H$ is the only $h$-vertex subgraph $H'$ appearing in $\ell + 1$ cards in $C \cup D$. This identifies $H$ and all $C_i$, the vertex $x_i$ in $C_i$, and the edges from $x_i$ to $V(H)$. 
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If $|V(H') \cap S| \geq 3$, then $H'$ appears in no card in $C \cap D$.

If $V(H') \cap S = \{x_i, x_j\}$, then $H'$ appears only in $D_{i,j}$.

If $V(H') \cap S = \{x_i\}$, then $H'$ is in one card in $C$ and can be in cards $D_{i,j}$ as $H' = D_{i,j} - x_j = G[V(H) + x_i - w_{i,j}]$. If $w_{i,j}$ is not the same for all $j$, then $H'$ is in $\leq \ell$ cards.

If $V(H') \cap S = \emptyset$, then $H' = H$, in all $\ell + 1$ cards of $C$. 
Reconstructing $G$ from $C \cup D$.

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- If $w_{i,j}$ is not the same for all $j$, then $H'$ is in $\leq \ell$ cards.
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**Idea:** $H$ can also be used to identify the card $D_{i,j}$, which is used to check whether $x_i x_j \in E(G)$. 
Reconstructing $G$ from $C \cup D$.

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**Idea:** $H$ can also be used to identify the card $D_{i,j}$, which is used to check whether $x_i x_j \in E(G)$.

Note $H = C_i - x_i$. For $w \in V(H)$, a card $D' \in D$ contains both $C_i - w$ and $C_j - w$ only when $D' = D_{i,j}$ and $w = w_{i,j}$.
Connectedness is $l$-Reconstructible for Large $n$

**Def.** Let $c(\mathcal{D}) = \#$ of connected cards in a deck $\mathcal{D}$.
Connectedness is $\ell$-Reconstructible for Large $n$

**Def.** Let $c(\mathcal{D}) = \#$ of connected cards in a deck $\mathcal{D}$.

Suppose $G$ connected, $H$ disconn., same $(n - \ell)$-deck $\mathcal{D}$. 
Connectedness is $\ell$-Reconstructible for Large $n$

**Def.** Let $c(D) = \# \text{ of connected cards in a deck } D$.

Suppose $G$ connected, $H$ disconn., same $(n - \ell)$-deck $D$.

$G \Rightarrow c(D) \geq 1$, so $H$ has component $C$ with $|V(C)| \geq n - \ell$. 
Connectedness is $\ell$-Reconstructible for Large $n$

**Def.** Let $c(\mathcal{D}) = \#$ of connected cards in a deck $\mathcal{D}$.

Suppose $G$ connected, $H$ disconn., same $(n - \ell)$-deck $\mathcal{D}$.

$G \Rightarrow c(\mathcal{D}) \geq 1$, so $H$ has component $C$ with $|V(C)| \geq n - \ell$.

Let $|V(C)| = n - p$, so $H \Rightarrow c(\mathcal{D}) \leq \binom{n-p}{\ell-p} \leq \binom{n-1}{\ell-1}$.

(Keep only vertices from $C$, discarding $\ell - p$ of them.)
Connectedness is $l$-Reconstructible for Large $n$

**Def.** Let $c(\mathcal{D}) = \#$ of connected cards in a deck $\mathcal{D}$.

Suppose $G$ connected, $H$ disconn., same $(n - l)$-deck $\mathcal{D}$. $G \Rightarrow c(\mathcal{D}) \geq 1$, so $H$ has component $C$ with $|V(C)| \geq n - l$.

Let $|V(C)| = n - p$, so $H \Rightarrow c(\mathcal{D}) \leq \binom{n-p}{l-p} \leq \binom{n-1}{l-1}$.

(Keep only vertices from $C$, discarding $l - p$ of them.)

Also $H \Rightarrow \hat{c}(\mathcal{D}) \geq \binom{n-1}{l}$, where $\hat{c}(\mathcal{D}) = \#$ cards having a component of order $\leq l$. (Keep a vertex $x$ outside $C$.)
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Also $H \Rightarrow \hat{c}(\mathcal{D}) \geq \binom{n-1}{l}$, where $\hat{c}(\mathcal{D}) = \#$ cards having a component of order $\leq l$. (Keep a vertex $x$ outside $C$.)

**Idea:** From $G$ get lower bd on $c(\mathcal{D})$ & upper bd on $\hat{c}(\mathcal{D})$, leads to upper bound on $n$. 
**Connectedness is \( l \)-Reconstructible for Large \( n \)**

**Def.** Let \( c(\mathcal{D}) = \# \) of connected cards in a deck \( \mathcal{D} \).

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\[ G \Rightarrow c(\mathcal{D}) \geq 1, \text{ so } H \text{ has component } C \text{ with } |V(C)| \geq n-l. \]

Let \(|V(C)| = n-p\), so \( H \Rightarrow c(\mathcal{D}) \leq \binom{n-p}{l-p} \leq \binom{n-1}{l-1}. \)

(Keep only vertices from \( C \), discarding \( l-p \) of them.)

Also \( H \Rightarrow \hat{c}(\mathcal{D}) \geq \binom{n-1}{l} \), where \( \hat{c}(\mathcal{D}) = \# \) cards having a component of order \( \leq l \). (Keep a vertex \( x \) outside \( C \).)

**Idea:** From \( G \) get lower bd on \( c(\mathcal{D}) \) & upper bd on \( \hat{c}(\mathcal{D}) \), leads to upper bound on \( n \).

Let \( T \) be a spanning tree of \( G \), and let \( \mathcal{D}' = \mathcal{D}_{n-l}(T) \).
Connectedness is $l$-Reconstructible for Large $n$

**Def.** Let $c(D) = \#$ of connected cards in a deck $D$.

Suppose $G$ connected, $H$ disconn., same $(n - l)$-deck $D$.

$G \Rightarrow c(D) \geq 1$, so $H$ has component $C$ with $|V(C)| \geq n - l$.

Let $|V(C)| = n - p$, so $H \Rightarrow c(D) \leq \binom{n-p}{l} \leq \binom{n-1}{l-1}$.

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**Idea:** From $G$ get lower bd on $c(D)$ & upper bd on $\hat{c}(D)$, leads to upper bound on $n$.

Let $T$ be a spanning tree of $G$, and let $\mathcal{D}' = \mathcal{D}_{n-l}(T)$.

- $c(D) \geq c(D')$ and $\hat{c}(D) \leq \hat{c}(D')$ (using same vertices).
Connectedness is \( \ell \)-Reconstructible for Large \( n \)

**Def.** Let \( c(\mathcal{D}) \) = \# of connected cards in a deck \( \mathcal{D} \).

Suppose \( G \) connected, \( H \) disconn., same \( (n - \ell) \)-deck \( \mathcal{D} \).

\[ G \Rightarrow c(\mathcal{D}) \geq 1, \text{ so } H \text{ has component } C \text{ with } |V(C)| \geq n - \ell. \]

Let \( |V(C)| = n - p \), so \( H \Rightarrow c(\mathcal{D}) \leq \binom{n-p}{\ell-p} \leq \binom{n-1}{\ell-1}. \)

(Keep only vertices from \( C \), discarding \( \ell - p \) of them.)

Also \( H \Rightarrow \hat{c}(\mathcal{D}) \geq \binom{n-1}{\ell} \), where \( \hat{c}(\mathcal{D}) = \# \text{cards having a component of order } \leq \ell \). (Keep a vertex \( x \) outside \( C \).)

**Idea:** From \( G \) get lower bd on \( c(\mathcal{D}) \) & upper bd on \( \hat{c}(\mathcal{D}) \), leads to upper bound on \( n \).

Let \( T \) be a spanning tree of \( G \), and let \( \mathcal{D}' = \mathcal{D}_{n-\ell}(T) \).

- \( c(\mathcal{D}) \geq c(\mathcal{D}') \) and \( \hat{c}(\mathcal{D}) \leq \hat{c}(\mathcal{D}') \) (using same vertices).

Get lower bd on \( c(\mathcal{D}') \) & upper bd on \( \hat{c}(\mathcal{D}') \) instead.
Cards in $D'$

Let $t$ be the number of leaves in $T$. 
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$. Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{k}$. 
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$.
Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{\ell}$.
Thus $\frac{t(\ell)}{\ell!} = \binom{t}{\ell} \leq c(\mathcal{D}) \leq \binom{n-1}{\ell-1} \leq \binom{n}{\ell-1} = \frac{n(\ell-1)}{(\ell-1)!}$. 
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$.
Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{l}$.

Thus $\frac{t(l)}{l!} = \binom{t}{l} \leq c(\mathcal{D}) \leq \binom{n-1}{l-1} \leq \binom{n}{l-1} = \frac{n(l-1)}{(l-1)!}$.

Hence $(t - l)^l < ln^{l-1}$, yielding $t < n\left(\frac{2l}{n}\right)^{1/l}$ for $n > l^2$. 
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$. Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{\ell}$.

Thus \( \frac{t(\ell)}{\ell !} = \binom{t}{\ell} \leq c(\mathcal{D}) \leq \binom{n-1}{\ell-1} \leq \binom{n}{\ell-1} = \frac{n(\ell-1)}{(\ell-1)!} \).

Hence $(t - \ell)^\ell < \ell n^{\ell-1}$, yielding $t < n \left(\frac{2\ell}{n}\right)^{1/\ell}$ for $n > \ell^2$.

Every card in $\mathcal{D}'$ counted by $\hat{c}(\mathcal{D}')$ has a tree component $F$ with $|V(F)| \leq \ell$, cut off by at most $\ell$ vertices.
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$. Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{l}$.

Thus $\frac{t(l)}{l!} = \binom{t}{l} \leq c(\mathcal{D}) \leq \binom{n-1}{l-1} \leq \binom{n}{l-1} = \frac{n(l-1)}{(l-1)!}$.

Hence $(t - l)^l < l n^{l-1}$, yielding $t < n(2l/n)^{1/l}$ for $n > l^{l^2}$.

Every card in $\mathcal{D}'$ counted by $\hat{c}(\mathcal{D}')$ has a tree component $F$ with $|V(F)| \leq l$, cut off by at most $l$ vertices.

If $F$ is cut off by $j$ vertices, then $F$ is a component in fewer than $\binom{n}{l-j}$ cards.
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$. Deleting leaves doesn’t disconnect, so $c(D) \geq c(D') \geq \binom{t}{l}$. Thus $\frac{t!}{l!} = \binom{t}{l} \leq c(D) \leq \binom{n-1}{l-1} \leq \binom{n}{l-1} = \frac{n(n-1)}{(l-1)!}$. Hence $(t - l)^l < ln^{l-1}$, yielding $t < n(2l/n)^{1/l}$ for $n > l^{l^2}$.

Every card in $\mathcal{D}'$ counted by $\hat{c}(\mathcal{D}')$ has a tree component $F$ with $|V(F)| \leq l$, cut off by at most $l$ vertices.

If $F$ is cut off by $j$ vertices, then $F$ is a component in fewer than $\binom{n}{l-j}$ cards. Let $b_j = \#$ such subtrees $F$. 
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$. Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{\ell}$. Thus $\frac{t(\ell)}{\ell!} = \binom{t}{\ell} \leq c(\mathcal{D}) \leq \binom{n-1}{\ell-1} \leq \binom{n}{\ell-1} = \frac{n(\ell-1)}{(\ell-1)!}$. Hence $(t - \ell)^{\ell} < \ell n^{\ell-1}$, yielding $t < n(2\ell/n)^{1/\ell}$ for $n > \ell^2$.

Every card in $\mathcal{D}'$ counted by $\hat{c}(\mathcal{D}')$ has a tree component $F$ with $|V(F)| \leq \ell$, cut off by at most $\ell$ vertices. If $F$ is cut off by $j$ vertices, then $F$ is a component in fewer than $\binom{n}{\ell-j}$ cards. Let $b_j = \#$such subtrees $F$. Hence $\hat{c}(\mathcal{D}) \leq \hat{c}(\mathcal{D}') \leq \sum_{j=1}^{\ell} b_j \binom{n}{\ell-j}$. 
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$. Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \ge c(\mathcal{D}') \ge \binom{t}{l}$. Thus \( \frac{t(l)}{l!} = \binom{t}{l} \le c(\mathcal{D}) \le \binom{n-1}{l-1} \le \binom{n}{l-1} = \frac{n(l-1)}{(l-1)!} \).

Hence \((t - l)^l < ln^{l-1}\), yielding \(t < n(2l/n)^{1/l}\) for \(n > l^{l^2}\).

Every card in $\mathcal{D}'$ counted by $\hat{c}(\mathcal{D}')$ has a tree component $F$ with $|V(F)| \le l$, cut off by at most $l$ vertices.

If $F$ is cut off by $j$ vertices, then $F$ is a component in fewer than $\binom{n}{l-j}$ cards. Let $b_j = \#$such subtrees $F$.

Hence $\hat{c}(\mathcal{D}) \le \hat{c}(\mathcal{D}') \le \sum_{j=1}^{l} b_j \binom{n}{l-j}$.

We claim: $b_j \binom{n}{l-j} \le \frac{l}{2} n^{l-1} t$. 
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$. Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{l}$. Thus $\frac{t(l)}{l!} = \binom{t}{l} \leq c(\mathcal{D}) \leq \binom{n-1}{l-1} \leq \binom{n}{l-1} = \frac{n(n-1)}{(l-1)!}$.

Hence $(t - l)^l < l n^{l-1}$, yielding $t < n(2l/n)^{1/l}$ for $n > l^{l^2}$.

Every card in $\mathcal{D}'$ counted by $\hat{c}(\mathcal{D}')$ has a tree component $F$ with $|V(F)| \leq l$, cut off by at most $l$ vertices.

If $F$ is cut off by $j$ vertices, then $F$ is a component in fewer than $\binom{n}{l-j}$ cards. Let $b_j = \#$such subtrees $F$.

Hence $\hat{c}(\mathcal{D}) \leq \hat{c}(\mathcal{D}') \leq \sum_{j=1}^{l} b_j \binom{n}{l-j}$.

**We claim:** $b_j \binom{n}{l-j} \leq \frac{l}{2} n^{l-1} t$. (see appendix at end)
Cards in $D'$

Let $t$ be the number of leaves in $T$.
Deleting leaves doesn’t disconnect, so $c(D) \geq c(D') \geq \binom{t}{l}$.

Thus $\frac{t(t(l))}{l!} = \binom{t}{l} \leq c(D) \leq \binom{n-1}{\ell-1} \leq \binom{n}{\ell-1} = \frac{n(n-1)}{(\ell-1)!}$.

Hence $(t - \ell)^l < l n^{l-1}$, yielding $t < n(2\ell/n)^{1/l}$ for $n > \ell^2$.

Every card in $D'$ counted by $\hat{c}(D')$ has a tree component $F$ with $|V(F)| \leq \ell$, cut off by at most $\ell$ vertices.

If $F$ is cut off by $j$ vertices, then $F$ is a component in fewer than $\binom{n}{\ell-j}$ cards. Let $b_j = \#$such subtrees $F$.

Hence $\hat{c}(D) \leq \hat{c}(D') \leq \sum_{j=1}^{\ell} b_j \binom{n}{\ell-j}$.

We claim: $b_j \binom{n}{\ell-j} \leq \frac{\ell}{2} n^{l-1} t$. (see appendix at end)

Thus $(\frac{n-\ell}{\ell})^l < \binom{n-1}{l} \leq \hat{c}(D) \leq \frac{\ell^2}{2} n^{l-1} t < \frac{\ell^2}{2} n^l \left(\frac{2\ell}{n}\right)^{1/l}$.
Cards in $\mathcal{D}'$

Let $t$ be the number of leaves in $T$.
Deleting leaves doesn’t disconnect, so $c(\mathcal{D}) \geq c(\mathcal{D}') \geq \binom{t}{l}$.
Thus $\frac{t(l)}{l!} = \binom{t}{l} \leq c(\mathcal{D}) \leq \binom{n-1}{l-1} \leq \binom{n}{l-1} = \frac{n(l-1)}{(l-1)!}$.
Hence $(t - l)^l < ln^{l-1}$, yielding $t < n(2l/n)^{1/l}$ for $n > l^{l^2}$.

Every card in $\mathcal{D}'$ counted by $\hat{c}(\mathcal{D}')$ has a tree component $F$ with $|V(F)| \leq l$, cut off by at most $l$ vertices.
If $F$ is cut off by $j$ vertices, then $F$ is a component in fewer than $\binom{n}{l-j}$ cards. Let $b_j = \#$ such subtrees $F$.
Hence $\hat{c}(\mathcal{D}) \leq \hat{c}(\mathcal{D}') \leq \sum_{j=1}^{l} b_j \binom{n}{l-j}$.

We claim: $b_j \binom{n}{l-j} \leq \frac{l}{2} n^{l-1} t$. (see appendix at end)

Thus $\left(\frac{n-l}{l}\right)^l < \binom{n-1}{l} \leq \hat{c}(\mathcal{D}) \leq \frac{l^2}{2} n^{l-1} t < \frac{l^2}{2} n^l \left(\frac{2l}{n}\right)^{1/l}$.
Requires $n < 2l^{(l+1)^2}$, roughly $l > \left(\frac{2 \log n}{\log \log n}\right)^{1/2}$. ■
Degree List is 3-Reconstructible (for $n \geq 7$)

Let $a_i$ be the number of vertices with degree $i$ in $G$. Let $\phi(j) =$ total #verts of degree exactly $j$ over all cards.
Degree List is 3-Reconstructible (for $n \geq 7$)

Let $a_i$ be the number of vertices with degree $i$ in $G$. Let $\phi(j)$ = total #verts of degree exactly $j$ over all cards.

**Lem.** Manvel ’74, Taylor ’90: Given $D_k(G)$ (and $\ell = n - k$),

$$\phi(j) = \sum_{i=j}^{j+\ell} a_i \binom{i}{j} \binom{n-1-i}{k-1-j}.$$
Degree List is 3-Reconstructible (for $n \geq 7$)

Let $a_i$ be the number of vertices with degree $i$ in $G$. Let $\phi(j) =$ total #verts of degree exactly $j$ over all cards.

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$$\phi(j) = \sum_{i=j}^{j+\ell} a_i \binom{i}{j} \binom{n-1-i}{k-1-j}.$$ 

**Pf.** To contribute to $\phi(j)$ in a $k$-card, an $i$-vertex must be chosen along with $j$ nbrs and $k - 1 - j$ nonneighbors. 

$\blacksquare$
Degree List is 3-Reconstructible (for $n \geq 7$)

Let $a_i$ be the number of vertices with degree $i$ in $G$. Let $\phi(j) =$ total #verts of degree exactly $j$ over all cards.

**Lem.** Manvel ‘74, Taylor ’90: Given $D_k(G)$ (and $\ell = n - k$),

$$\phi(j) = \sum_{i=j}^{j+\ell} a_i \binom{i}{j} \binom{n-1-i}{k-1-j}. \quad (*)$$

**Pf.** To contribute to $\phi(j)$ in a $k$-card, an $i$-vertex must be chosen along with $j$ nbrs and $k - 1 - j$ nonneighbors.  

**Idea:** If $G$ and $H$ with vertex counts $a_i$ and $b_i$ have same $k$-deck $D_k$, and $c_i = a_i - b_i$, then $0 = \sum_{i=j}^{j+\ell} c_i \binom{i}{j} \binom{n-1-i}{k-1-j} \quad (*)$. 

Degree List is 3-Reconstructible (for $n \geq 7$)

Let $a_i$ be the number of vertices with degree $i$ in $G$. Let $\phi(j) =$ total #verts of degree exactly $j$ over all cards.

**Lem.** Manvel ’74, Taylor ’90: Given $\mathcal{D}_k(G)$ (and $\ell = n - k$),

$$\phi(j) = \sum_{i=j}^{j+\ell} a_i \binom{i}{j} \binom{n-1-i}{k-1-j}.$$ 

**Pf.** To contribute to $\phi(j)$ in a $k$-card, an $i$-vertex must be chosen along with $j$ nbrs and $k-1-j$ nonneighbors. ■

**Idea:** If $G$ and $H$ with vertex counts $a_i$ and $b_i$ have same $k$-deck $\mathcal{D}_k$, and $c_i = a_i - b_i$, then $0 = \sum_{i=j}^{j+\ell} c_i \binom{i}{j} \binom{n-1-i}{k-1-j}$ (\*).

**Thm.** $c_i = 0$ for all $i$ when $n \geq 7$ and $k = n - 3$. 
Degree List is 3-Reconstructible (for $n \geq 7$)

Let $a_i$ be the number of vertices with degree $i$ in $G$. Let $\phi(j) =$ total # verts of degree exactly $j$ over all cards.

**Lem.** Manvel ’74, Taylor ’90: Given $\mathcal{D}_k(G)$ (and $\ell = n - k$),

$$\phi(j) = \sum_{i=j}^{j+\ell} a_i \binom{i}{j} \binom{n-1-i}{k-1-j}.$$

**Pf.** To contribute to $\phi(j)$ in a $k$-card, an $i$-vertex must be chosen along with $j$ nbrs and $k-1-j$ nonneighbors. ■

**Idea:** If $G$ and $H$ with vertex counts $a_i$ and $b_i$ have same $k$-deck $\mathcal{D}_k$, and $c_i = a_i - b_i$, then $0 = \sum_{i=j}^{j+\ell} c_i \binom{i}{j} \binom{n-1-i}{k-1-j}$ ($\ast$).

**Thm.** $c_i = 0$ for all $i$ when $n \geq 7$ and $k = n - 3$.

**Pf. Idea:** Use ($\ast$) for $j = n - 4$ and $j = n - 5$. 
Degree List is 3-Reconstructible (for $n \geq 7$)

Let $a_i$ be the number of vertices with degree $i$ in $G$.
Let $\phi(j)$ = total #verts of degree exactly $j$ over all cards.

**Lem.** Manvel '74, Taylor '90: Given $D_k(G)$ (and $l = n - k$),
$$\phi(j) = \sum_{i=j}^{j+l} a_i \binom{i}{j} \binom{n-1-i}{k-1-j}.$$  

**Pf.** To contribute to $\phi(j)$ in a $k$-card, an $i$-vertex must be
chosen along with $j$ nbrs and $k - 1 - j$ nonneighbors.  ■

**Idea:** If $G$ and $H$ with vertex counts $a_i$ and $b_i$ have same $k$-deck $D_k$, and $c_i = a_i - b_i$, then 0 = $\sum_{i=j}^{j+l} c_i \binom{i}{j} \binom{n-1-i}{k-1-j} \ (\ast)$.  

**Thm.** $c_i = 0$ for all $i$ when $n \geq 7$ and $k = n - 3$.

**Pf. Idea:** Use \((\ast)\) for $j = n - 4$ and $j = n - 5$.
Let $r$ be the largest index with $c_r \neq 0$ (consider cases).
Degree List is 3-Reconstructible (for \( n \geq 7 \))

Let \( a_i \) be the number of vertices with degree \( i \) in \( G \). Let \( \phi(j) = \text{total #verts of degree exactly } j \) over all cards.

**Lem.** Manvel ’74, Taylor ’90: Given \( D_k(G) \) (and \( \ell = n - k \)),
\[
\phi(j) = \sum_{i=j}^{j+\ell} a_i \binom{i}{j} \binom{n-1-i}{k-1-j}.
\]

**Pf.** To contribute to \( \phi(j) \) in a \( k \)-card, an \( i \)-vertex must be chosen along with \( j \) nbrs and \( k - 1 - j \) nonneighbors.

**Idea:** If \( G \) and \( H \) with vertex counts \( a_i \) and \( b_i \) have same \( k \)-deck \( D_k \), and \( c_i = a_i - b_i \), then \( 0 = \sum_{i=j}^{j+\ell} c_i \binom{i}{j} \binom{n-1-i}{k-1-j} \) (**).

**Thm.** \( c_i = 0 \) for all \( i \) when \( n \geq 7 \) and \( k = n - 3 \).

**Pf. Idea:** Use (**) for \( j = n - 4 \) and \( j = n - 5 \).
Let \( r \) be the largest index with \( c_r \neq 0 \) (consider cases).
May assume \( r \geq n - 3 \), since having \( D_k(G) \) and knowing \( a_i \) for \( i \geq k \) determines the degree list (Manvel [1974]).
Connectedness is 3-Reconstructible (for $n \geq 7$)

Consider $G$ and $H$ with $m$ edges & same $(n - 3)$-deck $\mathcal{D}$. 
Connectedness is 3-Reconstructible (for $n \geq 7$)

Consider $G$ and $H$ with $m$ edges & same $(n - 3)$-deck $\mathcal{D}$. $G$ connected $\Rightarrow \mathcal{D}$ has $\geq 2$ connected cards.
Connectedness is 3-Reconstructible (for $n \geq 7$)

Consider $G$ and $H$ with $m$ edges & same $(n - 3)$-deck $\mathcal{D}$.

$G$ connected $\Rightarrow \mathcal{D}$ has $\geq 2$ connected cards $\Rightarrow$

disconnected $H$ has component $C$ with $\geq n - 2$ vertices.
Connectedness is 3-Reconstructible (for $n \geq 7$)

Consider $G$ and $H$ with $m$ edges & same $(n - 3)$-deck $\mathcal{D}$.

$G$ connected $\Rightarrow \mathcal{D}$ has $\geq 2$ connected cards $\Rightarrow$
disconnected $H$ has component $C$ with $\geq n - 2$ vertices.

Since we know $G$ & $H$ have same degree list, $H = C + K_2$. 
Connectedness is 3-Reconstructible (for $n \geq 7$)

Consider $G$ and $H$ with $m$ edges & same $(n - 3)$-deck $\mathcal{D}$.

$G$ connected $\Rightarrow$ $\mathcal{D}$ has $\geq 2$ connected cards $\Rightarrow$
disconnected $H$ has component $C$ with $\geq n - 2$ vertices.

Since we know $G$ & $H$ have same degree list, $H = C + K_2$.

If $C$ has a 1-vertex, then $\mathcal{D}$ has a card with $m - 2$ edges, but $G$ cannot.
Connectedness is 3-Reconstructible (for $n \geq 7$)

Consider $G$ and $H$ with $m$ edges & same $(n-3)$-deck $\mathcal{D}$.

$G$ connected $\Rightarrow \mathcal{D}$ has $\geq 2$ connected cards $\Rightarrow$
disconnected $H$ has component $C$ with $\geq n-2$ vertices.

Since we know $G$ & $H$ have same degree list, $H = C + K_2$.

If $C$ has a 1-vertex, then $\mathcal{D}$ has a card with $m-2$ edges, but $G$ cannot. Thus $G$ & $H$ have exactly two 1-vertices.
Connectedness is 3-Reconstructible (for $n \geq 7$)

Consider $G$ and $H$ with $m$ edges & same $(n - 3)$-deck $D$.

$G$ connected $\Rightarrow D$ has $\geq 2$ connected cards $\Rightarrow$ disconnected $H$ has component $C$ with $\geq n - 2$ vertices.

Since we know $G$ & $H$ have same degree list, $H = C + K_2$.

If $C$ has a 1-vertex, then $D$ has a card with $m - 2$ edges, but $G$ cannot. Thus $G$ & $H$ have exactly two 1-vertices.

Let $x = \#2$-vertices.
Connectedness is 3-Reconstructible (for $n \geq 7$)

Consider $G$ and $H$ with $m$ edges & same $(n - 3)$-deck $\mathcal{D}$. $G$ connected $\Rightarrow \mathcal{D}$ has $\geq 2$ connected cards $\Rightarrow$ disconnected $H$ has component $C$ with $\geq n - 2$ vertices. Since we know $G$ & $H$ have same degree list, $H = C + K_2$. If $C$ has a 1-vertex, then $\mathcal{D}$ has a card with $m - 2$ edges, but $G$ cannot. Thus $G$ & $H$ have exactly two 1-vertices. Let $x = \#2$-vertices. $\mathcal{D}$ has $x$ cards with $m - 3$ edges.
Connectedness is 3-Reconstructible (for $n \geq 7$)

Consider $G$ and $H$ with $m$ edges & same $(n-3)$-deck $D$. $G$ connected $\Rightarrow D$ has $\geq 2$ connected cards $\Rightarrow$ disconnected $H$ has component $C$ with $\geq n-2$ vertices.

Since we know $G$ & $H$ have same degree list, $H = C + K_2$.

If $C$ has a 1-vertex, then $D$ has a card with $m-2$ edges, but $G$ cannot. Thus $G$ & $H$ have exactly two 1-vertices.

Let $x = \#2$-vertices. $D$ has $x$ cards with $m-3$ edges. Hence $G$ has one of these structures.

- $x = 1$
- $x = 4$
- $x = 3$
- $x = 1$
- $x = 2$
- $x = 2$
Connectedness is 3-Reconstructible (for \( n \geq 7 \))

Consider \( G \) and \( H \) with \( m \) edges & same \((n - 3)\)-deck \( \mathcal{D} \).

\( G \) connected \( \Rightarrow \) \( \mathcal{D} \) has \( \geq 2 \) connected cards \( \Rightarrow \) disconnected \( H \) has component \( C \) with \( \geq n - 2 \) vertices.

Since we know \( G \) & \( H \) have same degree list, \( H = C + K_2 \).

If \( C \) has a 1-vertex, then \( \mathcal{D} \) has a card with \( m - 2 \) edges, but \( G \) cannot. Thus \( G \) & \( H \) have exactly two 1-vertices.

Let \( x = \#2\)-vertices. \( \mathcal{D} \) has \( x \) cards with \( m - 3 \) edges. Hence \( G \) has one of these structures.

Cards of \( G \) have mindeg \( \leq 1 \) (except one case), so each vtx of \( C \) has 2-nbr.
Connectedness is 3-Reconstructible (for $n \geq 7$)

Consider $G$ and $H$ with $m$ edges & same $(n-3)$-deck $\mathcal{D}$.

$G$ connected $\Rightarrow$ $\mathcal{D}$ has $\geq 2$ connected cards $\Rightarrow$
disconnected $H$ has component $C$ with $\geq n-2$ vertices.

Since we know $G$ & $H$ have same degree list, $H = C + K_2$.

If $C$ has a 1-vertex, then $\mathcal{D}$ has a card with $m-2$ edges, but $G$ cannot. Thus $G$ & $H$ have exactly two 1-vertices.

Let $x = \#2$-vertices. $\mathcal{D}$ has $x$ cards with $m-3$ edges.
Hence $G$ has one of these structures.

Cards of $G$ have mindeg $\leq 1$ (except one case), so each vtx of $C$ has 2-nbr. This makes $C$ too small.
3-Reconstructibility of Trees

**Thm.** Kelly [1957] Trees with $n \geq 3$ are 1-reconstr’ble.

**Thm.** Giles [1976] Trees with $n \geq 6$ are 2-reconstr’ble.

**Thm.** K-N-W-Z’21+ Trees with $n \geq 22$ are 3-reconstr’ble.
3-Reconstructibility of Trees

**Thm.** Kelly [1957] Trees with \( n \geq 3 \) are 1-reconstr’ble.

**Thm.** Giles [1976] Trees with \( n \geq 6 \) are 2-reconstr’ble.

**Thm.** K-N-W-Z’21+ Trees with \( n \geq 22 \) are 3-reconstr’ble.

**Def.** For an \( n \)-vertex rooted tree \( T \), an rcl-card is an \((n - l)\)-vertex rooted tree \( T' \subseteq T \) with the same root.
3-Reconstructibility of Trees

**Thm.** Kelly [1957] Trees with $n \geq 3$ are $1$-reconstr’ble.

**Thm.** Giles [1976] Trees with $n \geq 6$ are $2$-reconstr’ble.

**Thm.** K-N-W-Z’21$^+$ Trees with $n \geq 22$ are $3$-reconstr’ble.

**Def.** For an $n$-vertex rooted tree $T$, an $rcl$-card is an $(n - l)$-vertex rooted tree $T' \subseteq T$ with the same root.

**Thm.** No two rooted trees have the same rc3-cards, with special exceptions.
3-Reconstructibility of Trees

**Thm.** Kelly [1957] Trees with $n \geq 3$ are 1-reconstr’ble.

**Thm.** Giles [1976] Trees with $n \geq 6$ are 2-reconstr’ble.

**Thm.** K-N-W-Z’21 Trees with $n \geq 22$ are 3-reconstr’ble.

**Def.** For an $n$-vertex rooted tree $T$, an $rcl$-card is an $(n - l)$-vertex rooted tree $T' \subseteq T$ with the same root.

**Thm.** No two rooted trees have the same rc3-cards, with special exceptions.

**Def.** The cost of a vertex $v$ in a tree $T$ is the max #vertices in a component of $T - v$.
The cost $c(T)$ is the minimum cost among the vertices. A centroid is a vertex of minimum cost.
3-Reconstructibility of Trees

**Thm.** Kelly [1957] Trees with $n \geq 3$ are 1-reconstructible.

**Thm.** Giles [1976] Trees with $n \geq 6$ are 2-reconstructible.

**Thm.** K-N-W-Z’21+ Trees with $n \geq 22$ are 3-reconstructible.

**Def.** For an $n$-vertex rooted tree $T$, an $rcl$-card is an $(n - l)$-vertex rooted tree $T' \subseteq T$ with the same root.

**Thm.** No two rooted trees have the same rc3-cards, with special exceptions.

**Def.** The cost of a vertex $v$ in a tree $T$ is the maximum number of vertices in a component of $T - v$.

The cost $c(T)$ is the minimum cost among the vertices.

A centroid is a vertex of minimum cost.

**Lem.** An $n$-vertex tree has one centroid, cost $< n/2$, or has two adjacent centroids, with cost $n/2$. 
Trees with Small Cost

**Lem.** Let $D$ be the $(n - \ell)$-deck of an $n$-vertex tree $T$. Let $c(D)$ be the max cost among connected cards in $D$.

$$c(D) = \begin{cases} c(T) & \text{if } c(T) \leq (n - \ell)/2, \\ \left\lfloor (n - \ell)/2 \right\rfloor & \text{if } c(T) > (n - \ell)/2. \end{cases}$$

Also, if $c(T) \leq (n - \ell)/2$, then the centroid of $T$ is a centroid in every connected card.
Trees with Small Cost

**Lem.** Let $\mathcal{D}$ be the $(n - \ell)$-deck of an $n$-vertex tree $T$. Let $c(\mathcal{D})$ be the max cost among connected cards in $\mathcal{D}$.

$$c(\mathcal{D}) = \begin{cases} c(T) & \text{if } c(T) \leq (n - \ell)/2, \\ \lceil (n - \ell)/2 \rceil & \text{if } c(T) > (n - \ell)/2. \end{cases}$$

Also, if $c(T) \leq (n - \ell)/2$, then the centroid of $T$ is a centroid in every connected card.

**Thm.** For $n \geq 7$, every $n$-vertex tree $T$ with cost at most $(n - 5)/2$ is 3-reconstructible.
**Lem.** Let $D$ be the $(n - \ell)$-deck of an $n$-vertex tree $T$. Let $c(D)$ be the max cost among connected cards in $D$.

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Also, if $c(T) \leq (n - \ell)/2$, then the centroid of $T$ is a centroid in every connected card.

**Thm.** For $n \geq 7$, every $n$-vertex tree $T$ with cost at most $(n - 5)/2$ is 3-reconstructible.

**Pf.** By the lemma, $c(T) = c(D)$, and in each connected card the centroid is the actual centroid of the reconstructed tree $T$. This expresses the connected cards as the rc3-cards of $T$ rooted at its centroid. By the rooted result, the reconstructed tree is unique.
Trees with Small Cost

**Lem.** Let $\mathcal{D}$ be the $(n - \ell)$-deck of an $n$-vertex tree $T$. Let $c(\mathcal{D})$ be the max cost among connected cards in $\mathcal{D}$.

$$c(\mathcal{D}) = \begin{cases} c(T) & \text{if } c(T) \leq (n - \ell)/2, \\ \lfloor (n - \ell)/2 \rfloor & \text{if } c(T) > (n - \ell)/2. \end{cases}$$

Also, if $c(T) \leq (n - \ell)/2$, then the centroid of $T$ is a centroid in every connected card.

**Thm.** For $n \geq 7$, every $n$-vertex tree $T$ with cost at most $(n - 5)/2$ is 3-reconstructible.

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• Trees with cost $\frac{n-4}{2}, \frac{n-3}{2}, \frac{n-2}{2}, \frac{n-1}{2}, \frac{n}{2}$: 46 more pages, many uses of rooted reconstruction.
Prob 11898 (Stanley [2016], Amer. Math. Monthly) If $G$ is an $n$-vertex graph whose components are cycles of length greater than $k$, show that the number of independent sets of size $k$ depends only on $n$ and $k$. 
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Thm. Let $G$ and $G'$ be $n$-vertex graphs with maximum degree 2 and $|E(G)| = |E(G')|$. If every component in each graph is a cycle with more than $k$ vertices or a path with at least $k - 1$ vertices, then $\mathcal{D}_k(G) = \mathcal{D}_k(G')$. 
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(1) $\mathcal{D}_k(C_{q+r}) = \mathcal{D}_k(C_q + C_r)$ if $q, r \geq k + 1$,
(2) $\mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$, and
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In fact, (1,2,3) suffice to prove the theorem, because:
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Thm. Let $G$ and $G'$ be $n$-vertex graphs with maximum degree 2 and $|E(G)| = |E(G')|$. If every component in each graph is a cycle with more than $k$ vertices or a path with at least $k - 1$ vertices, then $D_k(G) = D_k(G')$.

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In fact, (1, 2, 3) suffice to prove the theorem, because:

Lem. If $G$, $G'$, and $H$ are graphs, then $D_k(G) = D_k(G')$ if and only if $D_k(G + H) = D_k(G' + H)$. 
Sharpness and Key Idea

**Thm.** If $\Delta(G) = 2$, and two largest components have $m$ and $m'$ vertices, then $G$ is $k$-deck reconstructible iff $k \geq \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}$, where $\epsilon \in \{0, 1\}$ and $\epsilon' \in \{0, 1, 2\}$. ($\epsilon = 1$ if largest component is $P_m$.)
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Thm. If $\Delta(G) = 2$, and two largest components have $m$ and $m'$ vertices, then $G$ is $k$-deck reconstructible iff $k \geq \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}$, where $\epsilon \in \{0, 1\}$ and $\epsilon' \in \{0, 1, 2\}$. ($\epsilon = 1$ if largest component is $P_m$.)

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Let $s(G, H) = \#$ induced copies of $H$ in $G$.
Sharpness and Key Idea

Thm. If \( \Delta(G) = 2 \), and two largest components have \( m \) and \( m' \) vertices, then \( G \) is \( k \)-deck reconstructible iff \( k \geq \max\{\lfloor m/2 \rfloor + \varepsilon, m' + \varepsilon' \} \), where \( \varepsilon \in \{0, 1\} \) and \( \varepsilon' \in \{0, 1, 2\} \). (\( \varepsilon = 1 \) if largest component is \( P_m \)).

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Let \( s(G, H) = \# \) induced copies of \( H \) in \( G \).

Let \( s'(G', H) = \# \) induced copies of \( H + K_1 \) having a named vertex \( z \) of \( G' \) as an isolated vertex.
Sharpness and Key Idea

**Thm.** If $\Delta(G) = 2$, and two largest components have $m$ and $m'$ vertices, then $G$ is $k$-deck reconstructible iff $k \geq \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}$, where $\epsilon \in \{0, 1\}$ and $\epsilon' \in \{0, 1, 2\}$. ($\epsilon = 1$ if largest component is $P_m$.)

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$G'$

Let $s'(G', H) = \#$ induced copies of $H + K_1$ having a named vertex $z$ of $G'$ as an isolated vertex.

- $s'(P_n, H)$ is indep of $z$ when $z$ is far enough from ends.
Independent of the Named Vertex

**Lem.** Let $L$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L)$ is the same.
Independent of the Named Vertex

**Lem.** Let $L$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L)$ is the same.

**Pf.** Induction on $k$. For $k = 1$, any $w_h$ is in one $P_1$. 
Independent of the Named Vertex

**Lem.** Let $L$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L)$ is the same.

**Pf.** Induction on $k$. For $k = 1$, any $w_h$ is in one $P_1$. Compare $s'(P_n, L)$ to $s'(C_n, L)$ with edge $w_n w_1$ added.
Independent of the Named Vertex

**Lem.** Let $L$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L)$ is the same.

**Pf.** Induction on $k$. For $k = 1$, any $w_h$ is in one $P_1$. Compare $s'(P_n, L)$ to $s'(C_n, L)$ with edge $w_n w_1$ added. By symmetry, $s'(C_n, L)$ is independent of $h$. 
Independent of the Named Vertex

**Lem.** Let $L$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L)$ is the same.

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$s'(C_n, L)$ omits copies of $L$ in $P_n$ using $w_1$ and $w_n$. $s'(C_n, L)$ counts unwanted subgraphs using $w_n w_1$. 
Independent of the Named Vertex

**Lem.** Let $L$ be the linear forest $\sum_{i=1}^{p} m_i P_{\ell_i}$ with $k$ vertices, and let $P_n = \langle w_1, \ldots, w_n \rangle$. For all $z = w_h$ with $k \leq h \leq n + 1 - k$, the value $s'(P_n, L)$ is the same.

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With $L_i = L - V(P_{\ell_i})$ and $L_{i,j} = L - V(P_{\ell_i} + P_{\ell_j})$, we have

$$s'(P_n, L) = s'(C_n, L) + \sum_{i,j} s'(P_{n-(\ell_i+\ell_j+2)}, L_{i,j}) - \sum_i (\ell_i - 1)s'(P_{n-(\ell_i+2)}, L_i)$$
Independent of the Named Vertex

**Lem.** Let \( L \) be the linear forest \( \sum_{i=1}^{p} m_i P_{\ell_i} \) with \( k \) vertices, and let \( P_n = \langle w_1, \ldots, w_n \rangle \). For all \( z = w_h \) with \( k \leq h \leq n + 1 - k \), the value \( s'(P_n, L) \) is the same.

**Pf.** Induction on \( k \). For \( k = 1 \), any \( w_h \) is in one \( P_1 \).

Compare \( s'(P_n, L) \) to \( s'(C_n, L) \) with edge \( w_n w_1 \) added.

By symmetry, \( s'(C_n, L) \) is independent of \( h \).

\( s'(C_n, L) \) omits copies of \( L \) in \( P_n \) using \( w_1 \) and \( w_n \).

\( s'(C_n, L) \) counts unwanted subgraphs using \( w_n w_1 \).

With \( L_i = L - V(P_{\ell_i}) \) and \( L_{i,j} = L - V(P_{\ell_i} + P_{\ell_j}) \), we have

\[
s'(P_n, L) = s'(C_n, L) + \sum_{i,j} s'(P_{n-(\ell_i+\ell_j+2)}, L_{i,j}) - \sum_i (\ell_i - 1) s'(P_{n-(\ell_i+2)}, L_i)
\]

\( w_h \) is far enough from the ends to use induction hyp. □
Same \( k \)-deck

(3) \( \mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1}) \) if \( q, r \geq k \).

With \( q, r \geq k \), either index \( h \) for \( z = w_h \) satisfies \( k + 1 \leq h \leq (q + r + 3) - (k + 1) \), so \( s'(P_{q+r+2}, L) \) is the same for both when \( |V(L)| = k \).
(3) $\mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1})$ if $q, r \geq k$.

With $q, r \geq k$, either index $h$ for $z = w_h$ satisfies $k + 1 \leq h \leq (q + r + 3) - (k + 1)$, so $s'(P_{q+r+2}, L)$ is the same for both when $|V(L)| = k$.

(2) $\mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$. 
Same $k$-deck

(3) $\mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1})$ if $q, r \geq k$.

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(2) $\mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$.

Let $P_{q+r} = \langle w_1, \ldots, w_{q+r} \rangle$ and $C_q = [w_1, \ldots, w_q]$. 

Same $k$-deck

(3) $D_k(P_{q-1} + P_r) = D_k(P_q + P_{r-1})$ if $q, r \geq k$.

$$G'$$

With $q, r \geq k$, either index $h$ for $z = w_h$ satisfies $k + 1 \leq h \leq (q + r + 3) - (k + 1)$, so $s'(P_{q+r+2}, L)$ is the same for both when $|V(L)| = k$.

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Let $P_{q+r} = \langle w_1, \ldots, w_{q+r} \rangle$ and $C_q = [w_1, \ldots, w_q]$.

If $w_q$ not in copy of $L$, both cases give $s(P_{q-1} + P_r, L)$. 
**Same \( k \)-deck**

(3) \( \mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1}) \) if \( q, r \geq k \).

![Diagram]

With \( q, r \geq k \), either index \( h \) for \( z = w_h \) satisfies \( k + 1 \leq h \leq (q + r + 3) - (k + 1) \), so \( s'(P_{q+r+2}, L) \) is the same for both when \( |V(L)| = k \).

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Let \( P_{q+r} = \langle w_1, \ldots, w_{q+r} \rangle \) and \( C_q = [w_1, \ldots, w_q] \).

If \( w_q \) not in copy of \( L \), both cases give \( s(P_{q-1} + P_r, L) \).

If used, sum over position of \( w_q \) in which \( P_{\ell_i} \) in \( L \).
**Same $k$-deck**

(3) $\mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1})$ if $q, r \geq k$.

\[ q - 1 \quad z \quad r \]

\[ q \quad z \quad r - 1 \]

$G'$

With $q, r \geq k$, either index $h$ for $z = w_h$ satisfies $k + 1 \leq h \leq (q + r + 3) - (k + 1)$, so $s'(P_{q+r+2}, L)$ is the same for both when $|V(L)| = k$.

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Let $P_{q+r} = \langle w_1, \ldots, w_{q+r} \rangle$ and $C_q = [w_1, \ldots, w_q]$.

If $w_q$ not in copy of $L$, both cases give $s(P_{q-1} + P_r, L)$.

If used, sum over position of $w_q$ in which $P_{\ell_i}$ in $L$.

By (3), corresponding terms are equal.
Same $k$-deck

(3) $\mathcal{D}_k(P_{q-1} + P_r) = \mathcal{D}_k(P_q + P_{r-1})$ if $q, r \geq k$.

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
q-1 & z & r \\
G' \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
q & z & r-1
\end{array}
\]

With $q, r \geq k$, either index $h$ for $z = w_h$ satisfies $k + 1 \leq h \leq (q + r + 3) - (k + 1)$, so $s'(P_{q+r+2}, L)$ is the same for both when $|V(L)| = k$.

(2) $\mathcal{D}_k(P_{q+r}) = \mathcal{D}_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$.
Let $P_{q+r} = \langle w_1, \ldots, w_{q+r} \rangle$ and $C_q = [w_1, \ldots, w_q]$.
If $w_q$ not in copy of $L$, both cases give $s(P_{q-1} + P_r, L)$.
If used, sum over position of $w_q$ in which $P_{\ell_i}$ in $L$.
By (3), corresponding terms are equal.

(1) $\mathcal{D}_k(C_{q+r}) = \mathcal{D}_k(C_q + C_r)$ if $q, r \geq k + 1$. 
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(3) $D_k(P_{q-1} + P_r) = D_k(P_q + P_{r-1})$ if $q, r \geq k$. 

With $q, r \geq k$, either index $h$ for $z = w_h$ satisfies $k + 1 \leq h \leq (q + r + 3) - (k + 1)$, so $s'(P_{q+r+2}, L)$ is the same for both when $|V(L)| = k$. 

(2) $D_k(P_{q+r}) = D_k(C_q + P_r)$ if $q \geq k + 1$ and $r \geq k - 1$. Let $P_{q+r} = \langle w_1, \ldots, w_{q+r} \rangle$ and $C_q = [w_1, \ldots, w_q]$. If $w_q$ not in copy of $L$, both cases give $s(P_{q-1} + P_r, L)$.

By (3), corresponding terms are equal.

(1) $D_k(C_{q+r}) = D_k(C_q + C_r)$ if $q, r \geq k + 1$. Same idea, reducing to equalities given by (2).
Contents of Appendices

**Appendix 1:** Graphs with maximum degree 2. Proving $k$-deck reconstructibility, where $k$ is the lower bound from the result just discussed about common $k$-decks for such graphs.

**Appendix 2:** 3-regular graphs are 2-reconstructible.

**Appendix 3:** Recognize connectedness for $n \geq 2\ell^{(\ell+1)^2}$. Proving $b_j \leq \binom{t}{j} \binom{\ell+j-1}{j}$ for an $n$-vertex tree $T$ with $t$ leaves, where $b_j$ is the number of subtrees of $T$ having at most $\ell$ vertices and exactly $j$ outside neighbors.
Open Questions

**Kelly–Manvel Conj.** Find $M_\ell$ such that every graph $G$ with $|V(G)| \geq M_\ell$ is $\ell$-reconstructible.
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**Ques.** What is the least $N_\ell$ so that for $n \geq N_\ell$, $n$-vertex connected graphs are $\ell$-recognizable? $2\ell + 1$?
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**Conj.** All $n$-vertex trees $\ell$-reconstr’ble for $n \geq 2\ell + 1 \geq 7$.

**Ques.** Find least $k$ so that $D_k(G)$ determines $G$, $\forall G$ with $\Delta(G) \leq 3$. What $k$ suffices for all $n$-vertex such $G$?
Open Questions

**Kelly–Manvel Conj.** Find $M_\ell$ such that every graph $G$ with $|V(G)| \geq M_\ell$ is $\ell$-reconstructible.

**Ques.** What is the least $N_\ell$ so that for $n \geq N_\ell$, $n$-vertex connected graphs are $\ell$-recognizable? $2\ell + 1$?

**Conj.** All $n$-vertex trees $\ell$-reconstr’ble for $n \geq 2\ell + 1 \geq 7$.

**Ques.** Find least $k$ so that $D_k(G)$ determines $G$, $\forall G$ with $\Delta(G) \leq 3$. What $k$ suffices for all $n$-vertex such $G$? Note: for $r$-regular, not 2-connected $\Rightarrow (r+1)$-reconstr’ble.
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**Ques.** For $n$-vertex $G$, what $k$ suffices for $D_k(G)$ to fix connectivity, matching number, $\chi(G)$, planarity, etc.?
Appendix 1: Reconstructibility when $\Delta(G) = 2$

**Lem.** If all components with more than $k$ vertices are determined by $\mathcal{D}_k(G)$, then $G$ is determined by $\mathcal{D}_k(G)$. 
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$$\#F\text{-components} = [\#F\text{-cards in } \mathcal{D}_k(G)] - \sum_{i=1}^{r} s(H_i, F),$$

where $H_1, \ldots, H_r$ are the larger components.
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For $F$-components we have:

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Let $q$ be $\#$ path components with at least $k - 1$ vertices.

**Lem.** If $\Delta(G) = 2$, then $q = s(G, P_{k-1}) - s(G, P_k) - ks(G, C_k)$. 


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$$\#F\text{-components} = \left\lceil \#F\text{-cards in } D_k(G) \right\rceil - \sum_{i=1}^{r} s(H_i, F),$$

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Let $q$ be the number of path components with at least $k - 1$ vertices.

**Lem.** If $\Delta(G) = 2$, then $q = s(G, P_{k-1}) - s(G, P_k) - ks(G, C_k)$.

**Pf.** Each such path contributes 1 to $s(G, P_{k-1}) - s(G, P_k)$.

Each $k$-cycle contributes 0 to $s(G, P_{k-1}) - ks(G, C_k)$.

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**Lem.** If $\Delta(G) = 2$, then $D_k(G)$ determines $q$. 


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**Lem.** If all components with more than $k$ vertices are determined by $\mathcal{D}_k(G)$, then $G$ is determined by $\mathcal{D}_k(G)$.

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Let $q$ be the number of path components with at least $k-1$ vertices.

**Lem.** If $\Delta(G)=2$, then $q = s(G, P_{k-1}) - s(G, P_k) - ks(G, C_k)$.

**Pf.** Each such path contributes 1 to $s(G, P_{k-1}) - s(G, P_k)$. Each $k$-cycle contributes 0 to $s(G, P_{k-1}) - ks(G, C_k)$. Each longer cycle contributes 0 to $s(G, P_{k-1}) - s(G, P_k)$.

**Lem.** If $\Delta(G) = 2$, then $\mathcal{D}_k(G)$ determines $q$.

**Pf.** $s(G, P_k)$ and $s(G, C_k)$ just count cards. Each copy of $P_{k-1}$ is in $n-k+1$ cards, so $s(G, P_{k-1}) = \frac{\sum_{Q \in \mathcal{D}_k(G)} s(Q, P_{k-1})}{n-k+1}$.
How to Use the Lemmas

**Thm.** When the two largest components have $m$ and $m'$ vertices, $k \geq \max\{\left\lfloor m/2 \right\rfloor + \epsilon, m' + \epsilon'\} \Rightarrow D_k(G)$ determines $G$, where $\epsilon \in \{0, 1\}$ and $\epsilon' \in \{0, 1, 2\}$. ($\epsilon = 1$ when the largest component is $P_m$.)
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Let \( q \) be \#path components with at least \( k - 1 \) vertices.
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- If $q = 0$ and $0 < s(G, P_k) \leq 2k + 1$, then $G$ has one component with more than $k$ vertices, $C_{s(G, P_k)}$. 
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**Thm.** When the two largest components have \( m \) and \( m' \) vertices, \( k \geq \max\{\lfloor m/2 \rfloor + \varepsilon, m' + \varepsilon'\} \Rightarrow \mathcal{D}_k(G) \) determines \( G \), where \( \varepsilon \in \{0, 1\} \) and \( \varepsilon' \in \{0, 1, 2\} \). (\( \varepsilon = 1 \) when the largest component is \( P_m \).)

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- If \( q = 1 \) and \( 0 \leq s(G, P_k) \leq k \), then \( G \) has no cycle with more than \( k \) vertices, and its long path is \( P_{s(G, P_k)+k-1} \).
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This completes the proof except for small \( k \).
Appendix 2: 3-Regular $G$ are 2-Reconstr’ble

Let $G$ be cubic, $\mathcal{D}_{n-2}(H) = \mathcal{D}_{n-2}(G)$, and $H \not\cong G$. 
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Let $G$ be cubic, $D_{n-2}(H) = D_{n-2}(G)$, and $H \not\cong G$.

When $n \geq 6$, we know $H$ is cubic. Degree list yields:
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Girth 4: Delete opposite vertices on a 4-cycle.
Again cubic $\Rightarrow$ every reconstruction is $G$. $lacksquare$

**Lem.** From $G - x - y$, we know $d_G(x, y)$ is 1 or 2 or $> 2$. 

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**Appendix 2: 3-Regular $G$ are 2-Reconstr’ble**
Cubic Graphs: More Tools

**Lem.** General Kelly Lemma If $|V(F)| \leq n - \ell$, then # copies of $F$ in $G$ is $\ell$-reconstructible.

**Pf.** Each copy appears in exactly $\binom{n-|V(F)|}{\ell}$ cards.
Cubic Graphs: More Tools

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**Cor.** Every reconstruction has the same girth $g$, $g$-cycles, $(g + 1)$-cycles.
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**Lem.** **Key Fact:** If $d_G(x, y) \leq 2$ and $x, y$ lie on a shortest cycle $C$ in $G$, then the only possible reconstructions are $G$ and one other, $H$. 

![Graph Diagram]
**Cubic Graphs: More Tools**

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**Pf.** Other ways are cubic but have shorter cycles.
Excluding Short Cycle Structures

10 Lemmas: (Note each reconstruction has girth $g$.)

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**Lem.** No two cycles of length at most $g + 1$ share two consecutive edges.
The Final Lemmas
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**Lem.** No edge lies in a $g$-cycle and a $(g + 1)$-cycle.

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**Lem.** Every $g$-cycle shares an edge with a $(g+1)$-cycle.
Appendix 3 - Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is #subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside nbrs, then $b_j \leq \binom{t}{j} \binom{\ell+j-1}{j}$
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$t = 8$

$l = 11$

$j = 4$
Appendix 3 - Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is the number of subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside neighbors, then $b_j \leq {t \choose j} {\ell + j - 1 \choose j}$ (except $b_2 \leq nt\ell/2$).

$t = 8$
$\ell = 11$
$j = 4$

**Pf.** $j \geq 3$: Let $S$ be the set of outside vertices with neighbors in $F$. 
Appendix 3 - Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is the number of subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside nbrs, then $b_j \leq \binom{t}{j} \binom{\ell+j-1}{j}$ (except $b_2 \leq ntl/2$).

**Pf.** $j \geq 3$: Let $S$ be the set of outside vertices with nbrs in $F$. $F =$ component of $T - S$ having vertices between those of $S$. 

\[ t = 8 \quad \ell = 11 \quad j = 4 \]
Appendix 3 - Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq l$, and $b_j$ is the number of subtrees $F$ with $|V(F)| \leq l$ and exactly $j$ outside neighbors, then $b_j \leq \binom{t}{j} \binom{l+j-1}{j}$ (except $b_2 \leq ntl/2$).

$t = 8$
$l = 11$
$j = 4$

**Pf.** $j \geq 3$: Let $S$ be the set of outside vertices with neighbors in $F$. $F =$ component of $T - S$ having vertices between those of $S$. Paths from $F$ through $S$ reach leaves $S'$ of $T$. 
Appendix 3 - Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is the number of subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside nbrs, then $b_j \leq \binom{t}{j} \left( \frac{\ell+j-1}{j} \right)$ (except $b_2 \leq ntl/2$).

$t = 8$

$l = 11$

$j = 4$

**Pf.** $j \geq 3$: Let $S$ be the set of outside vertices with nbrs in $F$. $F = \text{component of } T - S$ having vertices between those of $S$. Paths from $F$ through $S$ reach leaves $S'$ of $T$. Bound the number of subgraphs $F$ generating fixed $S'$ of size $j$. 
Appendix 3 - Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is #subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside nbrs, then $b_j \leq \binom{t}{j} \binom{\ell+j-1}{j}$ (except $b_2 \leq ntl/2$).

$t = 8$

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**Pf.** $j \geq 3$: Let $S$ be the set of outside vertices with nbrs in $F$. $F =$ component of $T - S$ having vertices between those of $S$. Paths from $F$ through $S$ reach leaves $S'$ of $T$. Given $S'$ (in $\binom{t}{j}$ ways), let $T'$ be the tree generated by $S'$. 
Appendix 3 - Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is the number of subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside neighbours, then $b_j \leq \binom{t}{j}(\ell+j-1)$ (except $b_2 \leq nt\ell/2$).

$t = 8$
$l = 11$
$j = 4$

**Pf.** $j \geq 3$: Let $S$ be the set of outside vertices with neighbours in $F$. $F = \text{component of } T - S \text{ having vertices between those of } S$. Paths from $F$ through $S$ reach leaves $S'$ of $T$. Given $S'$ (in $\binom{t}{j}$ ways), let $T'$ be the tree generated by $S'$. The vertex $u \in S$ generating $v \in S'$ is on the path from $v$ to the nearest branch vertex $w$ in $T'$. Note $w \in V(F)$. Between $w$ and $u$ are fewer than $\ell$ vertices.
Appendix 3 - Counting Small Subtrees

**Thm.** If $T$ is an $n$-vertex tree with $t$ leaves, and $j \leq \ell$, and $b_j$ is the number of subtrees $F$ with $|V(F)| \leq \ell$ and exactly $j$ outside nbrs, then $b_j \leq \binom{t}{j} \binom{\ell+j-1}{j}$ (except $b_2 \leq ntl/2$).

- $t = 8$
- $\ell = 11$
- $j = 4$

**Pf.** $j \geq 3$: Let $S$ be the set of outside vertices with nbrs in $F$. $F$ = component of $T - S$ having vertices between those of $S$. Paths from $F$ through $S$ reach leaves $S'$ of $T$.

Given $S'$ (in $\binom{t}{j}$ ways), let $T'$ be the tree generated by $S'$. The vertex $u \in S$ generating $v \in S'$ is on the path from $v$ to the nearest branch vertex $w$ in $T'$. Note $w \in V(F)$. Between $w$ and $u$ are fewer than $\ell$ vertices.

The number of ways to place the break vertices $u \in S$ is at most the number of solutions to $x_1 + \cdots + x_j \leq \ell - 1$, which equals $\binom{\ell+j-1}{j}$. 
$j = 2$: Since $T$ has $t$ leaves, from each vertex $u$ there are at most $t$ vertices at distance $i$, for $2 \leq i \leq \ell + 1$. 
Smaller cases

$t = 8$
$\ell = 5$
$i = 3$

$j = 2$: Since $T$ has $t$ leaves, from each vertex $u$ there are at most $t$ vertices at distance $i$, for $2 \leq i \leq \ell + 1$.

Hence each vertex belongs to at most $t\ell$ sets $S$ of size 2 that can cut off desired subtrees; the bound is $nt\ell/2$. 


Smaller cases

\[ t = 8 \]
\[ \ell = 5 \]
\[ i = 3 \]

\[ j = 2: \] Since \( T \) has \( t \) leaves, from each vertex \( u \) there are at most \( t \) vertices at distance \( i \), for \( 2 \leq i \leq \ell + 1 \).

Hence each vertex belongs to at most \( t\ell \) sets \( S \) of size 2 that can cut off desired subtrees; the bound is \( nt\ell/2 \).

\[ j = 1: \] From a leaf move toward the centroid at most \( \ell \) steps to place the vertex \( u \) cutting off \( F \);
Smaller cases

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\( j = 2 \): Since \( T \) has \( t \) leaves, from each vertex \( u \) there are at most \( t \) vertices at distance \( i \), for \( 2 \leq i \leq \ell + 1 \).
Hence each vertex belongs to at most \( t\ell \) sets \( S \) of size 2 that can cut off desired subtrees; the bound is \( n\ell t/2 \).

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### Smaller cases

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\[ \ell = 5 \]
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Hence each vertex belongs to at most \( t\ell \) sets \( S \) of size 2 that can cut off desired subtrees; the bound is \( n\ell/2 \).

\( j = 1 \): From a leaf move toward the centroid at most \( \ell \) steps to place the vertex \( u \) cutting off \( F \); bound is \( t\ell \).

- \( b_j(\binom{n}{\ell-j}) \leq \binom{t}{j}(\binom{\ell+j-1}{j})(\binom{n}{\ell-j}) \leq \frac{\ell}{2} n^{\ell-1} t \) (biggest when \( j = 1 \))
Smaller cases

\[ t = 8 \quad \ell = 5 \quad i = 3 \]

\[ j = 2: \text{ Since } T \text{ has } t \text{ leaves, from each vertex } u \text{ there are at most } t \text{ vertices at distance } i, \text{ for } 2 \leq i \leq \ell + 1. \]

Hence each vertex belongs to at most \( tl \) sets \( S \) of size 2 that can cut off desired subtrees; the bound is \( ntl/2 \).

\[ j = 1: \text{ From a leaf move toward the centroid at most } \ell \text{ steps to place the vertex } u \text{ cutting off } F; \text{ bound is } tl. \]

- \( b_j \left( \frac{n}{\ell-j} \right) \leq \binom{t}{j} \binom{j-1}{j} \binom{n}{\ell-j} \leq \frac{\ell}{2} n^{\ell-1} t \) (biggest when \( j = 1 \))

- For \( \ell = 3 \), these computations imply that connectedness is 3-reconstructible for \( n > 86,000,000 \).
Smaller cases

\[ t = 8 \]
\[ \ell = 5 \]
\[ i = 3 \]

\( j = 2 \): Since \( T \) has \( t \) leaves, from each vertex \( u \) there are at most \( t \) vertices at distance \( i \), for \( 2 \leq i \leq \ell + 1 \). Hence each vertex belongs to at most \( t\ell \) sets \( S \) of size 2 that can cut off desired subtrees; the bound is \( n\ell t/2 \).

\( j = 1 \): From a leaf move toward the centroid at most \( \ell \) steps to place the vertex \( u \) cutting off \( F \); bound is \( t\ell \).

- \( b_j(\frac{n}{\ell-j}) \leq \binom{t}{j} (\frac{\ell+j-1}{j}) (\frac{n}{\ell-j}) \leq \frac{\ell}{2} n^{\ell-1} t \) (biggest when \( j = 1 \))
- For \( \ell = 3 \), these computations imply that connectedness is 3-reconstructible for \( n > 86,000,000 \).

For \( \ell = 3 \), Spinoza–West [2019] reduced this to \( n \geq 25 \). KNWZ’21+ improved to \( n \geq 7 \) using different methods.