On the bar visibility number of complete bipartite graphs

Weiting Cao* Douglas B. West† Yan Yang‡

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Abstract

A *t-bar visibility representation* of a graph assigns each vertex up to t horizontal bars in the plane so that two vertices are adjacent if and only if some bar for one vertex can see some bar for the other via an unobstructed vertical channel of positive width. The least t such that G has a t-bar visibility representation is the *bar visibility number* of G, denoted by $b(G)$. For the complete bipartite graph $K_{m,n}$, the lower bound $b(K_{m,n}) \geq \lceil \frac{mn+4}{2m+2n} \rceil$ from Euler's Formula is well known. We prove that equality holds.

Keywords: bar visibility number; bar visibility graph; planar graph; thickness; complete bipartite graph.
MSC Codes: 05C62, 05C10

1 Introduction

Many problems in computational geometry model visibility relations in the plane. For example, we may say that two vertices of a polygon "see" each other if the segment joining them lies inside the polygon. In the *visibility graph* on the vertex set, vertices are adjacent if they see each other. More complicated notions of visibility have been defined for families of rectangles and other geometric objects. Dozens of papers have been written concerning construction and recognition of visibility graphs and applications to search problems and motion planning. The book by Ghosh [9] gives an overview of algorithms for visibility problems.

*University of Illinois, Urbana, USA: caoweiting@gmail.com.
†Zhejiang Normal University, Jinhua, China, and University of Illinois, Urbana, USA: dwest@math.uiuc.edu. Supported by NNSF of China under Grant NSFC-11871439.
‡Tianjin University, Tianjin, China: yanyang@tju.edu.cn. Supported by NNSF of China under Grant NSFC-11401430.

We consider visibility among horizontal segments in the plane. A graph G is a *bar visibility graph* if each vertex can be assigned a horizontal line segment in the plane (called a *bar*) so that vertices are adjacent if and only if the corresponding bars can see each other along an unobstructed vertical channel having positive width. The assignment of bars is a *bar visibility representation* of G. The condition on positive width allows bars A and B to block visibility between bars C and D even though A and B cannot see each other, such as when $A = [(a, y), (x, y)]$, $B = [(x, z), (b, z)]$, C contains (x, u), and D contains (x, v), with $u < y < z < v$ and $a < x < b$.

Tamassia and Tollis [14] and Wismath [17] found a simple characterization of bar visibility graphs. Hutchinson [12] later gave another simple proof for the 2-connected case.

Theorem 1.1 ([14, 17]). *A graph G has a bar visibility representation if and only if for some planar embedding of G all cut-vertices appear on the boundary of one face.*

Thus bar visibility graphs are a subclass of planar graphs. Nevertheless, assigning multiple bars to vertices permits representations of all graphs and leads to a complexity parameter measuring how many bars are needed per vertex, introduced by Chang, Hutchinson, Jacobson, Lehel, and West [5].

Definition 1.2 ([5]). A *t-bar visibility representation* of a graph assigns to each vertex at most t horizontal bars in the plane so that vertices are adjacent if and only if some bar assigned to one sees some bar assigned to the other via an unobstructed vertical channel of positive width. The *bar visibility number* of a graph G, denoted by $b(G)$, is the least integer t such that G has a t-bar visibility representation.

Results in [5] include the determination of visibility number for planar graphs (always at most 2), a proof that $b(K_n) = \lceil n/6 \rceil$ for $n \geq 7$, the determination of $b(K_{m,n})$ within 1, and the general bound $b(G) \leq \lceil n/6 \rceil + 2$ for every n-vertex graph G (improved by 1 in [8]). Results on the visibility numbers for hypercubes [16] and an analogue for directed graphs [1] have also been obtained. For complete bipartite graphs, the result was as follows.

Theorem 1.3 ([5]). *$r \leq b(K_{m,n}) \leq r + 1$, where $r = \left\lceil \frac{mn+4}{2m+2n} \right\rceil$.*

For the lower bound, consider a t-bar representation with $t = b(K_{m,n})$, add vertical edges where visibilities produce edges of $K_{m,n}$, and then shrink the bars to single points. This produces a bipartite plane graph H with at most $t(m + n)$ vertices and at least mn edges. Hence $mn \leq 2t(m + n) - 4$ by Euler's Formula, so $b(K_{m,n}) \geq r$. Equality requires almost all faces in the given planar embedding of H to have length 4.

In this paper, we prove $b(K_{m,n}) = r$, constructively. Section 2 contains a short proof valid for $K_{n,n}$. For most cases, it suffices to decompose $K_{n,n}$ into r bar visibility graphs, where a *decomposition* of G is a set of edge-disjoint subgraphs whose union is G. When $n \equiv 3 \mod 4$, we use such a decomposition omitting one edge.

The remainder of the paper presents a different approach that solves the problem for all complete bipartite graphs. In Section 3 we describe the general approach and reduce the problem to the crucial cases. Details to complete the argument appear separately in Sections 4 and 5 for the cases of even and odd n, respectively. The proof uses the notion of a *t-split* of a graph G, which is a graph H in which each vertex of G is replaced by a set of at most t independent vertices in such a way that u and v are adjacent in G if and only if some vertex in the set representing u is adjacent in H to some vertex in the set representing v. The graph H used to prove the lower bound for Theorem 1.3 is an example of a t-split of $K_{m,n}$.

As defined by Eppstein et al. [7], the *planar split thickness* (or simply *split thickness*) of a graph G, which we denote by $\sigma(G)$, is the minimum t such that G has a t-split H that is a planar graph (we call H a *planar t-split* of G). As explained above, always $\sigma(G) \le b(G)$. If G has a planar $\sigma(G)$-split that is 2-connected, then $\sigma(G) = b(G)$. This rests on the following immediate corollary of Theorem 1.1 that is the basis for our constructions.

Corollary 1.4. *Every 2-connected planar graph is a bar visibility graph.*

In particular, a 2-connected planar t-split of G has a 1-bar visibility representation, which yields a t-bar visibility representation of G.

This connection was noted in the thesis of the first author [4], where planar split thickness was given the unfortunate name "split number", creating confusion with another concept. The *splitting number* of a graph is the minimum number of splittings of a vertex into two vertices (with each incident edge being inherited by one of the two new vertices) needed to produce a planar graph (the vertices resulting from a splitting can be split yet again).

The notion of t-split originated with Heawood [10], who proved that K_{12} has a planar 2-split. Later, Ringel and Jackson [13] proved in effect that K_n has a planar $\lceil n/6 \rceil$-split. A short proof of this by Wessel [15] was used in [5] to prove $b(K_n) = \lceil n/6 \rceil$.

The results in [7] that concern complete bipartite graphs determine which ones have planar 2-splits and obtain lower bounds on the split thickness of the others. The lower bounds on $\sigma(K_{m,n})$ in [7] use the same counting argument from Euler's Formula that yields the lower bounds for $b(K_{m,n})$ (see [5]).

In [7], the authors close the paper by asking whether graphs embeddable on the surface of genus k have planar $(k+1)$-splits, phrased as an open question. A positive answer follows from a recent result about the *thickness* $\theta(G)$ of a graph G, defined to be the minimum number of planar graphs needed to decompose G. A decomposition into k planar graphs is a planar k-split, so $\sigma(G) \le \theta(G)$; this motivated the term "split thickness". Xu and Zha [18] proved that $\theta(G) \le k + 1$ when G embeds on the surface of genus k, thereby providing a positive answer to the question in [7].

2 The bar visibility number of $K_{n,n}$

Usually $b(G) \le \theta(G)$. Beineke, Harary, and Moon [3] determined $\theta(K_{m,n})$ for most m and n; in particular, $\theta(K_{n,n}) = \lceil \frac{n+2}{4} \rceil$. When $\lceil \frac{n+2}{4} \rceil = \lceil \frac{n+1}{4} \rceil$, we decompose $K_{n,n}$ into $\theta(K_{n,n})$ bar visibility graphs. When $n \equiv 3 \mod 4$, the value $\lceil \frac{n+2}{4} \rceil$ exceeds the desired value $\lceil \frac{n+1}{4} \rceil$ for the bar visibility number, so we must use a different approach in that case.

Theorem 2.1. *$b(K_{n,n}) = \lceil \frac{n+1}{4} \rceil$, except for $b(K_{3,3}) = 2$.*

Proof. Both $K_{1,1}$ and $K_{2,2}$ are bar visibility graphs. Since $K_{3,3}$ is not planar, $b(K_{3,3}) \ge 2$; equality holds by decomposing $K_{3,3}$ into a 6-cycle and a matching, which are both bar visibility graphs. Hence we may assume $n \ge 4$. Let U and V be the parts of $K_{n,n}$, with $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$. Let $r = \lceil \frac{n+1}{4} \rceil$ and $p = \lfloor n/4 \rfloor$. Always $r = p + 1$.

Case 0: *$n \equiv 0 \mod 4$, so $n = 4p$.* For this case, Chen and Yin [6] provided a decomposition of $K_{n,n}$ into $p+1$ planar subgraphs $G_1, \dots, G_p$ and H. For $m \in \mathbb{N}$, let $[m] = \{1, \dots, m\}$. For $j \in [p]$, partition U by $\widehat{U}_j = \{u_{4j-3}, u_{4j-2}, u_{4j-1}, u_{4j}\}$, $U_1^j = \bigcup_{i \in [p]-\{j\}} \{u_{4i-3}, u_{4i-2}\}$, and $U_2^j = \bigcup_{i \in [p]-\{j\}} \{u_{4i-1}, u_{4i}\}$, and partition V by $\widehat{V}_j = \{v_{4j-3}, v_{4j-2}, v_{4j-1}, v_{4j}\}$, $V_1^j = \bigcup_{i \in [p]-\{j\}} \{v_{4i-3}, v_{4i-1}\}$, and $V_2^j = \bigcup_{i \in [p]-\{j\}} \{v_{4i-2}, v_{4i}\}$. Figure 1 shows the subgraph G_j, drawn to indicate that it is a 2-connected planar graph and hence is a bar visibility graph. It covers the vertices using four disjoint copies of $K_{2,2p-2}$, with parts $\{u_{4j-3}, u_{4j-2}\}$ and V_2^j, parts $\{u_{4j-1}, u_{4j}\}$ and V_1^j, parts $\{v_{4j-3}, v_{4j-1}\}$ and U_1^j, and parts $\{v_{4j-2}, v_{4j}\}$ and U_2^j. The subgraph induced by $\widehat{U}_j \cup \widehat{V}_j$ is $K_{4,4}$ minus the edges of the form $u_i v_i$. When drawn as shown, this subgraph accommodates the copies of $K_{2,2p-2}$ within faces of length 4. The copies of $K_{2,2p-2}$ are 2-connected, and $K_{4,4}$ minus a matching is 2-connected, so G_j is a 2-connected planar graph and hence a bar visibility graph. The graph H omitted by $\bigcup_j G_j$ is the matching consisting of $u_i v_i$ for $i \in [4p]$; again this is a bar visibility graph.

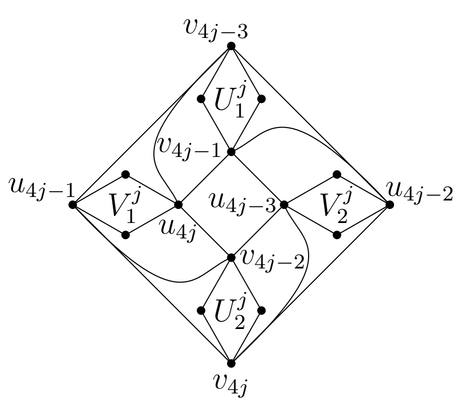

Figure 1: The graph G_j in a planar decomposition of $K_{4p,4p}$.

Case 1: $n \equiv 1 \mod 4$, *so* $n = 4p+1$. We add two adjacent vertices u_{4p+1} and v_{4p+1} to $K_{4p,4p}$, with u_{4p+1} also adjacent to V and v_{4p+1} also adjacent to U. Augmenting the decomposition of $K_{4p,4p}$ from Case 0, the edges incident to u_{4p+1} and v_{4p+1} can be added to the graph H of the previous case (see Figure 2). The resulting graph H' consists of the edge $u_n v_n$ and the paths of the form $\langle u_n, v_i, u_i, v_n\rangle$. It is planar and 2-connected, so $\{G_1, \dots, G_p\} \cup \{H'\}$ decomposes $K_{4p+1,4p+1}$ into $p+1$ bar visibility graphs.

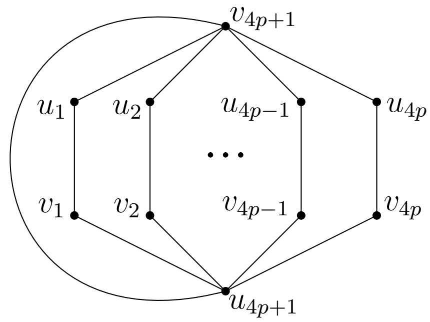

Figure 2: The graph H' in the planar decomposition of $K_{4p+1,4p+1}$

Case 2: $n \equiv 2 \mod 4$, *so* $n = 4p+2$. Begin again with the decomposition of $K_{4p,4p}$ in Case 0. Add $\{u_{4p+1}, u_{4p+2}\}$ to U and $\{v_{4p+1}, v_{4p+2}\}$ to V. Form H'' (shown in Figure 3) by adding to the matching H the 4-cycle $[u_{4p+1}, v_{4p+1}, u_{4p+2}, v_{4p+2}]$ plus, for $i \in [p]$, the edges joining u_{4p+1} to $\{v_{4i-2}, v_{4i}\}$, joining u_{4p+2} to $\{v_{4i-3}, v_{4i-1}\}$, joining v_{4p+1} to $\{u_{4i-2}, u_{4i-3}\}$, and joining v_{4p+2} to $\{u_{4i}, u_{4i-1}\}$. The four new vertices need more neighbors: for $j \in [p]$ and $i \in \{1,2\}$, form $\widehat{G}_j$ from G_j by adding u_{4p+i} to U_i^j and v_{4p+i} to V_i^j. Each new vertex thus gains the two neighbors among $\widehat{U}_j \cup \widehat{V}_j$ that it lacked in H''. Embed $\widehat{G}_j$ as in Figure 1, using the new U_i^j and V_i^j. We again have $p+1$ 2-connected planar graphs decomposing $K_{n,n}$.

Case 3: $n \equiv 3 \mod 4$, *so* $n = 4p+3$. Hobbs and Grossman [11] and Bouwer and Broere [2] independently gave decompositions of $K_{4p+3,4p+3}$ into planar subgraphs $H_1, \dots, H_{p+2}$. In each case, each H_i for $1 \le i \le p+1$ is a 2-connected maximal planar bipartite graph (hence a bar visibility graph), and H_{p+2} has only one edge. Label the vertices so that this extra edge is $u_i v_j$ (it is $u_1 v_1$ in [11] and $u_{4p+3} v_{4p-1}$ in [2]). We will combine bar visibility representations for two of the earlier graphs to represent the last edge without additional cost.

The bar visibility representation algorithm of [14] uses "s,t-numberings", allowing any vertex of a 2-connected bar visibility graph to be the unique lowest or highest bar in the representation. Since $n \ge 4$, we have $p+1 \ge 2$. Choose a representation of H_1 in which u_i is the unique lowest bar and a representation of H_2 in which v_j is the unique highest bar. Place the representation of H_1 above the representation of H_2 to incorporate the edge $u_i v_j$ without using an extra bar for u_i or v_j.

We must confirm that this can be done. Since the graph is bipartite, we may assume that bars for vertices of U have odd integral vertical coordinates and those for V having even

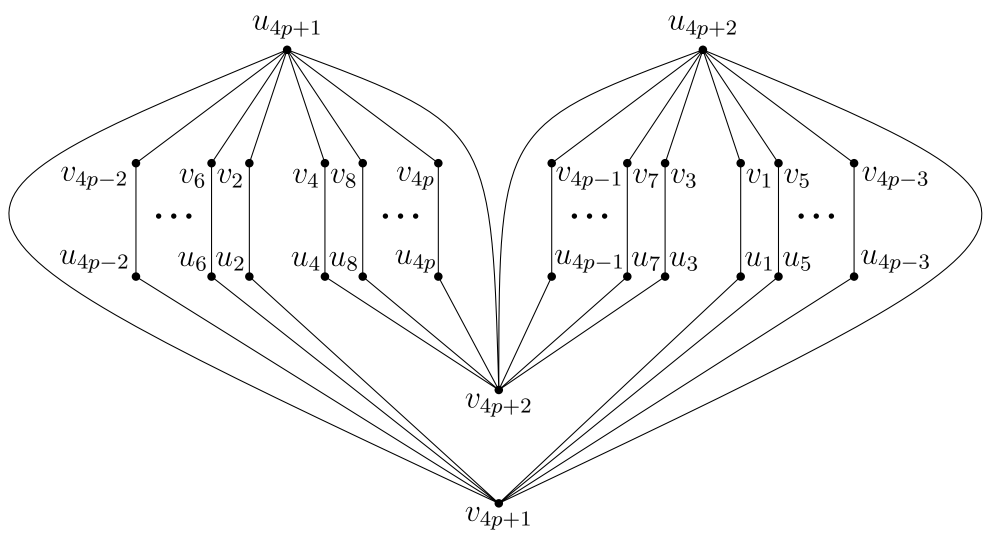

Figure 3: The graph H'' in the planar decomposition of $K_{4p+2,4p+2}$

coordinates. In addition, the bars on one horizontal line can extend to meet at endpoints to block visibility between higher and lower bars for the other part. This uses the requirement of positive width for visibility; for the same reason, sharing an endpoint does not allow bars on a single horizontal line to see each other. Extending bars in this way also does not introduce unwanted edges joining the two parts, because we are representing a complete bipartite graph, and redundant visibilities for edges are allowed. The bars also can extend so that the leftmost and rightmost occupied points on each horizontal line are the same, again because we are representing a complete bipartite graph. Now the two representations can be combined to obtain the extra edge as described above. □

3 General approach to computing $b(K_{m,n})$

Henceforth let $f(m,n) = \frac{mn+4}{2m+2n}$. Our proof of $b(K_{m,n}) \le \lceil f(m,n) \rceil$ for $m,n \in \mathbb{N}$ is independent of the short proof for $m = n$ in the previous section. That proof relies on decomposition results from earlier papers, but this proof uses only the observation in the introduction that it suffices to produce a 2-connected planar r-split of $K_{m,n}$, where $r = \lceil f(m,n) \rceil$.

We will consider various cases depending on the parities of m and n. In this section we explain the common aspects of the constructions. We assume $m \ge n$, by symmetry. Let the two parts of $K_{m,n}$ be X and Y with $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$.

Clearing fractions shows $f(m,n) \le n/2$ when $n \ge 2$. We first show $b(K_{m,n}) \le \lceil n/2 \rceil$. For $1 \le i \le \lfloor n/2 \rfloor$, let G_i be the subgraph induced by $X \cup \{y_{2i-1}, y_{2i}\}$, and when n is odd let $G_{(n+1)/2}$ be the subgraph induced by $X \cup \{y_n\}$. Since $K_{m,2}$ and $K_{m,1}$ are bar visibility graphs, this decomposes $K_{m,n}$ into $\lceil n/2 \rceil$ bar visibility graphs. For $n \le 3$ this settles all cases, since $K_{3,3}$ is nonplanar.

Hence we may assume $n \ge 4$ and $r < \lceil n/2 \rceil$. This restricts m, since clearing fractions yields $f(m,n) > n/2 - 1$ if and only if $m > \frac{1}{2}(n^2 - 2n - 4)$ and $f(m,n) > (n-1)/2$ if and only if $m > n^2 - n - 4$. Thus with $r < \lceil n/2 \rceil$ we may assume $m \le \frac{1}{2}(n^2 - 2n - 4)$ when n is even and $m \le n^2 - n - 4$ when n is odd.

To simplify computations in the constructions, we further reduce the cases (m, n) that need to be considered. For fixed n, the value of $\frac{mn+4}{2m+2n}$ increases with m. Fixing n and r, we show that it suffices to consider the maximum m such that $r = \lceil f(m,n) \rceil$. The next lemma allows us to obtain, from a 2-connected planar r-split of $K_{m,n}$, a 2-connected planar r-split of any 2-connected induced subgraph. Note that the requirement of $|X| \ge 3$ is needed: the conclusion does not hold when a vertex in the small part of $K_{2,n}$ is deleted.

The proof uses the notion of a *block* in a graph G, which is a maximal subgraph H having no cut-vertex. When G is connected with at least two vertices, its blocks can be single edges or be 2-connected. When two blocks intersect, they share only one vertex, which is a cut-vertex of G. When G is connected but not 2-connected, the blocks of G are arranged in a tree structure, the *leaf blocks* are the blocks containing only one cut-vertex of G.

Lemma 3.1. *Let H be a 2-connected bipartite planar graph with parts X and Y. If $v \in X$ and $|X| \ge 3$, then edges can be added to $H - v$ to obtain a 2-connected bipartite planar graph with the same vertex bipartition as $H - v$.*

Proof. If $H - v$ is 2-connected, then nothing need be done. Otherwise, consider a planar embedding of H with v on the unbounded face. When v is deleted, the leaf blocks of $H - v$ can be indexed cyclically in clockwise order in the embedding as $B_1, \ldots, B_k$ with $k \ge 2$.

If any B_i is not a single edge, then its unbounded face is a cycle C of length at least 4. Let x be a vertex of X on C that is not the cut-vertex of $H - v$ contained in B_i. All the neighbors of v in other blocks are on the unbounded face of $H - v$; make them all adjacent to x. By construction, the resulting graph H' is planar with the same bipartition as $H - v$, and it is 2-connected because all face boundaries are cycles.

In the remaining case, every leaf block is a single edge, consisting of a leaf that was adjacent to v and a cut-vertex in X. No vertex like that chosen above is available. Figure 4 illustrates such a graph H and the graph H' resulting from applying the proof below to $H - v$. If the leaf blocks all have the same cut-vertex, then $H - v$ is a star and H has the form $K_{2,n}$, which is forbidden by $|X| \ge 3$.

Hence we can group the leaf blocks by their cut-vertices, forming more than one group. All the leaves were neighbors of v. Form H' by making the leaves in each group adjacent to the cut-vertex for the next group. With the blocks indexed clockwise, this explicitly produces a planar embedding of a bipartite graph with the same bipartition as $H - v$. Also H' is 2-connected, because all face boundaries are cycles. □

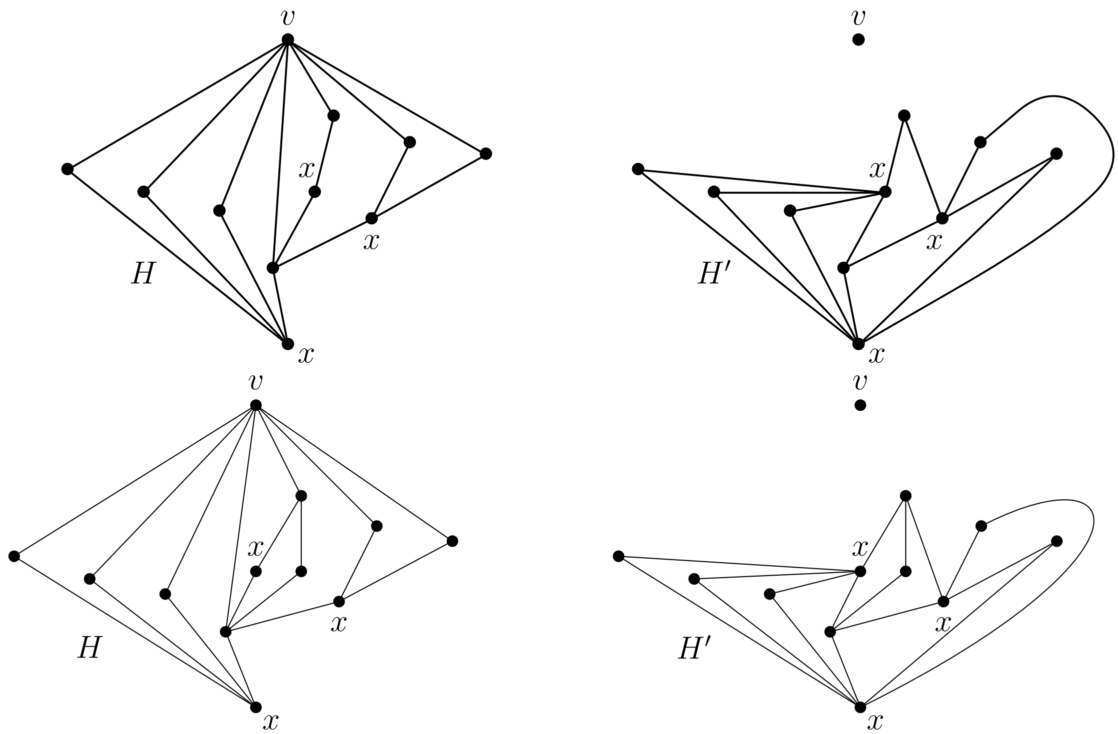

Figure 4: Deleting a vertex v

Lemma 3.2. *For $n, r \in \mathbb{N}$ with $r < n/2$, the largest m such that $\lceil f(m,n) \rceil = r$ is $\left\lfloor \frac{2rn-4}{n-2r} \right\rfloor$.*

Proof. If $\left\lceil \frac{mn+4}{2m+2n} \right\rceil = r$, then $r - 1 < \frac{mn+4}{2m+2n} \leq r$, equivalent to $\frac{2n(r-1)-4}{n-2(r-1)} < m \leq \frac{2nr-4}{n-2r}$. Thus $\left\lfloor \frac{2rn-4}{n-2r} \right\rfloor$ is the largest value of m such that $\lceil f(m,n) \rceil = r$. □

Lemma 3.3. *If $K_{m,n}$ has a 2-connected planar r-split whenever $r < n/2$ and $m = \left\lfloor \frac{2rn-4}{n-2r} \right\rfloor$, then $b(K_{m,n}) = \lceil f(m,n) \rceil$ for all $m, n \in \mathbb{N}$.*

Proof. Let G be such an r-split of $K_{m,n}$ when $m = \left\lfloor \frac{2rn-4}{n-2r} \right\rfloor$. In this case $\lceil f(m,n) \rceil = r$, so this graph G confirms the conclusion for this choice of m and n. For $\frac{2n(r-1)-4}{n-2(r-1)} < m \leq \frac{2nr-4}{n-2r}$, the value of $\lceil f(m,n) \rceil$ is the same number r, and Lemma 3.1 allows us iteratively to delete the copies in G of one vertex of X, restoring 2-connectedness after each vertex deletion by adding edges. The added edges joining the parts are not unwanted, because we are representing a complete bipartite graph, so the result is a 2-connected planar r-split of it. Hence $b(K_{m,n}) \leq r$ for all m where $b(K_{m,n}) \leq r$ is desired. □

Our goal is to construct a 2-connected planar r-split G of $K_{m,n}$ for the resulting pair (m, n). Each vertex in G will have a label in $X \cup Y$, with each label used at most r times. Whenever $(i, j) \in [m] \times [n]$, some vertex with label x_i will be adjacent to some vertex with label y_j. The vertices with labels in X will be on the horizontal axis, and those with labels in Y will be on the vertical axis; all edges join the two axes. Before describing the process, we suggest the aim by exhibiting in Figure 5 the graph G produced for $(m, n) = (8, 7)$. For clarity, we record only the subscripts of the vertex labels, using $x_1, \dots, x_m$ on the horizontal axis and $y_1, \dots, y_n$ on the vertical axis.

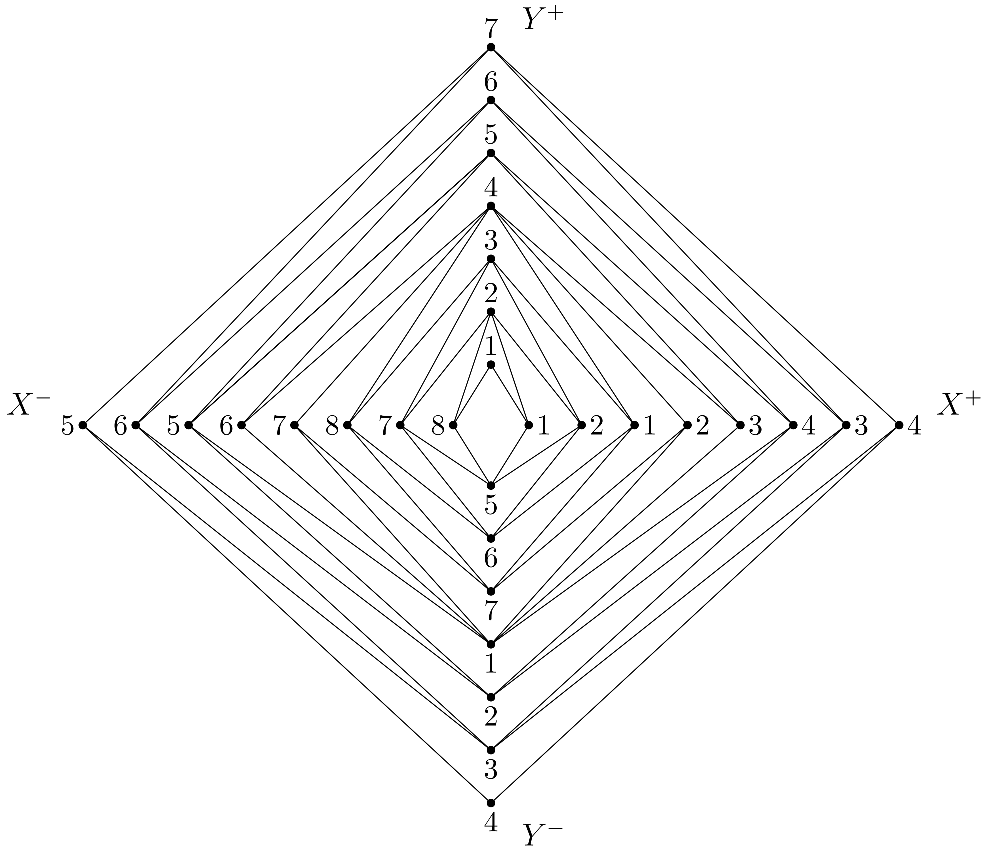

Figure 5: A bar visibility graph G that is a 2-split of $K_{8,7}$

Observe that in Figure 5 the graph G consists of four "quadrants" in the four quadrants of the plane. We show next that to guarantee that our planar r-split is 2-connected, it suffices to keep each quadrant connected. We refer to the positive and negative horizontal and vertical half-axes leaving the origin as the *rays* of the plane.

Lemma 3.4. *Let G be a plane graph whose vertex set is comprised of sets X^+ and X^- placed along the positive and negative horizontal axes and sets Y^+ and Y^- placed along the* *positive and negative vertical axes (as in Figure 5). If the four subgraphs induced by $X^+ \cup Y^+$, $Y^+ \cup X^-$, $X^- \cup Y^-$, and $Y^- \cup X^+$ are connected, then G is 2-connected.*

Proof. It suffices to show that any two vertices u and v in G are connected by two internally disjoint paths. Consider the four subgraphs combined as in Figure 5; the graph in each quadrant is connected. Suppose first that u and v are not on the same ray from the origin. Since the four subgraphs are connected, we can choose edges uu' and vv' such that u' and v' are on the remaining two rays. Now the four vertices are on distinct rays, and in all cases our two edges are chosen from the subgraphs in opposite quadrants. Because the remaining two quadrants are connected, we can choose a path in each to connect the vertices we have chosen from its rays. Now $\{u, u', v, v'\}$ lies on a cycle, which contains the two desired u, v-paths.

If u and v lie on the same ray, then we can choose uu' and vv' so that u' and v' are on the same ray neighboring it (possibly $u' = v'$). Now $\{u, u', v, v'\}$ lies in one of the four subgraphs. We find a u, v-path in one neighboring quadrant and a u', v'-path in the other neighboring quadrant (the latter may have length 0). Again we have completed a cycle through $\{u, u', v, v'\}$. □

Remark 3.5. *The Plan.* Recall that we have restricted the problem to $r < \lceil \frac{n}{2} \rceil$. Let $s = \lceil \frac{n}{2} \rceil - r$, so $s > 0$. The quantity s will be important, because the main step of the construction is to produce a 2-connected planar r-split $\widehat{G}$ of a subgraph of $K_{m,n}$. In $\widehat{G}$, each member of X will be represented by s vertices and each member of Y will be represented by r vertices. The graph $\widehat{G}$ will be 2-connected by virtue of Lemma 3.4. We will obtain the 2-connected planar r-split G of $K_{m,n}$ by adding at most $r - s$ vertices for each member of X. To make sense, this discussion requires $s \leq r$, which we confirm below.

Ideally, in the graph $\widehat{G}$ with $rn + sm$ vertices, the vertices representing $x_i \in X$ will together have edges to vertices representing $n - 2(r - s)$ members of Y. We hope to arrange that the members of Y having no representatives adjacent to representatives of x_i occur in $r - s$ pairs such that each pair lies on a face of length 4 in $\widehat{G}$. We can then add $r - s$ vertices representing x_i to turn $\widehat{G}$ into the desired graph G (it is well known that adding a vertex of degree at least 2 to a 2-connected graph yields a 2-connected graph). □

Next we confirm $s \leq r$.

Lemma 3.6. *Fix $m \geq n \geq 4$. With $r = \lceil f(m, n) \rceil$ and $s = \lceil \frac{n}{2} \rceil - r$, we have $s \leq r$, with equality if and only if $n \equiv 3 \mod 4$ and $m \in \{n, n+1\}$.*

Proof. Since $m \geq n$, we have $r \geq \left\lceil \frac{n^2+1}{4n} \right\rceil = \lceil \frac{n+1}{4} \rceil$, so $s \leq r$. Furthermore, the case $s = r$ requires $s = r = \frac{n+1}{4}$ and hence $n \equiv 3 \mod 4$. For $m, n \in \mathbb{N}$ with $m \geq n \geq 7$, the inequality $\frac{mn+4}{2m+2n} > \frac{n+1}{4}$ holds if and only if $m \geq n + 2$. Thus the case $s = r$ occurs if and only if $n \equiv 3 \mod 4$ and $m \in \{n, n+1\}$. Otherwise, $s < n/4 < r$. □

Since we often need to take vertices in pairs, it is not surprising that the construction depends on parity. We present the construction for even n in Section 4 and for odd n in Section 5, always with $m \geq n \geq 4$. Here we summarize the common aspects of both cases.

Remark 3.7. *Notation and terminology.* To distinguish between the members of X and Y and the vertices in $\widehat{G}$, we use A for the vertices on the horizontal axis (representing members of X) and B for those on the vertical axis (representing members of Y). We further break A and B into the subsets placed on the positive and negative rays from the origin point $(0, 0)$.

Let B^+ denote the set of the first $\lceil rn/2 \rceil$ integer points on the positive vertical ray, and let B^- denote the set of the first $\lfloor rn/2 \rfloor$ integer points on the negative vertical ray. Starting from the origin, label B^+ in order with $y_1, \ldots, y_n$, indices increasing cyclically modulo n. Similarly, label B^- outward from the origin, but start with $y_{\lceil n/2 \rceil + 1}$ and again increase cyclically through indices modulo n (this labeling is used in Figure 5, where only subscripts are shown). The vertices labeled y_j represent y_j in $\widehat{G}$. The labels farthest from the origin in B^+ and B^- are $\{y_n, y_{\lceil n/2 \rceil}\}$, with the last vertex of B^+ labeled y_n if r is even and the last vertex of B^- labeled y_n if r is odd. Each label y_j is used exactly r times, and $B = B^+ \cup B^-$.

The vertices with labels in X occupy integer points on the horizontal axis, with A^+ and A^- respectively denoting those on the positive and negative axes, with $A = A^+ \cup A^-$. The assignment of these labels will depend on the parity of n, so we postpone specifying them. We will introduce at most ms vertices into A.

For ease of describing and illustrating the constructions, we will decompose $\widehat{G}$ into two subgraphs, one in the right half-plane (positive horizontal coordinate) and one in the left half-plane (negative horizontal coordinate). The vertices of B lie in both subgraphs, so the right piece is the subgraph of $\widehat{G}$ induced by $B \cup A^+$, and the left piece is the subgraph induced by $B \cup A^-$. An example appears in Figure 6 for $K_{14,10}$, where $(r, s) = (3, 2)$. The subgraph in the right half-plane appears in the top piece; that in the left half-plane appears in the bottom piece.

In each of the two subgraphs, the vertices are put on three horizontal lines, with B^+ on the top line, A^+ or A^- on the middle line, and B^- on the bottom line. The vertices on the horizontal axis (A^+ or A^-) are listed in order from left to right as they appear in the plane (as in Figure 5). Thus in the top piece the vertices closest to the origin point $(0, 0)$ are on the left, while in the bottom piece those closest to the origin are on the right. Consistent with this, for the placement of B one can think of bending the vertical axis at the origin and laying its two rays down to the right around the positive horizontal axis or down to the left around the negative horizontal axis. Thus the subscripts of labels on B^+ and B^- increase cyclically from left to right in the top piece and from right to left in the bottom piece while moving away from the origin. □

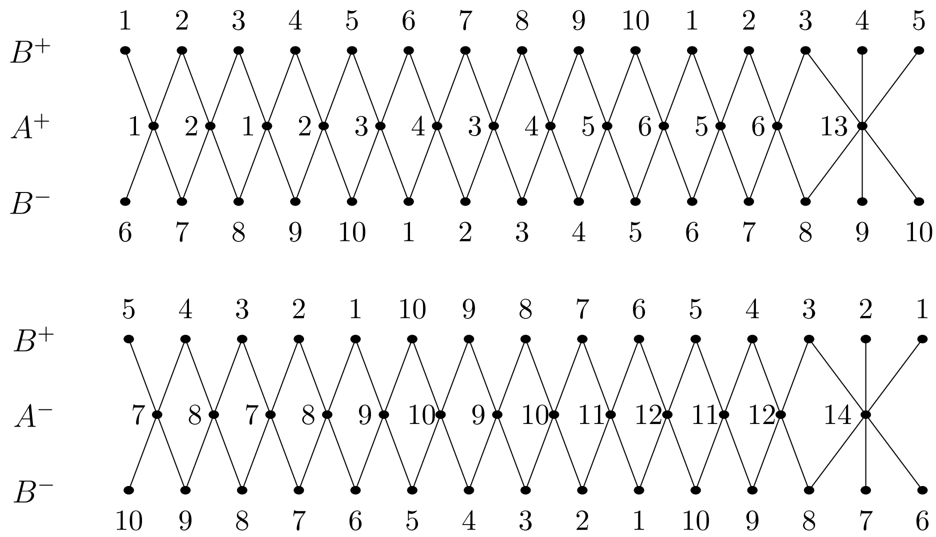

Figure 6: Pattern for even n, shown for $K_{14,10}$ with $(r, s) = (3, 2)$.

4 The case of even n

As seen in Figure 6, most vertices in A will have two consecutive neighbors in B^+ and two consecutive neighbors in B^- (ignore the edges there for x_{13} and x_{14}). We will produce the construction by adding edges in stages. When no vertices labeled x_i and y_j are yet adjacent, we say that x_i *misses* y_j; otherwise x_i *hits* y_j.

The main part of the construction consists of special building blocks that give s vertices in A to each of two members of X, enable them to hit $n - 2(r - s)$ labels in Y. For example, in Figure 6 the first of these structures, which we call "bricks", uses the first four vertices in A^+ and the first five vertices in each of B^+ and B^-. Note that $|B^+| = |B^-|$ when n is even.

Definition 4.1. An *opposite pair* is a pair of labels in Y whose subscripts differ by $n/2$, having the form $\{y_i, y_{i+n/2}\}$ with subscripts taken modulo n. The labels of vertices in B^+ and B^- at the same distance from the either end form an opposite pair.

For even n, an *i-brick* is a graph induced by $2s$ consecutive vertices in A^+ or A^- and $2s + 1$ consecutive vertices in each of B^+ and B^-, where the vertices in A alternate labels x_{2i-1} and x_{2i} and the vertices used from B form $2s + 1$ opposite pairs (see Figure 6). The edges join the jth of these vertices from A to the jth and $(j + 1)$th opposite pairs among these vertices from B. A *brick* is a subgraph that is an i-brick for some i.

Lemma 4.2. *When n is even, the labels from Y that lie on a 4-face in an i-brick form an opposite pair. Each label for a vertex of X in a brick hits two intervals of $2s$ cyclically* *consecutive labels in Y, forming $2s$ distinct cyclically consecutive opposite pairs. The labels missed by such a vertex of X thus also come in opposite pairs.*

Proof. Because n is even, $|B^+| = |B^-|$, and vertices in corresponding positions in the lists B^+ and B^- form opposite pairs, whether counted from the origin or from the outer end. The claims then follow immediately from the definition of i-brick, and the opposite pairs hit by x_{2i-1} and x_{2i} in an i-brick are distinct because $2s < n/2$. $\square$

Theorem 4.3. *If n is even, then $b(K_{m,n}) = \left\lceil \frac{mn+4}{2m+2n} \right\rceil = r$.*

Proof. By Lemma 3.3, we may assume $m = \left\lfloor \frac{2rn-4}{n-2r} \right\rfloor$. We have also reduced the problem to $r < n/2$. With $s = n/2 - r$, we have $s < n/4 < r$. Let $q = \frac{rn/2-1}{2s}$ and $t = \lfloor q \rfloor$. We have $m = \lfloor 4q \rfloor$, so $m = 4t + j$ for some $j \in \{0, 1, 2, 3\}$.

We begin by laying bricks. For $1 \leq i \leq t$, we form an i-brick using A^+, starting from the leftmost vertex of A^+ in Figure 6, which is the vertex of A^+ closest to the origin. Laying the i-brick assigns the labels x_{2i-1} and x_{2i} alternately to the next $2s$ vertices of A^+. For $i < t$, the last vertex of B^+ in the i-brick is also the first vertex in the $(i+1)$-brick (similarly for B^-). Thus these bricks use $1 + 2st$ vertices from B^+ (and from B^-). Since $|B^+| = rn/2 = 1 + 2sq$, enough vertices of B^+ and B^- are available to lay these bricks.

Similarly, working again from the left in the bottom piece (see Figure 6), we lay bricks using A^-. These bricks start with the vertices of A^- that are farthest from the origin in the planar embedding. For $t + 1 \leq i \leq 2t$, we insert an i-brick using A^-. With each brick starting at the last vertex of B^+ and B^- in the previous brick, again we use $1 + 2st$ vertices in those sets, which are available.

The rightmost vertices of B^+ and B^- in the t-brick and the $2t$-brick remain visible to further vertices of A^+ and A^-. Counting the last vertex of the t-brick, the number of vertices remaining visible at the right end of B^+ and B^- is $(rn/2) - 2st$, which equals $1 + 2s(q - t)$. Similarly, $1 + 2s(q - t)$ vertices of both B^+ and B^- are available at the right end of the bottom piece (these vertices are at the end near the origin).

By Lemma 4.2, all opposite pairs remain available on the 4-faces in the top piece using the first $n/2$ vertices in B^+ and B^-, the vertices $x_1, \ldots, x_{4t}$ now hit enough labels. Each such x_i has been used s times and has hit $4s$ labels in Y. Since $n - 4s = 2(r - s)$ and the $2(r - s)$ missed labels occur in opposite pairs (by Lemma 4.2), we can add $r - s$ vertices with this label in the appropriate faces to hit the remaining $2(r - s)$ labels in Y.

Since $m = 4t + j$, when $j > 0$ there remain j members of X that have hit nothing. Recall that $0 \leq j \leq 3$; the example in Figure 6 has $j = 2$. Note that the rightmost opposite pair in B^+ and B^- in the bottom piece (using vertices nearest the origin) cyclically follows the rightmost opposite pair in the top piece (using vertices farthest from the origin). Thus if a label in X hits consecutive opposite pairs in $B^+ \cup B^-$ using vertices in A^+, or in A^-, or from the right ends of A^+ and A^- together, then the labels in Y hit by that vertex will be distinct as long as the number of pairs is at most $n/2$.

When j is odd, x_m labels one vertex at the right end of A^+ and one at the right end of A^-. When $j \geq 2$, we use one vertex in A^+ for x_{m-j+1} and one vertex in A^- for x_{m-j+2} (as in Figure 6). This covers all possible j. With R_i denoting the portion devoted to the i-brick and $\varnothing$ denoting the position of the origin, the labeling of A from left to right on the horizontal axis is as indicated below, including A^- on the negative axis and A^+ on the positive axis.

$j = 0$:	$R_{t+1}, \dots, R_{2t}$		$\varnothing$	$R_1, \dots, R_t$	
$j = 1$:	$R_{t+1}, \dots, R_{2t}$	x_m	$\varnothing$	$R_1, \dots, R_t$	x_m
$j = 2$:	$R_{t+1}, \dots, R_{2t}$	x_m	$\varnothing$	$R_1, \dots, R_t$	x_{m-1}
$j = 3$:	$R_{t+1}, \dots, R_{2t}$	x_{m-1}, x_m	$\varnothing$	$R_1, \dots, R_t$	x_{m-2}, x_m

Let p be the number of vertices assigned to x_i, where $4t < i \leq m$; note that $p \in \{1, 2\}$. These p vertices for x_i need to hit $n - 2(r - p)$ labels in Y consisting of consecutive opposite pairs, so that the leftover vertices can hit the remaining opposite pairs. Since $n - 2(r - p) = 2s + 2p$, these vertices for x_i need to hit $s + p$ consecutive opposite pairs. For distinctness, we must ensure $s + p \leq n/2$. Since $p \leq 2$ and $s \leq (n - 2)/4$ when n is even, it suffices to have $(n - 2)/4 + 2 \leq n/2$, which is equivalent to $n \geq 6$. When $m \geq n = 4$ we always have $r = 2$ and completed that case in Section 3.

We must also show that $B^+ \cup B^-$ has enough opposite pairs available. For $j \in \{1, 2, 3\}$, we need in total to hit $s + 2$, $2s + 2$, or $3s + 4$ consecutive opposite pairs, respectively. We have observed that there are $1 + 2s(q - t)$ opposite pairs available at the right ends of both the top piece and the bottom piece. Since $q - t \geq j/4$, the $2 + 4s(q - t)$ pairs are at least $s + 2$, $2s + 2$, and $3s + 2$ for $j \in \{1, 2, 3\}$, respectively. However, when $j = 3$ we are using two vertices in each of A^+ and A^-, meaning that the last pair hit by one vertex can also be the first pair hit by the other. Thus in total the vertices in A can hit $3s + 4$ pairs instead of $3s + 2$, as needed.

Finally, as noted in Remark 3.5, it suffices to ensure that the graph $\widehat{G}$ produced before inserting the remaining vertices into 4-faces is 2-connected. By Lemma 3.4, it suffices to show that the four subgraphs induced by $B^+ \cup A^+$, $A^+ \cup B^-$, $B^- \cup A^-$, and $A^- \cup B^+$ are connected. Even after adding labels for $x_{m-j+1}, \dots, x_m$ in the last step, some vertices of B^+ and B^- may be unhit. Simply add edges joining such vertices to A in each of the four induced subgraphs to make them connected, while remaining planar and bipartite with the same bipartition. Because we are seeking a representation of a complete bipartite graph, extra edges joining vertices with labels from X and Y do not cause a problem. □

5 The case of odd n

Again we want to have s vertices for x_i hit $n - 2(r - s)$ labels in Y. When n is odd, this amount is odd, so in this case we define bricks somewhat differently. They will still use $2s$ vertices from A, but now they will use $2s$ vertices instead of $2s + 1$ in each of B^+ and B^-. Actually, the bricks we used before are too big to fit onto B^+ and B^- in this case.

Definition 5.1. A *skew pair* is a pair of labels in Y whose subscripts differ by $\lfloor n/2 \rfloor$, having the form $\{y_i, y_{i+(n-1)/2}\}$ with subscripts taken modulo n.

For odd n, an *i-brick* has $2s$ consecutive vertices in A^+ or A^- and $2s$ consecutive vertices in both B^+ and B^-; the vertices in A alternate labels x_{2i-1} and x_{2i} and the vertices in B form $2s$ skew pairs. A *brick* is an i-brick for some i. Figure 7 shows four bricks using A^+ and four using A^-. The vertices in a brick are indexed from the left, agreeing with the ordering of A along the horizontal axis. Thus for bricks using A^+, the jth vertices used from B^+ and B^- are the same distance from the origin, while for bricks using A^-, the jth vertices used from B^+ and B^- are the same distance from the ends away from the origin.

In an i-brick, edges join the jth x_{2i-1} to the $(2j-1)$th and $2j$th vertices used from B^+ and the $(2j-2)$th and $(2j-1)$th vertices used from B^-, except that the first x_{2i-1} hits in B^- only the first vertex. The jth x_{2i} hits the $2j$th and $(2j+1)$th vertices used from B^+ and the $(2j-1)$ and $2j$th vertices used from B^-, except that the last x_{2i} hits in B^+ only the last vertex. This holds in both the top piece and the bottom piece when r is odd, but when r is even we reverse the roles of B^+ and B^- in defining the edges of the bricks using A^-.

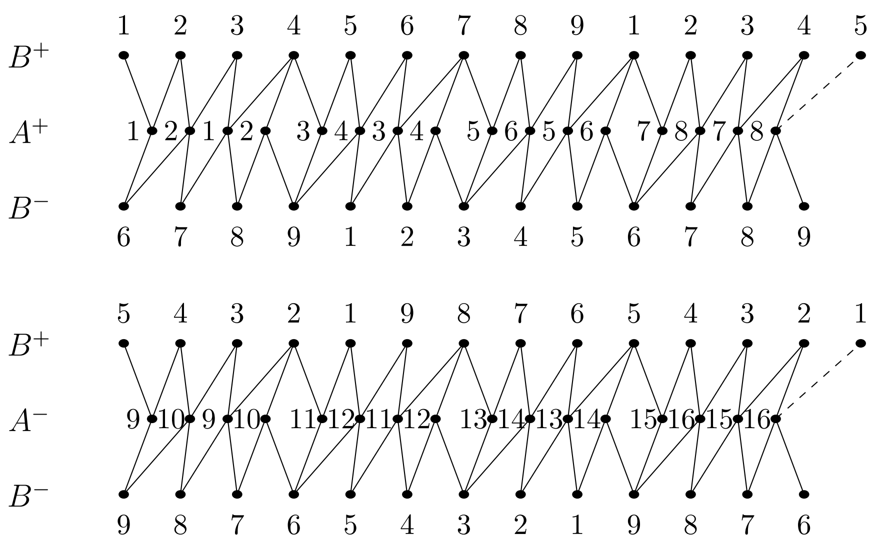

Figure 7: Pattern for odd n, shown for $K_{16,9}$ with $(r, s) = (3, 2)$.

Laying the bricks will label the vertices of A^+ and A^-.

Lemma 5.2. *When n is odd, the labels from Y that lie on a 4-face in a brick form a skew pair. Each label for a vertex of X in a brick hits two intervals of $2s-1$ cyclically consecutive labels in Y, forming $2s-1$ cyclically consecutive skew pairs, plus one more label at the end of one of those intervals. These labels are distinct. The labels missed by such a vertex of X also come in skew pairs.*

Proof. Since B^+ and B^- respectively start with labels y_1 and $y_{(n+3)/2}$ moving away from the origin, the labels of the jth vertices in B^+ and B^- from the left in the top piece form a skew pair, as do the $(j+1)$th vertex of B^+ and the jth in B^- (see Figure 7). The latter are the pairs lying on 4-faces in the top piece.

In the bottom piece, we start from the left with the labels $\{y_n, y_{(n+1)/2}\}$ in B^+ and B^- farthest from the origin (see Remark 3.7). These form a skew pair. Also $\{y_n, y_{(n-1)/2}\}$ is a skew pair. Since y_n ends B^+ when r is even and B^- when r is odd, these shifted skew pairs will be the $(j+1)$th vertex of B^+ and the j vertex of B^- when r is odd (as in the top piece), but when r is even they consist of the jth vertex of B^+ and the $(j+1)$th vertex of B^-. Therefore, to keep each such shifted skew pair on a common 4-face in the bottom piece, in defining the edges of the bricks using A^- we reverse the roles of B^+ and B^- when r is even.

Since corresponding positions in B^+ and B^- (from either end) are labeled by skew pairs, the $4s$ labels in Y occurring in a brick are distinct unless $2s = (n+1)/2$, which can occur when $n \equiv 3 \mod 4$ and $s = r = (n+1)/4$. In this case, the first label from B^+ is the same as the last label from B^- in a brick. However, as constructed in Definition 5.1, the first label from B^+ is hit only by x_{2i-1}, and the last label from B^- is hit only by x_{2i} in the brick, so each label from X still hits $4s-1$ distinct labels in Y (reversing B^+ and B^- in the argument for the bottom piece when r is even).

These $4s-1$ distinct labels in Y hit by a single label from X group into $2s-1$ cyclically consecutive skew pairs plus one more label. The two intervals of labels hit by the pairs leave two intervals of labels missed, and the lengths of the intervals of missed labels are $(n+1)/2 - s + 1$ and $(n-1)/2 - s + 1$. The extra label hit by x_i is at the end of the longer interval. No matter which end of the longer interval it shortens, the remaining missed labels match up as skew pairs. ☐

The approach to the construction is the same as for even n in Theorem 4.3, but the technical details are different.

Theorem 5.3. *If n is odd, then $b(K_{m,n}) = \left\lceil \frac{mn+4}{2m+2n} \right\rceil = r$.*

Proof. By Lemma 3.3, we may assume $m = \left\lfloor \frac{2rn-4}{n-2r} \right\rfloor$. We have reduced to $\frac{n+1}{4} \le r < \frac{n}{2}$. With $s = \frac{n+1}{2} - r$, we have $s \le \frac{n+1}{4} \le r$. Also $n - 2r = 2s - 1$. Let $q = \frac{rn/2-1}{2s-1}$ and $t = \lfloor q \rfloor$. We have $m = \lfloor 4q \rfloor$, so $m = 4t + j$ for some $j \in \{0, 1, 2, 3\}$.

We begin by laying bricks. For $1 \le i \le t$, we form an i-brick using A^+, starting from the leftmost vertex of A^+ in Figure 7, which is the vertex of A^+ closest to the origin. Laying the i-brick assigns the labels x_{2i-1} and x_{2i} alternately to the next $2s$ vertices of A^+. For $i < t$ the last vertex of B^+ in the i-brick is also the first vertex in the $(i+1)$-brick (similarly for B^-). Thus these bricks use $1+(2s-1)t$ vertices from B^+ (and B^-). Since B^+ and B^- each have at least $(rn-1)/2$ vertices, and $(rn-1)/2 = (1/2)+(2s-1)q \ge (1/2)+(2s-1)t$, enough vertices of B^+ and B^- are available to lay these bricks.

Similarly, working again from the left in the bottom piece of Figure 7, we lay i-bricks using A^-. These start with the vertex of A^- farthest from the origin in the plane. For $t+1 \le i \le 2t$ in order, we add an i-brick using A^-. Again we use $1+(2s-1)t$ vertices from B^+ and B^-, which are available.

The rightmost vertices of B^+ and B^- can be used again. Counting the last vertex of the t-brick, the number of vertices remaining visible to unused vertices of A^+ at the right end of B^+ and B^- together is $rn - 2(2s-1)t$, which equals $2+2(2s-1)(q-t)$. Similarly, this many vertices are visible to unused vertices of A^- at the right end of the bottom piece.

By Lemma 5.2, all skew pairs remain available on the faces with A^+ involving the first n vertices in B^+ and B^-, we have now satisfied the vertices $x_1, \ldots, x_{4t}$. Each such label has been used s times and hit $4s-1$ labels in Y. Since $n+1-4s = 2(r-s)$ and the $2(r-s)$ missed labels occur in skew pairs (by Lemma 5.2), we can add $r-s$ vertices with this label in the appropriate faces to hit the remaining $2(r-s)$ labels in Y.

Since $m = 4t+j$, there remain j vertices in X to be considered, where $j \le 3$ (none if $j = 0$; as in Figures 5 and 7). The labels at the right end of B^+ and B^- in the top piece are $\{y_n, y_{(n+1)/2}\}$, a skew pair. Those at the right end of B^+ and B^- in the bottom piece are $\{y_1, y_{(n+3)/2}\}$, the next skew pair. Thus if a label in X hits consecutive skew pairs in $B^+ \cup B^-$ using vertices in A^+, or in A^-, or from the ends of both, then the labels in Y hit by that vertex will be distinct as long as the number of pairs is at most $n/2$.

When j is odd, x_m labels one vertex at the right end of A^+ and one at the right end of A^-. When $j \ge 2$, we use one vertex in A^+ for x_{m-j+1} and one vertex in A^- for x_{m-j+2}. This covers all possible j. With R_i denoting the portion devoted to the i-brick and $\varnothing$ denoting the position of the origin, the labeling of A from left to right on the horizontal axis is exactly as it was in the case of even n, as shown in the display in the proof of Theorem 4.3.

Let p be the number of vertices assigned to x_i, where $4t < i \le m$; note that $p \in \{1, 2\}$. These p vertices for x_i need to hit a set of $n - 2(r-p)$ labels in Y that omits $2(r-p)$ labels comprising $r-p$ skew pairs, so that the leftover vertices can be added in the 4-faces of the bricks to hit the remaining labels. Since $n - 2(r-p) = 2s - 1 + 2p$, it suffices for x_i to hit $s+p-1$ consecutive skew pairs and one label from the next pair.

Since $p \le 2$, ensuring that the labels hit are distinct requires $s+1 \le (n-1)/2$. Because

$s \leq (n+1)/4$, it suffices to have $(n+1)/4 \leq (n-3)/2$, which is equivalent to $n \geq 7$. Since we have reduced to $n \geq 4$, and $s \leq (n-1)/4$ when $n \equiv 1 \mod 4$, all cases are covered.

It remains only to show that $B^+ \cup B^-$ has enough vertices available. For $j \in \{1, 2, 3\}$, we need the labels in $\{x_{4t+1}, \ldots, x_{4t+j}\}$ to hit $2s+3$, $4s+2$, or $6s+5$ labels, respectively. We have observed that there are in total $2+2(2s-1)(q-t)$ vertices of B visible at the right ends of each of A^+ A^-. Since $q - t \geq j/4$, the total number of vertices is at least $4 + (2s-1)j$, which is enough when $j \leq 2$. When $j = 3$ we are using two vertices in each of A^+ and A^-, meaning that the last pair seen by one vertex can also be the first pair seen by the other. This provides four additional visibilities to reach the needed $6s + 5$.

Finally, we must ensure that the graph $\widehat{G}$ produced before inserting the remaining vertices into 4-faces is 2-connected. Here the argument applying Lemma 3.4 to the subgraphs induced by $B^+ \cup A^+$, $A^+ \cup B^-$, $B^- \cup A^-$, and $A^- \cup B^+$ is exactly the same as in Theorem 4.3. $\square$

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