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A NOTE ON THE INTERVAL NUMBER OF A GRAPH

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Three results on the interval number of a graph on n vertices are presented. (1) The interval number of almost every graph is between $n/4\lg n$ and $n/4$ (this also holds for almost every bipartite graph). (2) There exist $K_{m,m}$-free bipartite graphs with interval number at least $c(m)n^{1-2/(m+1)}/\lg n$, which can be improved to $\sqrt{n}/4+o(\sqrt{n})$ for $m=2$ and $(n/2)^{\frac{2}{3}}/\lg n$ for $m=3$. (3) There exists a regular graph of girth at least g with interval number at least $\frac{1}{2}((n-1)/2)^{1/(g-2)}$.

In this note, we apply counting arguments and results on graph decomposition to obtain inequalities concerning the interval number of a graph G. The interval number $i(G)$, first appearing in [7], is the minimum t such that G is the intersection graph of sets consisting of at most t intervals on the real line. Such a description of G is called a *t-representation* of G.

Extremal results on the interval number of a graph have given us upper bounds on $i(G)$ in terms of other graph parameters; see Table 1.

Since the maximum interval number for a graph on n vertices is attained by $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$, we are motivated to ask how large the interval number can be if we forbid induced copies of $K_{m,m}$. The following lemma is applicable to this question and also yields an immediate bound on the interval number of a random graph. Since the arguments extend to higher dimensions, we consider $i_d(G)$, which is the 'd-dimensional' interval number: a *d-dimensional t-representation* of G expresses it as the intersection graph of collections of at most t d-dimensional boxes (sides

Table 1

Parameter	Upper bound on $i(G)$	Attained by	Reference
Number of vertices n	$\lceil (n+1)/4\rceil$	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$	[3]
Maximum degree Δ	$\lceil (\Delta+1)/2\rceil$	any triangle-free regular graph	[4]
Genus	3 for planar graphs	adding pendant vertices to $K_{2,9}$	[6]
	(for higher genus		see [5])
Number of edges e	$\sqrt{e}^{\,*}$		[4]
	$(\frac{1}{2}\lceil\sqrt{e}+1\rceil$ conjectured	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil})$	

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parallel to axes), and $i_d(G)$ is the minimum t such that G has a d-dimensional t-representation. See [5, 8] for further discussion of this generalized parameter.

All logarithms are taken with base 2. A property holds for *almost every graph* if its probability goes to 1 as n goes to ∞ in the probability spaces defined for each n by letting the $2^{\binom{n}{2}}$ labeled graphs on n vertices be equally likely.

Lemma 1. *For fixed t and d, the logarithm of the total number of graphs with $i_d(G) \leq t$ is bounded by a function asymptotic to $(2nt+\frac{1}{2})d \lg n-(n-\frac{1}{2})d \lg(4\pi t)$. For triangle-free graphs and $d=1$, the corresponding bound is $nt \lg n+\mathrm{O}(nt)$.*

Proof. In each dimension, we count the possible arrangements of t intervals per vertex. A representation is determined by the ordering of the endpoints of the intervals. We may assume the endpoints of the intervals are distinct. The representation is then determined by assigning $2t$ of the sequence of $2nt$ endpoints to each vertex. This can be done in $\binom{2nt}{2t, \ldots, 2t}$ ways. Counting the t-representations certainly overcounts the graphs. In d dimensions, we use the dth power of this. Using Stirling's approximation, the logarithm of this is asymptotic to $(2nt+\frac{1}{2})d \lg n-(n-\frac{1}{2})d \lg(4\pi t)$.

The argument for triangle-free graphs is somewhat more delicate. Intervals for three distinct vertices cannot intersect, which greatly restricts the possible t-representations. To obtain an upper bound, consider the length of the intervals, where we have assumed the endpoints lie at $x=1,2, \ldots, 2nt$. Since the 'depth' of the representation is at most 2 and there must always be a unit lost between the end of an interval and the start of another, the total length of intervals is at most $3nt-2$. Representations with shorter total length can be obtained from these by shrinking intervals, so we need only count representations with maximum total length.

Let v_j be the jth interval assigned to vertex v. The number of ways to distribute the total length among the v_j, for all j and v, is $\binom{3(nt-1)}{nt-1}$. Since v_1 appears before v_2, etc., the entire representation is now determined by specifying the order of the left endpoints, i.e., an arrangement of t copies of each v. When one interval ends, the interval to start at the next point is specified by this ordering. There are $\binom{nt}{t, \ldots, t}$ such orderings. Many of these orderings give rise to non-representations, but at least this gives an upper bound. The logarithm of this bound $\binom{3(nt-1)}{nt-1}\binom{nt}{t, \ldots, t}$ is asymptotic to $nt \lg n+\mathrm{O}(nt)$. As before, the number of graphs allowed is at most the number of legal t-representations. $\square$

Theorem 1. *Almost every graph G has $i(G) \geq n/4 \lg n$. More generally, for fixed d almost every graph has $i_d(G) \geq n/4d \lg n$. The lower bound of $n/4 \lg n$ also holds asymptotically for the interval number of almost every bipartite graph.*

Proof. Let $t=n/4d \lg n$. From Lemma 1, the logarithm of the number of graphs on n vertices with $i_d(G) \leq t$ is bounded by $n^2/2-nd \lg n+\mathrm{O}(n \lg \lg n)$. The logarithm of the total number of graphs is $\binom{n}{2}$. The number of graphs grows exponentially faster than the number with $i_d(G) \leq t$, so almost all graphs have $i_d(G) > t$.

The same argument works with random bipartite graphs when $d = 1$. Given a partition of the vertices into two equal-sized parts, the logarithm of the number of bipartite graphs that can be formed is $n^2/4$. Considering all bipartite graphs adds only a linear term to this logarithm. By Lemma 1 and the same rate-of-growth argument as before, if $t = (1-\varepsilon)n/4 \lg n$, where $\varepsilon > 1/\lg n$, then almost every bipartite graph has $i(G) \geq r$. □

Scheinerman [5] used a less detailed version of the first counting argument in Lemma 1 to show that $i_d(G)$ is unbounded. If t grows faster than $n/4 \lg n$, then there are many more t-representations on n vertices than there are labeled graphs, according to this count. Hence an upper bound on $i(G)$ can be given if almost every graph has not too many t-representations. More precisely, let $t = (1+\varepsilon)n/4 \lg n$. If almost every graph has at most $\varepsilon n^2/2$ t-representations, then $i(G) \leq t$ for almost every graph. If almost every graph has at most $(2c-1)nd \lg n$ t-representations, then $i(G) \leq c + n/4 \lg n$ for almost every graph.

The edge bound on interval number suggests a more direct approach to an $O(n/\lg n)$ upper bound. We need only show that almost every graph has cliques that cover almost all the edges but do not use any vertex too many times. More precisely, if a set of cliques in which each vertex appears at most $c_1 n/\lg n$ times covers all but $(c_2 n/\lg n)^2$ edges in G, then $i(G) \leq (c_1 + c_2)n/\lg n$, by the bound $i(G) \leq \sqrt{e}$ in [4].

Next we apply Lemma 1 to the interval number of $K_{m,m}$-free bipartite graphs. The most interesting case is $m = 2$, $d = 1$. It is known that a graph with no induced 4-cycle has at most $\sqrt{n}(n-1)/2 + n/2$ edges (see [1, p. 310]). The edge bound thus yields an upper bound of $O(n^{\frac{3}{4}})$ for the interval number of such graphs, noticeably smaller than the bound for arbitrary graphs. However, we can show only the existence of $K_{2,2}$-free bipartite graphs with $i(G) > n^{\frac{1}{2}}$.

Theorem 2. *There exist $K_{m,m}$-free bipartite graphs with interval number at least $t \geq (1 - 1/(m!^2)(n/2)^{1-2/(m+1)}/4d \lg n$ plus lower-order terms. This can be improved to $\sqrt{n}/4 + O(\sqrt{n})$ for $m = 2$ and $(n/2)^{\frac{2}{3}}/4d \lg n$ for $m = 3$ and certain values of n. For $d = 1$ and $m \geq 3$, these bounds can be improved by a factor of 2.*

Proof. Let $z(n, m)$ be the largest number of edges in a $K_{m,m}$-free bipartite graph with $n/2$ points in each part, and let t be the maximum of $i_d(G)$ over all $K_{m,m}$-free bipartite graphs. It is known that $z(n, m) \geq \lfloor (1 - 1/(m!^2)(n/2)^{2-2/(m+1)} \rfloor$ (see [1, p. 316]). The extremal graph has $2^{z(n,m)}$ subgraphs, each of which is $K_{m,m}$-free. By Lemma 1, we must have $2ntd \lg n \geq z(n, m)$. Thus $t \geq (1 - 1/(m!^2)(n/2)^{1-2/(m+1)}/4d \lg n$ plus lower-order terms. The same argument can be applied to get lower bounds on the maximum interval number of $K_{s,t}$-free bipartite graphs. The bound for $m=3$ follows from a better bound for $z(n,3)$. When $n/2$ is the cube of an odd prime, $z(n,3) \geq (n/2)^{\frac{5}{3}}$ (see [1, p 314]). If $d=1$, we can use the triangle-free version of Lemma 1 to save a factor of 2.

For $m=2$ and $d=1$, the most interesting case, we can do better than this argument. Asymptotically, $z(n,2)$ is about $(n/2)^{\frac{3}{2}}$, which would yield a lower bound of $\sqrt{(n/2)}/2d \lg n + O(n^{\frac{1}{3}})$ for $m=2$ for the maximum d-dimensional interval number on $K_{2,2}$-free bipartite graphs. When $d=1$, we can dispose of the $\lg n$ factor.

It is easy to see that the interval number of a union of graphs is at most the sum of the interval numbers of the graphs united, by using optimal representations for each. Thus, for any decomposition of the edges of $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ into k graphs, one of them must have interval number at least $(n+1)/4k$. We can show there is a decomposition of $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ into a small number of $K_{m,m}$-free graphs by obtaining a lower bound on the k-color Ramsey number of $K_{m,m}$. If $r_k(K_{m,m}) > f(k)$, then K_n, and with it $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$, can be decomposed into $f^{-1}(n)$ $K_{m,m}$-free graphs.

Graham and Chung [2] showed that $r_k(K_{s,t}) > (2/e^2)mk^{m/2}$. This guarantees a $K_{m,m}$-free bipartite graph with interval number at least $n^{1-2/m}/4$, which is not as good as the result above. However, they conjectured that $r_k(K_{s,t}) \sim (t-1)k^s + o(k^s)$ for $t \geq s \geq 2$, which would produce a $K_{s,t}$-free graph with interval number at least $n^{1-1/s}(t-1)^{1/s}/4 + o(n^{1-1/s})$. This result would be uniformly better that that above. For $m=2$, the result is available; Graham and Chung showed $r_k(K_{2,2}) > k^2-k+1$ when $k-1$ is a prime power. By considering the next prime power, we get $r_k(K_{2,2}) > k^2 + o(k^2)$. Inverting this yields the existence of a $K_{2,2}$-free graph with interval number $\sqrt{n}/4 + o(\sqrt{n})$. Of course, better bounds for all m would be obtained by using lower bounds on the k-color *bipartite* Ramsey numbers for $K_{s,t}$, when such become available. □

Finally, we consider bounds on the maximum interval number for another way of forbidding the 4-cycles that are so prevalent in $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$, namely increasing the girth. Consider graphs on n vertices with girth at least g. We obtain the result below, but setting $g=4$ shows that it is not best possible.

Theorem 3. *Among the regular graphs on n vertices with girth at least g, there exists a graph with interval number at least* $\frac{1}{2}((n-1)/2)^{1/(g-2)}$.

Proof. In [1], Bollobas summarizes results on the minimum number of vertices in a graph with girth g and minimum degree δ. One such result is particularly applicable here. If $m \geq [(d-1)^{g-1}-1]/(d-2)$, then there exists a d-regular graph on $2m$ vertices with girth at least g. If $g \geq 4$, the interval number of such a graph is exactly $\lceil (d+1)/2 \rceil$ [4]. Setting $n=2m$ and inverting this relationship as in the proof of Theorem 2 yields the result claimed. □

Note added in proof

The bound $i(G) \leqslant \sqrt{e}$ for graphs with e edges has been improved to $i(G) \leqslant \sqrt{e/2}$ by J. Spinrad, G. Vijayan and D. West.

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References

[1] B. Bollobas, Extremal Graph Theory (Academic Press, New York, 1980).

[2] R.L. Graham and F.R.K. Chung, On multicolor Ramsey numbers for complete bipartite graphs, J. Combin. Theory (B) 18 (1975) 164–169.

[3] J.R. Griggs, Extremal values of the interval number of a graph, II. Discrete Math. 28 (1979) 37–47.

[4] J.R. Griggs and D.B. West, Extremal values of the interval number of a graph, I. SIAM J. Algebraic Discrete Meths. 1 (1980) 1–8.

[5] E.R. Scheinerman, Intersection representations of graphs, Ph.D. Thesis, Princeton Univ., 1984.

[6] E.R. Scheinerman and D.B. West, The interval number of a planar graph: Three intervals suffice, J. Combin. Theory (B) 35 (1983) 224–239.

[7] W.T. Trotter, Jr. and F. Harary, On double and multiple interval graphs, J. Graph Theory 3 (1979) 205–211.

[8] D.B. West, Parameters of partial orders and graphs: Packing, covering, and representation, in: I. Rival, ed., Graphs and Orders, Proc. Symp. Banff 1984 (Reidel, Dordrecht, 1985).