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The Interval Number of a Planar Graph: Three Intervals Suffice

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Suppose each vertex of a graph G is assigned a subset of the real line consisting of at most t closed intervals. This assignment is called a t-interval representation of G when vertex v is adjacent to vertex w if and only if some interval for v intersects some interval for w. The interval number $i(G)$ of a graph G is the smallest number t such that G has a t-interval representation. It is proved that $i(G) \leqslant 3$ whenever G is planar and that this bound is the best possible. The related concepts of displayed interval number and depth-r interval number are discussed and their maximum values for certain classes of planar graphs are found.

1. INTRODUCTION

In this paper we are interested in the interval number of planar graphs. The interval number of a simple undirected graph (hereafter "graph") was introduced to generalize interval graphs. A graph on n vertices is an *interval graph* if it is the intersection graph of a collection of n intervals on the real line. In other words, each vertex of G is assigned an interval, and two vertices are adjacent if and only if the corresponding intervals intersect. Trotter and Harary [TH] introduced the following generalization. Given an integer t, a mapping f that assigns each vertex a subset of $\mathbf{R}$ is called a *t-representation* if (1) for each vertex v of G, $f(v)$ is the union of at most t finite closed intervals, and (2) v and w are adjacent in G if and only if $f(v)$ intersects $f(w)$. (We may assume that different intervals do not share endpoints. Hence it is also irrelevant whether the intervals are open or closed; they may even consist of single points, as in [HTW].) The interval

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number of G, denoted $i(G)$, is the smallest integer t such that G has a t-representation. Interval graphs are those with $i(G) \leqslant 1$. Bounds on $i(G)$ in terms of other parameters appear in [HT, HTW, G, GW, and TH]. Recently, West and Shmoys [WS] showed that for a fixed value of $t \geqslant 2$ it is NP-complete to determine whether $i(G) \leqslant t$.

Several authors have remarked that interval graphs and interval numbers are useful in scheduling and allocation problems. Here we describe only an application where "multiple-interval" graphs may turn out to be more appropriate than interval graphs. Benzer [B] used interval graphs to model the linear arrangement of nucleotides in genes encoded in DNA. DNA is a linear molecule composed of a sequence of sub-units, each of which is one of four possible nucleotide combinations. For many years biologists thought that the sequence of nucleotides encoding a given gene appear in a single unbroken sequence, covering a single "interval" on the chain of nucleotides. Recently, Chambon [C] and others showed that many genes are not represented as single unbroken sequences but rather as a collection of unbroken sequences on the DNA strand. This is analogous to a family of intervals on the real line. Thus, interval numbers may be a useful tool for studying the structure of genes. Perhaps some upper bound can be given on the number of intervals used for each gene, in which case interval numbers could be used to test gene compositions.

Restricting our attention to planar graphs, our primary concern in this paper is to prove

THEOREM 1. *If G is a planar graph, then $i(G) \leqslant 3$, and this is the best possible bound.*

In Section 2 we discuss the related notion of displayed interval number, introduced by Trotter and Harary [TH]. A *displayed t-representation f* of a graph G is one in which for every vertex v in G there is some open interval in $f(v)$ that belongs to no other $f(w)$. The *displayed interval number* $\hat{\imath}(G)$ is the smallest t for which G has a displayed t-representation.

In Section 3 we use the relationship between $\hat{\imath}$ and i to obtain planar graphs with $i(G) = 3$. In Section 4 we show that three intervals always suffice. When the question of determining the interval number of planar graphs first arose, Trotter noted that the existence of a vertex with degree at most 5 immediately gives an inductive upper bound of 6 on the displayed interval number of planar graphs. Subsequently, Saks, Trotter, and the present authors independently gave short proofs that planar graphs have interval number at most four. (Theorem 4 in Section 5 is a stronger result, also with a short proof.) The simplest proof is that of Trotter. He noted that the interval number of a graph is at most one more than its arboricity, which is the minimum number of spanning forests needed to partition its edges. It is known that the arboricity is at most two for triangle-free planar graphs or for outerplanar graphs, and at most three for any planar graph. Thus, arbitrary planar graphs have $i(G) \leqslant 4$, and triangle-free planar graphs have $i(G) \leqslant 3$. Hence, the main content of this paper is to reduce the upper bound for arbitrary planar graphs from four to three and to construct triangle-free planar graphs with $i(G) = 3$.

Section 5 contains results on the depth-r interval numbers for planar graphs.

2. Displayed Representations

Given a t-representation $f(G)$, a *displayed interval* for v is an interval in $f(v)$ containing an open interval intersecting no other $f(w)$. So, a t-representation is displayed if every vertex has a displayed interval. Displayed representations are useful because they behave well in induction proofs, since intervals for new vertices can be placed within their neighbors' displayed intervals to obtain new representations. Many optimal upper bound results for $i(G)$ use this technique (see [GW, G, HT, HTW, TH]); we will use it also. Its application rests on several straightforward remarks.

Lemma 1. $i(G) \leqslant \hat{\imath}(G) \leqslant i(G) + 1$.

Proof. Any displayed t-representation is a t-representation, and any $i(G)$-representation is made displayed by adding one displayed interval for each vertex. ∎

Lemma 2. *Let G be a graph with n vertices $\{v_i\}$. Let G^+ be the graph obtained by adding n vertices $\{w_i\}$ and the n edges (v_i, w_i). Then $\hat{\imath}(G) = i(G^+)$.*

Proof. t-Representations of G^+ and displayed t-representations of G can be transformed into one another simply by deleting or adding intervals for $\{w_i\}$. ∎

Theorem 2. *If $\mathbf{G}$ is a family of graphs such that $G \in \mathbf{G}$ implies $G^+ \in \mathbf{G}$, then*

$$\sup_{G \in \mathbf{G}} i(G) = \sup_{G \in \mathbf{G}} \hat{\imath}(G).$$

Proof. Immediate from Lemmas 1 and 2. ∎

If G is planar, then clearly G^+ is also planar. We will prove the following equivalent form of Theorem 1.

THEOREM 1′. *If G is planar, then $\hat{\imath}(G) \leqslant 3$, and this is the best possible bound.*

3. PLANAR GRAPHS WITH INTERVAL NUMBER 3

In this section we demonstrate that $\hat{\imath}(K_{2,9}) = 3$; we believe $K_{2,9}$ is the smallest such planar graph. For this we discuss two phenomena that reduce the number of edges representable in a given representation. These are of more interest than $K_{2,9}$ itself, because they may lead to a characterization of the planar graphs with $i(G) = 3$. Also, the first is used in the upper bound proof in Section 4.

Fix a t-representation f for a graph G. We say that a vertex v has a *broken end* in f whenever some endpoint of an interval in $f(v)$ belongs to no other interval of f. If $f(v)$ consists of fewer than t intervals, adding the missing ones in an unused portion of the real line would add two broken ends for each additional interval. Hence the "*effective*" *number of broken ends for v* is $b(v) = j + 2(t - s)$, where j is the number of actual broken ends in $f(v)$, and $f(v)$ consists of s disjoint intervals. Let $b^* = \Sigma b(v)$.

Suppose that vertices v and w are adjacent in G, so that $f(v)$ and $f(w)$ must meet. This may happen in more than one place, making the representation "redundant." Let the *redundancy* of v, denoted $r(v)$, be the number of *extra* times intervals for v meet other intervals. Then $r(v) = k(v) - d(v)$, where $k(v)$ is the total number of intersections between intervals for v and intervals for other vertices, and $d(v)$ is the degree of v. Let $r^* = \Sigma r(v)$. $\Sigma k(v)$ counts each intersection twice, so the total number of intersections of intervals in f is $|E| + \frac{1}{2}r^*$, where E is the edge set of the graph.

Note that r^* and b^* must both be even. Whenever an interval for v meets an interval for w, an interval for w meets one for v. Similarly, the broken ends come in pairs that "face each other," except for the leftmost and rightmost, which also form a pair.

A t-representation has *depth* 2 if no point on the real line lies in more than 2 intervals. (See also Section 5.) Note that any representation of a triangle-free graph has depth 2. In a depth-2 t-representation, the edges can be counted using the broken ends and redundancy.

LEMMA 3. *If G is a graph on n vertices that has a depth-2 t-representation f, then*

$$|E| = nt - \frac{b^* + r^*}{2}.$$

Proof. If any vertex is assigned fewer than t intervals, add intervals in an unused portion of the line so the representation will use t intervals per vertex.

This does not change b^* or r^*. Let the intervals be $I_1,\ldots,I_{nt}$, labeled in increasing order of left endpoints.

Count the total number of intersections among these nt intervals. In a representation with depth 2, each I_j intersects at most one interval with lower subscript, since each such interval contains the left endpoint of I_j. I_j intersects no interval of lower subscript if and only if its left endpoint is a broken end. Since every intersection pairs some interval with one of lower subscript, the number of intersections is $nt - \frac{1}{2}b^*$. As remarked above, this also equals $|E| + \frac{1}{2}r^*$, so $|E| = nt - \frac{1}{2}(b^* + r^*)$. ∎

LEMMA 4. *In any t-representation, $r(v) + b(v) \geqslant t - d(v)$ for any vertex v. In a displayed t-representation, the inequality must be strict.*

Proof. Again augment f so that $f(v)$ consists of t disjoint intervals, without changing $b(v)$ or $r(v)$. Assume that $r(v) + d(v) \leqslant t$, since otherwise the claim holds. As before, the number of intersections involving intervals in $f(v)$ is $r(v) + d(v)$. Let $j = t - d(v) - r(v)$; there are at least $j \geqslant 0$ intervals for v that intersect no other interval. Hence $b(v) \geqslant 2j$. Since $r(v) + b(v) \geqslant (t - d(v) - j) + 2j = t - d(v) + j$, the desired inequality holds except possibly when $j = 0$, $b(v) = 0$, and the representation was required to be displayed. In this case, there are exactly t intersections between intervals for v and other intervals. If any interval in $f(v)$ does not intersect another then $b(v) \geqslant 2$. Therefore, $f(v)$ consists of t disjoint intervals, each of which intersects exactly one other interval. If $b(v) = 0$, then each of the intervals in $f(v)$ is entirely contained within the interval it intersects, and v has no displayed interval. ∎

Now we apply these results to $K_{2,9}$.

LEMMA 5. $\hat{\imath}(K_{2,9}) = 3$.

Proof. Assume a displayed 2-representation for $K_{2,9}$; it must have depth 2. Applying Lemma 4 to the 9 vertices of degree 2 yields $b^* + r^* \geqslant 9$. Applying Lemma 3,

$$18 = |E| = nt - \tfrac{1}{2}(b^* + r^*) \leqslant 11 \times 2 - \tfrac{1}{2} \times 9 = 17.5,$$

so $\hat{\imath}(K_{2,9}) \leqslant 2$ is impossible. A displayed 3-represssentation of $K_{2,9}$ is shown in Fig. 1. ∎

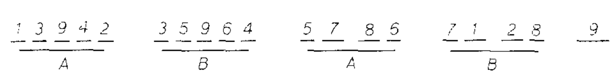

FIG. 1. A displayed 3-representation of $K_{2,9}$.

4. Three Intervals Suffice

We will obtain the bound inductively, but the induction requires a much more elaborate hypothesis than simply $\hat{\imath}(G) \leqslant 3$. This is typical of proofs about displayed interval numbers. The examples we give to motivate the terminology and specifications appearing in the induction hypothesis are in fact cases that arise in the proof, and we will cite them again there.

A *plane graph* is a particular planar embedding of a planar graph. The vertices on the unbounded face of a plane graph are called *external* vertices or *outervertices*; the others are *internal* vertices or *innervertices*. An edge between external vertices is called an *external edge* if it is an edge of the unbounded face; otherwise, it is called a *chord*. The *innerneighbors* or *outerneighbors* of a vertex are its neighbors that are inner- or outervertices. Denote by G^0 the subgraph of G induced by its outervertices. The edges of G^0 are the chords and external edges of G. We use $x \sim y$ to read "x neighbors y" or "x is adjacent to y." If $f(v)$ consists of s disjoint closed intervals, then we say that v *appears* s *times* in the representation f.

To perform induction we delete external vertices. Note that the innerneighbors of any external vertex v are external vertices in the induced subgraph $G - v$. If they appear only twice in a representation for $G - v$, we can add a third interval for each of them within a displayed interval for v. Accordingly, we will construct a 3-representation of each plane graph in which each external vertex appears at most twice.

This does not yet represent G since v also has outerneighbors, which must not be assigned a third interval. To overcome this problem, the induction hypothesis must specify *how* the edges of G^0 must be represented.

We say that an edge uv is *displayed* in f if $f(u) \cap f(v)$ contains some open interval that intersects no other $f(w)$. This interval is the *displayed portion for* uv. Suppose v is an external vertex of G with outerneighbors x and y, where xy is a chord. Let f be a representation for $G - v$, and add a displayed interval for v containing new intervals for the innerneighbors of v. If the chord xy is displayed, place the second interval available for v in the displayed portion for xy, as illustrated in Fig. 2. If v has no other outerneighbors, then this yields a representation of G. (In the illustrations for

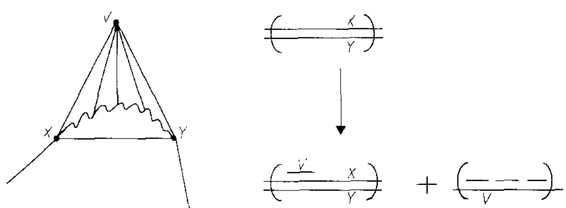

Fig. 2. Adding an outervertex joined to the endpoints of a displayed chord.

these constructions, we indicate displayed portions of the representation by parentheses.)

Unfortunately, $G - v$ may have no representation in which all chords are displayed. For example, in the inductive construction described above the new edges vx and vy are not displayed, and they may become chords as more vertices are added. So, we must allow another way to represent edges of G^0.

Suppose v, x, y are as above, but xy is not displayed. Suppose instead that $f(x)$ contains a point b which is either a broken end for v (contained in no other intervals) or belongs to an interval for exactly one other vertex u, an innervertex in G. The modification we make to f to add v in this case is shown in Fig. 3. The vertex u must be external in $G - v$, so it appears at most twice in f. Split the interval in $f(u)$ that covers b into two pieces, baring b. Add the displayed interval for v so that it overlaps $f(x)$ at b. Add intervals for the innerneighbors of v as before, except that if $v \sim u$ extend the displayed interval for v that intersects $f(x)$ far enough to intersect $f(u)$ also. This optional overlap is indicated by brackets in the figure. We complete the representation for G by adding a second interval for v in the displayed portion of $f(y)$. If b is a broken end in f, simply delete u from the discussion and illustration.

Note that in this construction the new edges vx and vy are displayed. Also, in the first construction (Fig. 2) the new edges vx and vy are represented in the second manner, with the endpoints of the displayed interval for v serving as broken endpoints for these edges. This suggests that allowing this much flexibility in f may produce an induction hypothesis that works. Hence we formalize these ideas with definitions.

Let f be a t-representation of G, and let xy be an edge of G^0. An endpoint b of an interval in $f(x)$ is called a *reusable endpoint* for the edge xy if either

(1) b is a broken end, or

(2) b belongs to $f(u)$ for only one other vertex u, and ux is an external edge.

In the latter case, we say that u *covers* b.

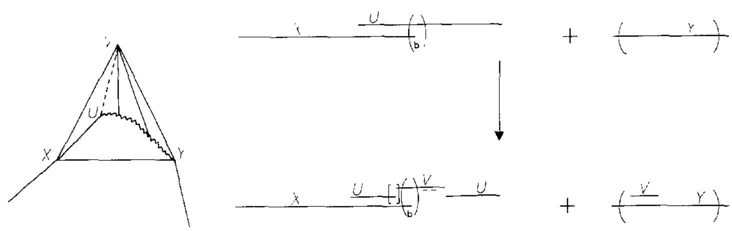

FIG. 3. Adding an outervertex joined to the endpoints of a non-displayed chord.

Given a representation f for G, we want to assign reusable endpoints to the edges of G^0 that are not displayed in f. A *proper assignment of reusable endpoints* assigns a reusable endpoint to every non-displayed edge of G^0 such that:

(1) no endpoint is assigned as the reusable endpoint for more than one edge of G^0,

(2) no outervertex u covers more than one assigned reusable endpoint, and

(3) any non-displayed *external* edge has a broken end as its reusable endpoint.

Note that the requirement on external edges is slightly stronger than the requirement on chords, but requiring that external edges be displayed would be too strong.

In the construction in Fig. 3, it was necessary not only that the covering vertex u be external in $G-v$, but also that it be internal in G, so that we could split the covering interval in $f(u)$ without creating too many intervals for u. This requires that the edges vx and vy enclose u.

To formalize this, let xy be a chord (*not* an external edge) of a plane graph G, and let $\{z, u\}$ be two other external vertices of the same component. Deleting $\{x, y\}$ disconnects this component. If $\{z, u\}$ lie in the same component of $G-x-y$, we say that u is *on the* z*-side* of xy in G. If they lie in different components, we say that u is *on the* z^**-side* of xy *in* G. We are particularly interested in the case where $\{z, x, y\}$ lie in the same block, where a *block* of G is a maximal 2-connected subgraph of G. We say that a reusable endpoint for xy is z^**-reusable* if it is a broken endpoint or is covered by a vertex u which lies on the z^*-side of xy and in the same block as xy. (Recall that in the latter case ux must be an external edge.)

Restricting vertices to a specified side of a chord suggests consideration of "rooted" plane graphs, which will yield a natural choice of vertices to delete for the induction. To specify these we need further definitions.

A planar graph is *outerplanar* if it can be embedded in the plane so that every vertex is on the unbounded face; for outerplanar graphs we discuss only embeddings of that type, in which case $G^0 = G$. Given a plane graph G, its *dual* graph G^* has a vertex for each face of G and an edge between two vertices if and only if the corresponding faces of G share an edge. The *weak dual* G^w is obtained by deleting from G^* the vertex corresponding to the unbounded face of G. Fleischner, Geller, and Harary [FGH] noted that the weak dual of an outerplanar graph is a forest. Moreover, the weak dual of each block of an outerplanar graph is a tree. Blocks containing at most one cutvertex (articulation point) of G are *endblocks* of G. The external vertices we delete for induction will come from endblocks of G^0.

We define a *rooted plane graph* by choosing as a root for a plane graph G (in each component) an external vertex z_0 that belongs to only one block H_0 of G^0. In other words, z_0 is not a cutvertex of G. (For example, z_0 can be a leaf of any spanning tree of any component of G.) For a block H in a rooted plane graph, let z_H be the first cutvertex encountered in any path from any vertex of H to the root of the component containing H; z_H is a natural "initial" vertex in H. The choice of the vertices z_H is illustrated in Fig. 4.

We will pull external vertices from the very "tips" of endblocks to do induction, so the reusable endpoints in block H of G^0 will have to be z_H^*-reusable. However, z_H^*-reusable makes no sense for edges containing z_H itself. Fortunately, these edges are even more nicely behaved than external edges; we can require them to be displayed.

We define a *fan* F to be a 2-connected outerplanar graph in which all chords contain a single vertex z, as illustrated in Fig. 4. Note that the external edges of a fan with more than two vertices form a cycle, but that not all the vertices need be joined to z. If H is a block of G^0 in a rooted plane graph G, we define $F(H)$ to be the largest fan in H containing z_H. It contains all the outerneighbors of z_H in H. (See Fig. 4.)

Now we are ready to prove that $\hat{\imath}(G) \leqslant 3$ for any plane graph. We prove a stronger theorem that implies Theorems 1 and 1′. Given a rooted plane graph G, a *P-special representation* ("P" for "plane") of G is a displayed 3-representation f with a proper assignment of reusable endpoints such that:

(1) each root appears once and every other external vertex appears at most twice,

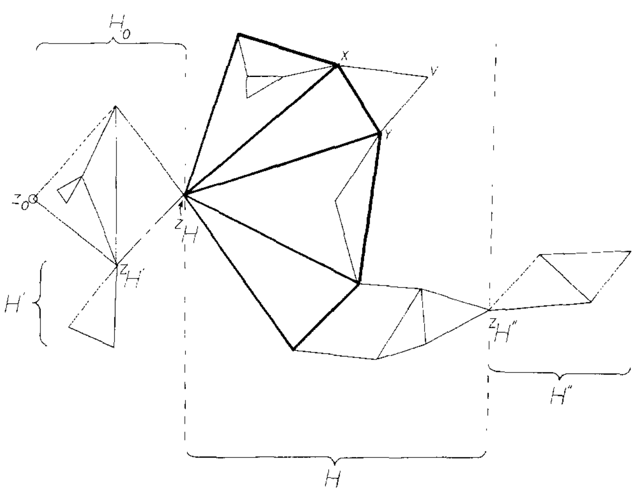

FIG. 4. A rooted plane graph.

(2) every edge of H containing z_H is displayed, where H is any block of G^0, and

(3) any edge of H that is not displayed is assigned a z_H^*-reusable endpoint.

THEOREM 1″. *Every rooted plane graph G has a P-special representation.*

Proof. The claim is trivial if G consists of isolated vertices. Suppose it has been shown for all rooted plane graphs on fewer than n vertices. Let G be a rooted plane graph on n vertices having at least one edge.

Choose an endblock H of G^0 that contains at least one edge, but do not use a root block unless there is no other block in that component of G^0. Let $z = z_H$. Now either $H = F(H)$ or $H \supset F(H)$. In either case we will delete one or more vertices from H and form a P-special representation for the resulting induced subgraph of G. Then we will show how to extend that representation to a P-special representation for G.

Case I. $H = F(H)$

This case is illustrated in Fig. 5. Label the vertices of H as $z, v_1, \dots, v_k$ in order on the external face of H. Let $z = v_{k+1}$. Let $d_H(v)$ be the degree in H of a vertex v. Then $d_H(v_1) = d_H(v_k) = 2$, and $d_H(v_i)$ may be 2 or 3 for $1 < i < k$.

Let p be the smallest integer with $2 \leqslant p \leqslant k$ such that $d_H(v_p) = 2$, if $k > 1$; otherwise, set $p = 1$. Let $V = \{v_1, \dots, v_p\}$. Delete the vertices of V from G, and form a P-special representation f for the induced subgraph $G - V$. We modify f to obtain a P-special representation for G as indicated in Fig. 5.

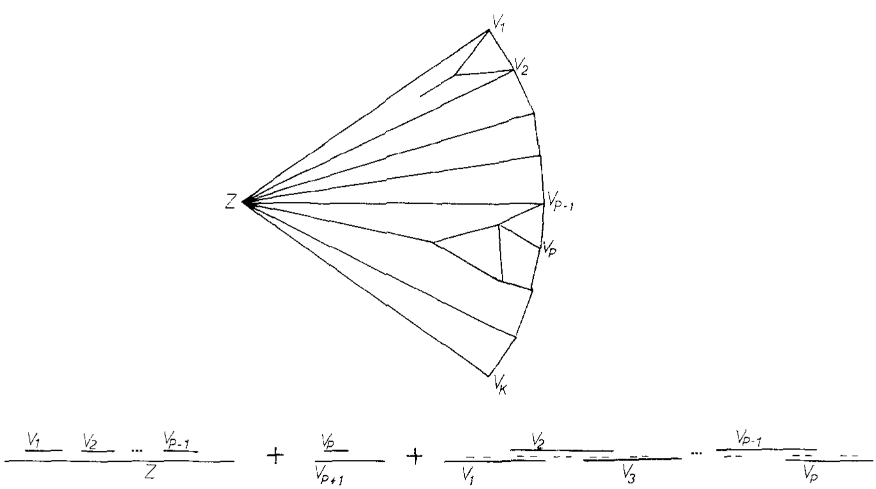

FIG. 5. Performing induction when $H = F(H)$.

Insert intervals for $v_1,\ldots, v_{p-1}$ into the displayed interval for z. Insert an interval for v_p into the displayed interval for v_{p+1}. Place displayed intervals for the vertices of V in an unused portion of the line, overlapping consecutively to create intersections representing the edges of the path $v_1,\ldots, v_p$. Finally, consider possible innerneighbors of V. If wv_i and wv_j were edges with $j > i+1$ and $i < p$, then one of those edges would cross zv_{i+1}. So, other vertices neighbor at most two *consecutive* v's. These innerneighbors w are external in $G - V$, so insert an interval for w in the displayed portion of v_i or of v_iv_{i+1} to represent these edges.

Note that this construction is valid even when $p = 1$ $(H = zv_1)$, or $p = k$ $(v_{p+1} = z)$. To prove that the modified f is P-special for G, consider the edges of G^0. The added vertices and edges within V are displayed, as are the edges zv_i and v_pv_{p+1}. Other edges of G^0 were properly represented in f and remain so, since their representation and the assignment of reusable endpoints has not been changed in any way. In particular, vertices that cover reusable endpoints for such chords remain external vertices on the z^*-side in the appropriate block. Chords of $(G - V)^0$ that are not chords of G^0 no longer have reusable endpoints, but they do not need them. Indeed, deleting V may disconnect the component, but this causes no problem. The requirements of a P-special representation concerning the representation of chords make $d_H(v_p) = 2$ important, and also make it difficult to delete all of the v_i in one step. Hence the choice of p as specified.

Case II. $H \neq F(H)$

We wish to delete a vertex or two from a "leaf face" of the endblock H of G^0. Such a face is illustrated by the triangle vxy in Figs. 2 and 3. Observe that the weak dual of $F(H)$ is always a path, and the weak dual of H is a tree consisting of this path and subtrees growing from it, since the vertices of $F(H)$ are all external. Since H contains more than $F(H)$ and is 2-connected, we can choose an endvertex (leaf) of H^w that does not belong to $F(H)^w$. We call the corresponding face of H a *leaf face* of H. A leaf face of H contains exactly one chord of G, and all of its remaining edges are external edges of G. Call this unique chord xy. Now x and/or y may lie in $F(H)$, but neither can be z. Also, no vertices of the leaf face other than $\{x, y\}$ can be in $F(H)$.

Choose a leaf face C from the endblock H. We must consider two cases: either (a) C is a triangle, or (b) C has 4 or more vertices.

Subcase IIa. *C is a triangle.* This is the case we described in developing the induction hypothesis. Let v be the third vertex of C, and let f be a P-special representation for the induced subgraph $G - v$. If the edge xy is displayed in f, modify f as indicated in Fig. 2. Otherwise, xy has a z^*-reusable endpoint in f, in which case modify it as indicated in Fig. 3.

If xy is displayed in f, then in the modification xy is still displayed, but the

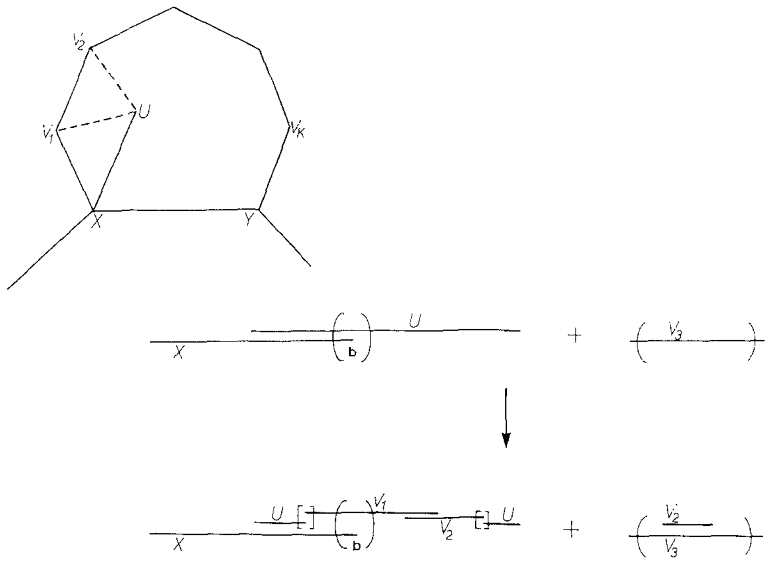

FIG. 6. Adding two vertices to complete a leaf face.

new external edges vx and vy are not displayed. Assign the left endpoint of the displayed interval for v as the z^*-reusable endpoint for vx and the right endpoint as the z^*-reusable endpoint for vy. This extension of f gives a P-special representation for G, since the new external edges are assigned broken ends and the other edges of G^0 remain properly represented.

If xy is not displayed in f, then in the modification the edges vx and vy are displayed. The change is that v becomes the external vertex covering b, which is still z^*-reusable for xy, since vx is an external edge in the block containing xy, and v covers no other reusable endpoint. As usual, since we have made no other changes, the new representation has a proper assignment of reusable endpoints and is P-special for G.

Subcase IIb. *C is not a triangle.* Let the vertices of C be (in cyclic order) $x, v_1, v_2, \ldots, v_k, v_{k+1} = y$, where xy is the chord of G and the other edges of C are external. Delete the vertices $V = \{v_1, v_2\}$ to form the induced subgraph $G - V$, and let f be a P-special representation for $G - V$. To obtain a P-special representation for G we procede as follows (Fig. 6 illustrates the most complicated case).

(1) Assign v_1 an interval in the displayed interval for x, and assign v_2 an interval in the displayed interval for v_3.

(2) The next step in the construction depends on how xy is represented in f.

(2a) *Edge xy is displayed or is assigned a z^*-reusable broken end b.* Place overlapping displayed intervals for v_1 and v_2 in an unused portion of the line.

(2b) *Edge xy is assigned a z^*-reusable covered endpoint $b \in f(x)$.* The covering vertex u must be an outerneighbor of x in $G - V$, but it must be internal in G, since C is a leaf face of G and has no chords. Hence, there is an extra interval available for u; use it by splitting the covering interval for u to expose b as a broken end. Place overlapping displayed intervals for v_1 and v_2 in the gap between b and the new broken end for u, with the interval for v_1 nearest to b. If $v_1 \sim u$ or $v_2 \sim u$, extend the corresponding interval to intersect $f(u)$, as illustrated by the brackets in Fig. 6.

(2c) *Edge xy is assigned a z^*-reusable covered endpoint $b \in f(y)$.* If $k = 2$, interchange the roles of x and y and procede as in step (2b). If $k > 2$, then the covering vertex u must be v_k, since uy must be an external edge of $G - V$ in the same block as xy. Furthermore, $v_1 \nsim v_k$, since C has no chords. Hence we can procede as in step (2a), with v_k continuing to cover the still z^*-reusable b.

(3) Finally, add intervals for the innerneighbors of v_1, v_2 in the displayed portions for v_1, v_2, and $v_1 v_2$, according to which subset of these two vertices they neighbor. As before, these innervertices are external in $G - V$ and appear at most twice in f.

It is routine to verify that we have constructed a P-special representation for G. The new vertices $\{v_1, v_2\}$ and the external edges xv_1, $v_1 v_2$, and $v_2 v_3$ are all displayed. The edge xy remains displayed if it was before. Otherwise, b remains a z^*-reusable endpoint for xy. It is either a broken end or is covered by v_1 or v_k as described above, and both xv_1 and yv_k are external edges of G in the same block as xy. No changes have been made in the representation of other edges, so again the new representation has a proper assignment of reusable endpoints and is P-special for G.

Finally, note that a vertex is assigned more than one interval only when added to a smaller graph, when it becomes internal by the addition of other vertices, or when it covers a reusable endpoint and is split. Therefore, the roots always appear only once. This completes the proof. ∎

The fact that we can require roots to appear only once was not necessary to the induction, but it may be useful in inductive applications of this theorem. An immediate consequence of the theorem is

COROLLARY 1. *If G is an outerplanar graph, then $\hat{\imath}(G) \leqslant 2$ and $i(G) \leqslant 2$, and these bounds are best possible.*

Proof. Note that if G is outerplanar, then so is G^+, hence Theorem 2 applies here. In the P-special representation of G all outervertices, hence all vertices, appear at most twice, hence $\hat{\imath}(G) \leqslant 2$. The 4-cycle shows this is best possible. ∎

The proof of Theorem 1″ gives an $O(|V(G)|)$ algorithm for finding a 3-representation of a planar graph. However, it is not guaranteed to find a 2-representation if one exists, so it does not determine the interval number. The tractability of determining the interval number of a planar graph remains an open question.

5. Depth-r Interval Number of Planar Graphs

In this section we study interval representations of bounded depth. A t-representation has *depth r* if no point on the real line lies in more than r intervals of the representation. The [*displayed*] *depth-r interval number* of G, denoted $[\hat{\imath}_r(G)]\, i_r(G)$, is the least t for which G has a [displayed] t-representation of depth r. Since any depth-r representation is also a depth-$(r+1)$ representation, we have $i(G) \leqslant i_{r+1}(G) \leqslant i_r(G)$ and $\hat{\imath}(G) \leqslant \hat{\imath}_{r+1}(G) \leqslant \hat{\imath}_r(G)$. Mimicking the earlier proofs, we can replace i and $\hat{\imath}$ by i_r and $\hat{\imath}_r$ in Lemmas 1 and 2 and in Theorem 2.

Since a planar graph has no 5-clique, its representations have depth at most 4. Actually, the P-special representation constructed in the proof of Theorem 1″ has depth 3. This yields

THEOREM 3. *Suppose G is planar and $r \geqslant 3$. Then $\hat{\imath}_r(G) \leqslant 3$ and $i_r(G) \leqslant 3$, and these are the best possible bounds.*

Setting $r = 2$ may be more restrictive if the graph has triangles. For triangle-free planar graphs $i_2(G) = i(G) \leqslant 3$ and $\hat{\imath}_2(G) = \hat{\imath}(G) \leqslant 3$, but in general

THEOREM 4. *If G is a planar graph, then $i_2(G) \leqslant 4$ and $\hat{\imath}_2(G) \leqslant 4$, and these bounds are best possible.*

Proof. We proceed by induction using the following hypothesis:

($*$) If G is a plane graph on n vertices, then G has a displayed 4-representation of depth 2 in which outervertices appear at most 3 times.

Clearly ($*$) holds for graphs with no edges. Let G be a plane graph on n vertices with at least one edge. G must have an outervertex v with $d_{G^0}(v) \leqslant 2$. Let f be a 4-representation of $G - v$ satisfying ($*$). Augment that representation by assigning v a displayed interval and an interval in the displayed portion of each of its outerneighbors. The innerneighbors of v are outervertices in $G-v$ and therefore appear at most 3 times in f. Assign each innerneighbor of v an interval in the new displayed interval for v. The resulting representation of G satisfies $(*)$.

This displayed bound is achieved by the triangulated planar graph G on 22 vertices shown in Fig. 7. Graph G is obtained by triangulating both the inside and outside faces of a 7-cycle by a single vertex of degree 7, and then triangulating each of the 13 resulting bounded faces by a single vertex of degree 3. By Euler's formula, G has 60 edges. If G has a displayed 3-interval representation of depth-2, then by Lemma 4 each vertex v of degree 3 satisfies $r(v)+b(v)>0$. Hence $r^*+b^*\geqslant 13$, and Lemma 3 yields the contradiction

$$60=E=3V-\tfrac{1}{2}(b^*+r^*)\leqslant 66-\tfrac{1}{2}13=59.5. \quad \blacksquare$$

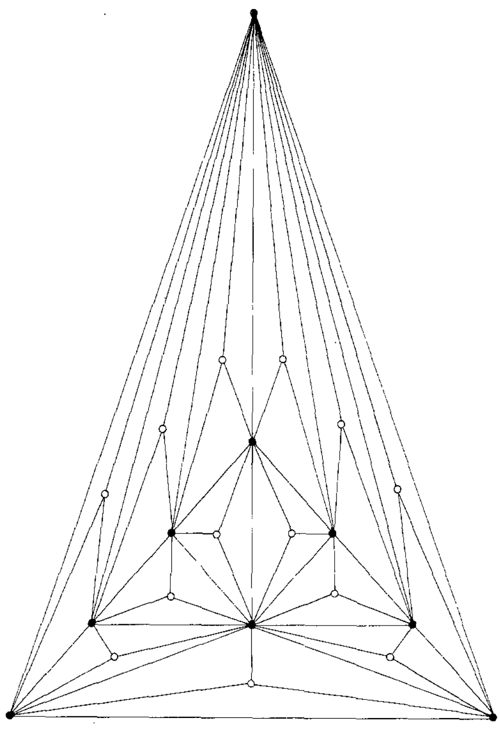

FIG. 7. A planar graph with $\hat{t}_2(G)=4$.

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