An Improved Edge Bound on the Interval Number of a Graph

Jeremy R. Spinrad[1]
G. Vijayan[2]
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GEORGIA
Douglas B. West[3]
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS 61801

ABSTRACT

The upper bound on the interval number of a graph in terms of its number of edges is improved. Also, the interval number of graphs in hereditary classes is bounded in terms of the vertex degrees.

A representation of a graph as an intersection graph assigns each vertex a set such that vertices are adjacent if and only if the sets intersect. The *interval number* $i(G)$ is the minimum t such that G is the intersection graph of sets on the real line consisting of at most t intervals. A representation by intervals is *displayed* if the set assigned to each vertex has an open interval disjoint from the other assigned sets; such an interval is a *displayed segment*. Extremal results on interval number in terms of other graph parameters or for special classes of graphs appear in [1–9]. In particular, it is shown in [5] that $i(G) \leq \lfloor \sqrt{e} \rfloor$ for a graph with e edges. This bound is twice the conjectured optimum, which for $e \geq 4$ is $\lceil \frac{1}{2}(\sqrt{e} + \sqrt{e^{-1}}) \rceil$, achieved by $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$. In this note, we improve the upper bound by a factor of $\sqrt{2}$, to $i(G) \leq 1 + \lceil \sqrt{e/2} \rceil$. We use the result in [4] that every graph on n vertices has an interval representation

[1]Present address Vanderbilt Univ., Nashville, TN 37235
[2]Present address IBM Corp., Yorktown Heights, NY
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with at most $\lceil (n+1)/4 \rceil$ intervals per vertex (this is best possible, again by $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$).

Theorem. Every graph with e edges has a displayed interval representation with at most $1 + \lceil \sqrt{e/2} \rceil$ intervals for each vertex.

Proof. We construct a representation of G with at most this many intervals per vertex. First, partition the vertices into those with degree at least $(\sqrt{2e}+1)/2$ and those with degree at most $(\sqrt{2e}-1)/2$. The number of high-degree vertices is at most $\sqrt{8e}-1$, since when there are k vertices of degree at least α we have $\alpha k \leq 2e$. The vertex bound of [4] guarantees that the subgraph induced by the high-degree vertices has an interval representation with at most $\lceil (\sqrt{8e}-1+1)/4 \rceil = \lceil \sqrt{e/2} \rceil$ intervals for each vertex.

To guarantee that the representation is displayed, add an isolated interval for each vertex. Finally, for each edge not joining two high-degree vertices, take an endpoint having low degree and place a small interval for it in a displayed segment for its neighbor. This step adds at most $(\sqrt{2e}-1)/2$ intervals for each low-degree vertex and none for each high-degree vertex, so we have a representation for G using at most $1 + \lceil \sqrt{e/2} \rceil$ intervals for each vertex. ∎

By taking a careful look at how close the proof of [4] is to guaranteeing displayed representations with at most $\lceil (n-1)/4 \rceil$ intervals per vertex when $n \geq 4$ (the induction argument preserves this, but a large and tedious basis case would be required because the triangle K_3 fails the bound), it should be possible to shave the additive constant somewhat smaller. However, since the multiplicative constant is believed nonoptimal, such an effort seems unjustified.

The idea of inserting a vertex of low degree at the end of the construction also yields the following remark. If **F** is a hereditary family of graphs (closed under vertex deletions) such that every member of **F** on n vertices has a vertex of degree at most $f(n)$ for some nondecreasing function $f(n)$, then every member of **F** on n vertices has a displayed interval representation with at most $f(n)+1$ intervals per vertex. This generalizes the bound in [9] on the interval number of forests, but in general it is not tight.

Nevertheless, this remark shows that one of the results of [3] has the correct rate of growth. Let $z(n,m)$ be the largest number of edges in a $K_{m,m}$-free bipartite graph (no induced $K_{m,m}$) with $n/2$ vertices in each part. In [3], a count of the possible representations shows there exist $K_{m,m}$-free bipartite graphs on n vertices with interval number at least $z(n,m)/(2n \lg n)$. For $m=2$ this existence bound can be improved to at $\sqrt{(n/4)} + o(\sqrt{n})$. Since $z(n,2) \in O(n^{3/2})$, the minimum degree for graphs in the hereditary class of C_4-free bipartite graphs is $O(n^{1/2})$, and thus every C_4-free bipartite graph has interval number bounded by some constant times $\sqrt{n}$. For $m > 2$ the existence bound is weaker, and the upper bound $z(n,m)/n$ on the interval number of $K_{m,m}$-free bipartite graphs is a factor of $c \lg n$ from the current lower bound for the worst case.

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