A Characterization of Influence Graphs of a Prescribed Graph

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Abstract. For a given graph G, and subset X of the vertex set, the closed influence graph, I*(G, X), of G with respect to X, has vertex set X with uv an edge if and only if the distance in G from u to v is at most the sum of the distances in G from u to its closest neighbor in X and v to its closest neighbor in X.

In this paper, the graphs H that arise as closed influence graphs, I*(G, X) are completely characterized, thus answering a question of Harary, Jacobson, Lipman and McMorris.

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1. Introduction.

In trying to capture the perceptual relevance of a given set of points in the plane, which might represent a (possibly very sketchy or inaccurate) dot picture, Toussaint [5] defined two new types of proximity graphs. Let S be a finite set of at least two points in the plane. For each point x in S, let r_x denote the smallest distance from x to any other point in S. Let B_x and C_x be the open and closed disks of radius r_x centered at x, respectively. The *sphere-of-influence graph of S* is the intersection graph of the B_x's. That is, it has vertex set S and two vertices x and y are adjacent if and only if B_x and B_y have a non-empty intersection. The *closed sphere-of-influence graph of S* is defined similarly using closed disks. For convenience we will refer to these as SIGs and closed SIGs or CSIGs, respectively. A graph G is an abstract SIG if it is isomorphic to some SIG, G*, which is then said to realize G. Several results for SIGs and abstract SIGs are given in [1], while trees realizable by SIGs or CSIGs are characterized in [4]. It was also shown that for any triangle-free SIG with n vertices contains at most 4.5n edges.

In [2], this idea was generalized by using the natural distance metric induced by a graph. For a graph G, and $x,y \in V(G)$ in the same component of G, the distance in G from x to y, denoted $d_G(x,y)$, is the number of edges in a shortest path in G from x to y. If G contains no path from x to y then we will say that the distance is infinite. When no confusion will result, we simply use d(x,y). We now introduce the concept of a SIG of a graph.

Let G be a graph and X a nonempty subset of V(G). For each $x \in X$, let c(x) be a vertex in X - {x} whose distance to x is as small as possible. Note, c(x) may be chosen to be any one of x's closest neighbors in X. The *influence graph of G with respect to X*, denoted I(G, X), is the graph with

$$V(I(G, X)) = X \quad \text{and}$$

$$\text{for } x,y \in X,\ xy \in E(I(G, X)) \text{ if and only if}$$

$$d_G(x,y) < d_G(x,c(x)) + d_G(y,c(y))$$

This generalizes the idea of SIG's by incorporating a metric distinct from the Euclidean metric. As in the original model, these graphs can be considered to be intersection graphs, where the set corresponding to each vertex x in X, is precisely the subset of vertices in G a distance at most $d_G(x,c(x))$ from x. We might think of this as the sphere of influence of x in G, with respect to X.

We also can generalize the idea of a closed SIG. The *closed influence graph of G with respect to X,* denoted I*(G, X), is the graph with

$$V(I^*(G, X)) = X \quad \text{and}$$

$$\text{for } x,y \in X,\ xy \in E(I^*(G, X)) \text{ if and only if}$$

$$d_G(x,y) \le d_G(x,c(x)) + d_G(y,c(y)).$$

For convenience, we say that H is a (closed) influence of a graph when there exists a graph G and subset X so that H is isomorphic to I(G, X) (I*(G, X)). In this case we simply say H = I(G, X) (I*(G, X)). In keeping with the terminology used in [2], we will also say that H is *realized* by G and X. For any undefined terms or notation the reader is refered to [3].

It is easily seen that all graphs with no isolated vertices can be realized by an "open" influence graph of a graph as shown in [2]. Several examples of graphs that are and aren't closed influence graphs are also given in [2]. In this paper we answer the question, which graphs are realized as closed influence graphs of a prescribed graph.

A Characterization of Closed Influence Graphs of a Graph.

Before giving the characterization, some additional notation would be helpful. For a graph G, a set of cliques $\mathcal{K}$ in G is said to be a *(edge) clique cover* if each vertex (edge) of G is in at least one clique in $\mathcal{K}$. A set of cliques is a *clique partition* if it is a clique cover and each vertex is in a single clique. Note that every graph has a trivial clique partition, by considering each vertex as a clique by itself. We will call a clique partition *non-trivial* if each clique has order at least two. Finally, we *subdivide* an edge in a graph by introducing vertices of degree 2 on the edge. To *subdivide n times*, will mean that n new vertices are inserted.

Theorem. A graph H is realizable as the closed influence graph of a prescribed graph G, with respect to some subset of vertices X, if and only if H contains a non-trivial clique partition.

Proof. Let H be a graph with a non-trivial clique partition $\mathcal{K}$. Let G be the graph obtained from H by subdividing one time each edge not in $\mathcal{K}$. Let X be the vertices in G that correspond to the original vertices of H. Observe that since $\mathcal{K}$ was a non-trivial clique partition, $d(x,c(x)) = 1$ for each x in X. Also, any two vertices u and v of H joined by an edge which is not in an element of $\mathcal{K}$ has $d_G(u, v) = 2$ while all other pairs of vertices, i.e. not adjacent in H, are a distance of at least three apart in G. Consequently, it follows that $H = I^*(G, X)$, and thus every graph with a non-trivial clique partition is a closed influence graph.

To show that every closed influence graph $H = I^*(G, X)$ contains a non-trivial clique partition, we proceed by induction. It is easy to see that this is true for small order graphs, so let $H = I^*(G, X)$ have order n, and assume all closed influence graphs of order less than n have a non-trivial clique partition. We begin by making two observations. First, let x be any vertex in X, then the subset of vertices in X which have x as a closest neighbor induces a complete graph. This follows since if y and z have x as one of their closest neighbors then $d(y,c(y)) = d(y,x)$ and $d(z,c(z)) = d(z,x)$

and hence $d(y,z) \leq d(y,x) + d(x,z) = d(y,c(y)) + d(z,c(z))$. Also note that x is also adjacent to all of these vertices. Second, seleet u so that $d(u,c(u))$ is as small as possible. Let U be the set of vertices in X closest to u. Clearly, U is non-empty and as above U induces a complete graph. If v is in U and V is the subset of vertices, other than u, that have v as its closest neighbor in X, then u is adjacent to all those vertices. With these observations we are ready to proceed.

Let u be an element in X so that $d(u,c(u))$ is as small as possible. Let U be the set of vertices in X closest to u. Suppose $U = \{u_1, u_2, \ldots, u_k\}$. Let U_1, be the subset of vertices having u_1 as one of its closest neighbors in X. Let U_2, be the subset of unselected vertices having u_2 as one of its closest neighbors and so forth to U_k being the subset of unselected vertices having u_k as one of its closest neighbors. Note some or all of the U_i's may be empty, although U is definitely non-empty. Continue this process for all the elements in all the subsets U_i over and over until no new elements of X can be selected. Let Z' be the last non-empty subset of elements of X selected in this manner, say having z as their closest neighbor. For convenience, let $Z = Z' \cup \{z\}$.

If z is not in U, then for every element y in X-Z, there is an element y' in X-Z so that $d(y,y') = d(y,c(y))$, hence y' could be chosen as c(y) without disrupting the structure of the graph. That is to say, no vertex in X-Z depends on a vertex in Z to determine its closest neighbor. Thus, $I^*(G, X\text{-}Z) = H\text{-}\langle Z\rangle$, and since $H\text{-}\langle Z\rangle$ has order less than n, $H\text{-}\langle Z\rangle$ has a non-trivial clique partition, which with $\langle Z\rangle$ gives a non-trivial clique partition of H. If z is in U, but not the only element of U, then the same argument as above applies. If z is the only element in U, then $\langle Z \cup \{u\}\rangle$ is the subset to remove from X to arrive at a smaller closed influence graph. Finally if z = u then the original argument yields the non-trivial clique partition. ❑

In [2], it was shown that K_3-free graph, that is graphs with girth at least four, that are influence graphs must contain a perfect matching. Since the only non-trivial clique partitions in K_3-free graphs are perfect matchings, using the theorem, we get the following:

Corollary. Let H be a K_3-free graph, $H = I^*(G, X)$ for some G and subset X if and only if H contains a perfect matching.

We also note that in [2] the authors posed the problem for requiring X to be an independent set. By considering the construction of subdivision in the theorem, and now subdividing each edge in one of the non-trivial cliques once and all other edges twice, and by choosing X to be the original set of vertices, this set is now independent, and it is easy to see the desired graph is achieved. Hence restricting X to independent sets in fact is no restriction at all.

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