A Characterization of Influence Graphs of a Prescribed Graph

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Abstract. For a given graph G, and subset X of the vertex set, the closed influence graph, I*(G, X), of G with respect to X, has vertex set X with uv an edge if and only if the distance in G from u to v is at most the sum of the distances in G from u to its closest neighbor in X and v to its closest neighbor in X.

In this paper, the graphs H that arise as closed influence graphs, I*(G, X) are completely characterized, thus answering a question of Harary, Jacobson, Lipman and McMorris.

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1. Introduction.

In trying to capture the perceptual relevance of a given set of points in the plane, which might represent a (possibly very sketchy or inaccurate) dot picture, Toussaint [5] defined two new types of proximity graphs. Let $S$ be a finite set of at least two points in the plane. For each point $x$ in $S$, let $r_x$ denote the smallest distance from $x$ to any other point in $S$. Let $B_x$ and $C_x$ be the open and closed disks of radius $r_x$ centered at $x$, respectively. The sphere-of-influence graph of $S$ is the intersection graph of the $B_x$'s. That is, it has vertex set $S$ and two vertices $x$ and $y$ are adjacent if and only if $B_x$ and $B_y$ have a non-empty intersection. The closed sphere-of-influence graph of $S$ is defined similarly using closed disks. For convenience we will refer to these as SIGs and closed SIGs or CSIGs, respectively. A graph $G$ is an abstract SIG if it is isomorphic to some SIG, $G^*$, which is then said to realize $G$. Several results for SIGs and abstract SIGs are given in [1], while trees realizable by SIGs or CSIGs are characterized in [4]. It was also shown that for any triangle-free SIG with $n$ vertices contains at most $4.5n$ edges.

In [2], this idea was generalized by using the natural distance metric induced by a graph. For a graph $G$, and $x, y \in V(G)$ in the same component of $G$, the distance in $G$ from $x$ to $y$, denoted $d_G(x,y)$, is the number of edges in a shortest path in $G$ from $x$ to $y$. If $G$ contains no path from $x$ to $y$ then we will say that the distance is infinite. When no confusion will result, we simply use $d(x,y)$. We now introduce the concept of a SIG of a graph.

Let $G$ be a graph and $X$ a nonempty subset of $V(G)$. For each $x \in X$, let $c(x)$ be a vertex in $X - \{x\}$ whose distance to $x$ is as small as possible. Note, $c(x)$ may be chosen to be any one of $x$'s closest neighbors in $X$. The influence graph of $G$ with respect to $X$, denoted $I(G,X)$, is the graph with
\[ V(I(G, X)) = X \text{ and} \]

for \( x, y \in X \), \( xy \in E(I(G, X)) \) if and only if
\[ d_G(x, y) < d_G(x, c(x)) + d_G(y, c(y)) \]

This generalizes the idea of SIG's by incorporating a metric distinct from the Euclidean metric. As in the original model, these graphs can be considered to be intersection graphs, where the set corresponding to each vertex \( x \) in \( X \), is precisely the subset of vertices in \( G \) a distance at most \( d_G(x, c(x)) \) from \( x \). We might think of this as the sphere of influence of \( x \) in \( G \), with respect to \( X \).

We also can generalize the idea of a closed SIG. The *closed influence graph of \( G \) with respect to \( X \)*, denoted \( I^*(G, X) \), is the graph with
\[ V(I^*(G, X)) = X \text{ and} \]

for \( x, y \in X \), \( xy \in E(I^*(G, X)) \) if and only if
\[ d_G(x, y) \leq d_G(x, c(x)) + d_G(y, c(y)). \]

For convenience, we say that \( H \) is a (closed) influence of a graph when there exists a graph \( G \) and subset \( X \) so that \( H \) is isomorphic to \( I(G, X) \) (\( I^*(G, X) \)). In this case we simply say \( H = I(G, X) \) (\( I^*(G, X) \)). In keeping with the terminology used in [2], we will also say that \( H \) is realized by \( G \) and \( X \). For any undefined terms or notation the reader is referred to [3].

It is easily seen that all graphs with no isolated vertices can be realized by an "open" influence graph of a graph as shown in [2]. Several examples of graphs that are and aren't closed influence graphs are also given in [2]. In this paper we answer the question, which graphs are realized as closed influence graphs of a prescribed graph.

**A Characterization of Closed Influence Graphs of a Graph.**

Before giving the characterization, some additional notation would be helpful. For a graph \( G \), a set of cliques \( \mathcal{X} \) in \( G \) is said to be a *(edge) clique cover* if each vertex
(edge) of $G$ is in at least one clique in $\mathcal{K}$. A set of cliques is a \textit{clique partition} if it is a clique cover and each vertex is in a single clique. Note that every graph has a trivial clique partition, by considering each vertex as a clique by itself. We will call a clique partition \textit{non-trivial} if each clique has order at least two. Finally, we \textit{subdivide} an edge in a graph by introducing vertices of degree 2 on the edge. To \textit{subdivide n times}, will mean that $n$ new vertices are inserted.

**Theorem.** A graph $H$ is realizable as the closed influence graph of a prescribed graph $G$, with respect to some subset of vertices $X$, if and only if $H$ contains a non-trivial clique partition.

**Proof.** Let $H$ be a graph with a non-trivial clique partition $\mathcal{K}$. Let $G$ be the graph obtained from $H$ by subdividing one time each edge not in $\mathcal{K}$. Let $X$ be the vertices in $G$ that correspond to the original vertices of $H$. Observe that since $\mathcal{K}$ was a non-trivial clique partition, $d(x,c(x)) = 1$ for each $x$ in $X$. Also, any two vertices $u$ and $v$ of $H$ joined by an edge which is not in an element of $\mathcal{K}$ has $d_G(u,v) = 2$ while all other pairs of vertices, i.e. not adjacent in $H$, are a distance of at least three apart in $G$. Consequently, it follows that $H = I^*(G, X)$, and thus every graph with a non-trivial clique partition is a closed influence graph.

To show that every closed influence graph $H = I^*(G, X)$ contains a non-trivial clique partition, we proceed by induction. It is easy to see that this is true for small order graphs, so let $H = I^*(G, X)$ have order $n$, and assume all closed influence graphs of order less than $n$ have a non-trivial clique partition. We begin by making two observations. First, let $x$ be any vertex in $X$, then the subset of vertices in $X$ which have $x$ as a closest neighbor induces a complete graph. This follows since if $y$ and $z$ have $x$ as one of their closest neighbors then $d(y,c(y)) = d(y,x)$ and $d(z,c(z)) = d(z,x)$
and hence \(d(y, z) \leq d(y, x) + d(x, z) = d(y, c(y)) + d(z, c(z))\). Also note that \(x\) is also adjacent to all of these vertices. Second, select \(u\) so that \(d(u, c(u))\) is as small as possible. Let \(U\) be the set of vertices in \(X\) closest to \(u\). Clearly, \(U\) is non-empty and as above \(U\) induces a complete graph. If \(v\) is in \(U\) and \(V\) is the subset of vertices, other than \(u\), that have \(v\) as its closest neighbor in \(X\), then \(u\) is adjacent to all those vertices. With these observations we are ready to proceed.

Let \(u\) be an element in \(X\) so that \(d(u, c(u))\) is as small as possible. Let \(U\) be the set of vertices in \(X\) closest to \(u\). Suppose \(U = \{u_1, u_2, \ldots, u_k\}\). Let \(U_1\), be the subset of vertices having \(u_1\) as one of its closest neighbors in \(X\). Let \(U_2\), be the subset of unselected vertices having \(u_2\) as one of its closest neighbors and so forth to \(U_k\) being the subset of unselected vertices having \(u_k\) as one of its closest neighbors. Note some or all of the \(U_i\)'s may be empty, although \(U\) is definitely non-empty. Continue this process for all the elements in all the subsets \(U_i\) over and over until no new elements of \(X\) can be selected. Let \(Z'\) be the last non-empty subset of elements of \(X\) selected in this manner, say having \(z\) as their closest neighbor. For convenience, let \(Z = Z' \cup \{z\}\).

If \(z\) is not in \(U\), then for every element \(y\) in \(X-Z\), there is an element \(y'\) in \(X-Z\) so that \(d(y, y') = d(y, c(y))\), hence \(y'\) could be chosen as \(c(y)\) without disrupting the structure of the graph. That is to say, no vertex in \(X-Z\) depends on a vertex in \(Z\) to determine its closest neighbor. Thus, \(G_{X-Z} = H(Z)\), and since \(H(Z)\) has order less than \(n\), \(H(Z)\) has a non-trivial clique partition, which with \(Z\) gives a non-trivial clique partition of \(H\). If \(z\) is in \(U\), but not the only element of \(U\), then the same argument as above applies. If \(z\) is the only element in \(U\), then \(Z \cup \{u\}\) is the subset to remove from \(X\) to arrive at a smaller closed influence graph. Finally if \(z = u\) then the original argument yields the non-trivial clique partition.

\[\Box\]

In [2], it was shown that \(K_3\)-free graph, that is graphs with girth at least four, that
are influence graphs must contain a perfect matching. Since the only non-trivial clique partitions in $K_3$-free graphs are perfect matchings, using the theorem, we get the following:

**Corollary.** Let $H$ be a $K_3$-free graph, $H = \text{I}^*(G, X)$ for some $G$ and subset $X$ if and only if $H$ contains a perfect matching.

We also note that in [2] the authors posed the problem for requiring $X$ to be an independent set. By considering the construction of subdivision in the theorem, and now subdividing each edge in one of the non-trivial cliques once and all other edges twice, and by choosing $X$ to be the original set of vertices, this set is now independent, and it is easy to see the desired graph is achieved. Hence restricting $X$ to independent sets in fact is no restriction at all.
References


