Matching extendibility in hypercubes

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Abstract

In a bipartite graph G, a set $S \subseteq V(G)$ is *deficient* if $|N(S)| < |S|$. A matching M (with vertex set U) is k-*suitable* if $G - U$ has no deficient set of size less than k. Let $f_k(d)$ be the maximum r such that in the d-dimensional hypercube Q_d every k-suitable matching having size at most r extends to a perfect matching. We generalize results of Limaye and Sarvate by proving that $f_k(d) = k(d-k) + \binom{k-1}{2}$ for $k \leq d-3$. To this end we prove lower bounds on the sizes of neighborhoods of vertex sets in Q_d. We also prove that every induced matching in Q_d extends to a perfect matching.

1 Introduction

A graph G is r-*extendible* if every matching of size r in G extends to a perfect matching. The *extendibility* of G is the largest k such that G is k-extendible. Plummer [4, 5] surveyed early results in this area, and more than 50 papers involving this notion have since appeared.

Some papers have studied the extendibility of matching in terms of not only size but also structural conditions on the matching, often in special types of graphs. For example, one can study what lower bound on the distance between edges in the matching is needed for extendibility. Aldred and Jackson [1] studied this for cubic bipartite graphs, Tseng and Anstee [8] for grids, Qian [6] for "square" graphs (formed by adding edges to join vertices at distance 2 in another graph). Yuan [11] characterized the 3-regular graphs in which every induced matching (distance at least 2 between edges) extends to a perfect matching.

In this paper, we take a similar approach to matchings in the hypercube, extending results of Limaye and Sarvate [3] about the size of matchings guaranteed to extend to perfect matchings when small obstructions are forbidden.

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In particular, in the d-dimensional hypercube Q_d, a matching with d edges may cover the neighborhood of a vertex that itself is not covered. Such a matching cannot extend to a perfect matching, so Q_d is not d-extendible. Limaye and Sarvate [3] proved that Q_d is r-extendible when $r < d$. Furthermore, they showed that matchings of size d that do not cover the neighborhood of an uncovered vertex also extend to perfect matchings. In Q_4, they presented a matching of size 5 that does not cover the neighborhood of an uncovered vertex and does not extend.

We generalize and strengthen these results via a simpler proof. In a bipartite graph G, a set $S \subseteq V(G)$ is *deficient* if $|N(S)| < |S|$. A matching M (with vertex set U) is *k-suitable* if $G-U$ has no deficient set of size less than k. Since no set of size 0 is deficient, all matchings are 1-suitable. The 2-suitable matchings are those that do not cover the neighborhood of an uncovered vertex. Let $f_k(d)$ be the maximum r such that every k-suitable matching in Q_d having size at most r extends to a perfect matching.

In this language, the results of Limaye and Sarvate [3] are that $f_1(d) = d-1$ and $f_2(d) \geq d$ (and specifically $f_2(4) = 4$). For $k \leq d-3$, we show that every k-suitable matching in Q_d that has at most $k(d-k)+\binom{k-1}{2}$ edges extends to a perfect matching. For $k \leq d-1$, we construct a k-suitable matching of size $k(d-k)+\binom{k-1}{2}+1$ that does not extend. Together, these results imply that $f_k(d) = k(d-k)+\binom{k-1}{2}$ for $k \leq d-3$. In particular, $f_2(d) = 2d-4$. For $d \in \{k+1, k+2\}$ with $k > 2$, we construct examples showing that $f_k(d) < k(d-k)+\binom{k-1}{2}$.

Another way to restrict partial matchings is to require that the edges be distance at least s apart, for some s greater than 1. Tseng and Anstee [8] proved that in cartesian products of d copies of a given path, every matching in which edges are separated by distance at least 3 extends to a perfect matching. Balogh (private communication) independently proved the special case for Q_d, using a cover of $V(Q_d)$ by 4-cycles.

For $s = 2$, the restriction is to induced matchings. For Q_d, we strengthen the result of [8] by proving that every induced matching in the hypercube extends to a perfect matching. This strengthening does not hold for products of paths with at least three vertices, since it is easy to isolate a corner vertex by deleting the vertices of an induced matching with d edges. Graphs in which every induced matching extends to a perfect matching have also been studied in [6] and [11].

Another direction in which one can generalize matching extendibility is to view it as extending spanning subgraphs with maximum degree 1 to spanning 1-regular graphs. In [10], we study extension of subgraphs with maximum degree 2 to spanning 2-regular subgraphs.

2 Neighborhood sizes in the hypercube

An X, Y-*bigraph* is a bipartite graph G with partite sets X and Y. The fundamental tool for matchings in bipartite graphs is Hall's Theorem [2], stating that an X, Y-bigraph has a matching that covers X if and only if $|N(S)| \ge |S|$ whenever $S \subseteq X$, where $N(S)$ denotes the union of the neighborhoods of vertices in S. Thus it suffices to have no deficient set.

In order to apply Hall's Theorem to matching extension in the hypercube, we need a lower bound on the size of the neighborhood of a vertex set in Q_d. If each set has enough "extra" neighbors, then no set will become deficient when the endpoints of the initial matching are deleted from the X, Y-bigraph Q_d. Our goal, reached in Theorem 2.4, is to prove for $S \subseteq X$ that $|N(S)| - |S| \ge k(d-k) + \binom{k-1}{2}$ when $|S| \ge k$ and $k \le d-3$, except when S is almost all of X. Note that this is the desired value of $f_k(d)$; deleting this many edges will not make any set of size at least k deficient (except possibly for very large sets), and the hypothesis will forbid smaller deficient sets.

Our first tool toward the bound is a lower bound on neighborhood sizes in Q_d that extends a theorem of Somani and Peleg [7]. They proved Lemma 2.1 for $|S| \le d+1$, and in this range it is sharp. We remove the restriction on $|S|$ and provide a much simpler proof.

Viewing Q_d as the set of binary d-tuples (that is, $v = (v_1, \ldots, v_d)$ when $v \in V(Q_d)$, with $V(Q_d) = \{0,1\}^d$), we often split vertex sets according to whether the last coordinate is 0 or 1. Doing this with the full vertex set yields a partition of $V(Q_d)$ into two sets inducing $(d-1)$-dimensional hypercubes, which we generally call Q^0 and Q^1.

Lemma 2.1. *If $\varnothing \ne S \subseteq V(Q_d)$ and $d \ge 1$, then $|N(S) - S| > d|S| - \binom{|S|+1}{2}$.*

Proof. The inequality is immediate when $|S| = 1$, so henceforth consider $|S| > 1$. We use induction on d. When $d = 1$, the set S has two vertices and no outside neighbors, and $d|S| - \binom{|S|+1}{2} = -1$. For $d > 1$, since S has distinct vertices, we may assume by symmetry that two differ in the last coordinate. For $i \in \{0,1\}$, let $S_i = \{v \in S\colon\ v_d = i\}$.

By applying the induction hypothesis to the subcubes Q^0 and Q^1, each S_i has at least $(d-1)|S_i| - \binom{|S_i|+1}{2} + 1$ outside neighbors in Q^i. These sets are disjoint, so

$$|N(S) - S| \ge (d-1)(|S_0| + |S_1|) - \left[\binom{|S_0|+1}{2} + \binom{|S_1|+1}{2}\right] + 2.$$

Since $|S_0| + |S_1| = |S|$ and each $|S_i| \ge 1$, the sum $\binom{|S_0|+1}{2} + \binom{|S_1|+1}{2}$ is maximized when $\{|S_0|, |S_1|\} = \{|S|-1, 1\}$. The computation below now completes the proof. □

$$|N(S) - S| \ge d(|S|) - |S| - \left[\binom{(|S|-1)+1}{2} + \binom{1+1}{2}\right] + 2 = d|S| - \binom{|S|+1}{2} + 1.$$

When S is contained in a single partite set, Lemma 2.1 provides a lower bound on $|N(S)|$. We apply the proof technique of Lemma 2.1 for further results about neighborhood sizes, restricting the given set S to sets S_0 and S_1 within disjoint $(d-1)$-dimensional subcubes Q^0 and Q^1, respectively. The induction provides lower bounds on $|N_{Q^i}(S_i)| - |S_i|$ (where $N_{Q^i}(S_i)$ denotes the neighborhood of S_i in Q^i) unless S_0 or S_1 does not satisfy the induction hypothesis. The next lemma handles some of these difficult cases.

Lemma 2.2. *Let S be a subset of one partite set in Q_d. Choose k with $k \le \min\{|S|, d-3\}$. For $i \in \{0,1\}$, let $S_i = \{v \in S\colon\ v_d = i\}$.*
(a) If $|S| \le d$, then $|N(S)| - |S| \ge k(d-k) + \binom{k-1}{2}$, with equality only when $|S| = k$.
(b) If $1 \le |S_i| \le k-1$ for each i, then again $|N(S)| - |S| \ge k(d-k) + \binom{k-1}{2}$.

Proof. Let $s = |S|$.

(a) By Lemma 2.1, $|N(S)| - |S| \ge (d-1)s - \binom{s+1}{2} + 1$. Hence it suffices to show that $(d-1)s - \binom{s+1}{2} + 1 \ge k(d-k) + \binom{k-1}{2}$. Since $(d-1)s - \binom{s+1}{2} = \frac{1}{2}s(2d-s-3)$ and $k(d-k) + \binom{k-1}{2} - 1 = \frac{1}{2}k(2d-k-3)$, we have

$$(d-1)s - \binom{s+1}{2} + 1 - k(d-k) - \binom{k-1}{2} = \frac{1}{2}s(2d-s-3) - \frac{1}{2}k(2d-k-3).$$

The expression $\frac{1}{2}x(2d-3-x)$ is increasing with x for $x < d - \frac{3}{2}$. Since $k \le d-3$ and $k \le s \le d$, we conclude that the desired inequality holds. In fact, for the allowed values of s and k, it is strict unless $s = k$.

(b) Let $s_i = |S_i|$ for $i \in \{0,1\}$. If $s \le d$, then (a) implies the result. Otherwise, Lemma 2.1 yields $|N_{Q^i}(S_i)| \ge (d-1)s_i - \binom{s_i+1}{2} + 1$. Since $N_{Q^i}(S_i) \subseteq V(Q^i)$, we have $|N(S)| \ge (d-1)s - \binom{s_0+1}{2} - \binom{s_1+1}{2} + 2$, and hence $|N(S)| - |S| \ge (d-2)s - (\binom{s_0+1}{2} + \binom{s_1+1}{2}) + 2$. Since $s_0 + s_1 = s$ with $\max\{s_0, s_1\} \le k-1$, the quantity $\binom{s_0+1}{2} + \binom{s_1+1}{2}$ is maximized when $s_0 = k-1$. Hence

$$|N_{Q_d}(S)| - s \ge (d-2)s + 2 - \binom{k}{2} - \binom{s-k+2}{2}.$$

It suffices to show that $(d-2)s + 2 - \binom{k}{2} - \binom{s-k+2}{2} \ge k(d-k) + \binom{k-1}{2}$. Expanding and collecting terms yields

$$(d-2)s + 2 - \binom{k}{2} - \binom{s-k+2}{2} - k(d-k) - \binom{k-1}{2} = \frac{1}{2}(s-k)(2d-7-(s-k)).$$

Since $s \ge k$ and $s \le 2k-2 \le 2d-8$, all factors are nonnegative, which suffices. □

To study neighborhoods of very large sets, we will need a well-known elementary exercise.

Lemma 2.3. *Let G be an X,Y-bigraph with $|X| = |Y|$. For all $S \subseteq X$,*

$$|S| - |N(S)| \leq |T| - |N(T)|,$$

where $T = Y - N(S)$.

Proof. By the definition of T, the sets $N(T)$ and S are disjoint. Now $|N(S)| + |T| = |Y|$ and $|S| + |N(T)| \leq |X|$ imply the inequality. □

Theorem 2.4. *Fix positive integers k and d with $k \leq d-3$, and let $g_k(d) = 2^{d-1} - k(d-k) - \binom{k}{2}$. If S is a subset of one partite set in Q_d such that $k \leq |S| \leq g_k(d)$, then $|N(S)| - |S| \geq k(d-k) + \binom{k-1}{2}$.*

Proof. Let X and Y be the partite sets, and suppose that $S \subseteq X$. If $|S| \leq d$, then part (a) of Lemma 2.2 suffices. Suppose that $|S| > 2^{d-2}$. Since Q_d satisfies Hall's Condition, $|N(S)| \geq |S|$. Let $T = Y - N(S)$, so $|T| < 2^{d-2}$. By Lemma 2.3, $|N(S)| - |S| \geq |N(T)| - |T|$. Hence it suffices to prove the claim when $d < |S| \leq 2^{d-2}$. There are no such instances for $d \leq 4$, so we may assume that $d \geq 5$. Let $s = |S|$.

We proceed by induction on d. Since $|S| > d \geq 5$, we may assume by permuting coordinates that the $(d-1)$-dimensional hypercubes Q^0 and Q^1 each have at least one vertex of S. As before, let $S_i = \{v \in S\colon\ v_d = i\}$ and $s_i = |S_i|$. By symmetry, we may assume that $s_0 \geq s_1$. When $k \geq 2$, the induction hypothesis applies to S_i if $k-1 \leq s_i \leq g_{k-1}(d-1)$. Thus our cases depend on whether s_0 and s_1 lie within those bounds and on whether $k = 1$.

Case 1: $k = 1$. We can apply the induction hypothesis to both S_0 and S_1 if $1 \leq s_1 \leq s_0 \leq g_1(d-1)$; in this case let $S_0' = S_0$. Otherwise, let S_0' be a set of $g_1(d-1)$ vertices in S_0, and let $S' = S_1 \cup S_0'$.

In both cases, $s_1 \leq g_1(d-1)$, since $2(1 + g_1(d-1)) > 2^{d-2}$. The induction hypothesis applies to S_1 and S_0' to yield $|N(S')| - |S'| \geq 2d - 4$. This suffices if $S' = S$. Otherwise, since $g_1(d-1) < s_0 \leq 2^{d-2}$, we have $|S_0 - S_0'| \leq d-3$. With $S' \subseteq S$, we now have $|N(S)| - |S| \geq 2d - 4 - (|S| - |S'|) \geq d - 1$, as desired.

Case 2: $1 \leq k-1 \leq s_1 \leq s_0 \leq g_{k-1}(d-1)$. Since we have checked the case $k = 1$, and since $1 \leq k-1 \leq (d-1) - 3$, the induction hypothesis in each Q^i yields $|N_{Q^i}(S_i)| - |S_i| \geq (k-1)(d-k) + \binom{k-2}{2}$. The two inequalities sum to

$$|N(S)| - |S| \geq k(d-k) + \binom{k-1}{2} + (k-2)(d-k) - (k-2) + \binom{k-2}{2}.$$

If $d - 3 \geq k \geq 2$, then $(k-2)(d-k) - (k-2) + \binom{k-2}{2} \geq 0$, so the desired inequality holds.

Case 3: $1 \le s_1 < k-1 \le s_0 \le g_{k-1}(d-1)$. By the induction hypothesis,

$$|N_{Q^0}(S_0)| - |S_0| \ \ge \ (k-1)(d-k) + \binom{k-2}{2}.$$

By Lemma 2.1, and using $1 \le s_1 < k-1 \le d-4$,

$$|N_{Q^1}(S_1)| - |S_1| \ \ge \ (d-2)s_1 - \binom{s_1+1}{2} + 1 \ = \ s_1\left(d-2-\frac{s_1+1}{2}\right) + 1 \ \ge \ d-2.$$

Summing the inequalities yields

$$|N(S)| - |S| \ \ge \ (k-1)(d-k) + \binom{k-2}{2} + d - 2 \ = \ k(d-k) + \binom{k-1}{2}.$$

Case 4: $1 \le s_1 \le s_0 \le k-1$. The desired inequality $|N(S)| - |S| \ge k(d-k) + \binom{k-1}{2}$ in this case is the statement of Lemma 2.2(b).

Case 5: $s_0 > g_{k-1}(d-1)$. Let S_0' be a subset of S_0 with size $g_{k-1}(d-1)$. By the induction hypothesis, $|N_{Q^0}(S_0')| - |S_0'| \ge (k-1)(d-k) + \binom{k-2}{2}$. Using the formula for $g_{k-1}(d-1)$, we have $|N_{Q^0}(S_0)| \ge |N_{Q^0}(S_0')| \ge 2^{d-2} - (k-2)$. Also, S_0 has s_0 neighbors in Q^1, so $|N(S)| \ge |N(S_0)| \ge 2^{d-2} - (k-2) + |S| - s_1$, which yields $|N(S)| - |S| \ge 2^{d-2} - (k-2) - s_1$. We want $|N(S)| - |S| \ge k(d-k) + \binom{k-1}{2}$; if this fails, then $s_1 \ge 2^{d-2} - (k-3) - k(d-k) - \binom{k-1}{2}$.

On the other hand, $s \le 2^{d-2}$ and $s_0 \ge g_{k-1}(d-1)+1$ yield $s_1 \le (k-1)(d-k) + \binom{k-1}{2} - 1$. Together, the two inequalities on s_1 require $(2k-1)(d-k) + 2\binom{k-1}{2} + (k-4) \ge 2^{d-2}$. The left side simplifies to $d(2k-1) - (k^2+k+2)$. For $k \le d-3$, this expression increases with k. Setting $k = d-3$, we have completed the proof for d unless $d^2 - 2d - 8 \ge 2^{d-2}$.

For integer d, this inequality requires $d = 6$, where it holds with equality. We may thus assume that every inequality along the way holds with equality. That is, $d = 6$, $k = 3$, $s_0 = g_2(5) + 1 = 10$, and $s_1 = 2(6-3) + 1 - 1 = 6$. This is the final case to consider. We have $|S| = 16$ and want $|N(S)| - |S| \ge 10$, so we seek $|N(S)| \ge 26$. It suffices to show that $|N_{Q^1}(S_1)|$ is larger than the 10 vertices counted previously for $|N(S_0) \cap V(Q^1)|$. Since $2 \le s_1 \le 9 = g_2(5)$, the induction hypothesis with $k = 2$ and $d = 5$ applies to yield $|N_{Q^1}(S_1)| - |S_1| \ge 2 \cdot 3 + 0 = 6$. Hence $|N_{Q^1}(S_1)| \ge 12$ and $|N(S)| \ge 27$. □

3 Extendibility of k-suitable matchings in Q_d

We apply Theorem 2.4 to matching extendibility in the hypercube. When M is a matching, U_M denotes the set of endpoints of edges in M, which are the vertices *covered* by M.

Recall that a matching M is *k-suitable* if $Q_d - U_M$ does not contain a deficient set of size less than k; such a deficient set is an immediate obstacle to extension to a perfect matching. Limaye and Sarvate [3] restrict attention to matchings in Q_d that do not cover the neighborhood of an uncovered vertex; our "k-suitability" is a further restriction. As we prohibit larger deficient sets, we can guarantee that larger matchings extend.

Theorem 3.1. *If $d-3 \geq k \geq 1$, then every k-suitable matching in Q_d with at most $k(d-k)+\binom{k-1}{2}$ edges extends to a perfect matching in Q_d.*

Proof. Let M be a k-suitable matching in Q_d. It suffices to show that $Q_d - U_M$ has a perfect matching, which we do by verifying Hall's Condition. In fact, we show that $|N(S) - U_M| \geq |S|$ for all $S \subseteq X$, where X and Y are the partite sets of Q_d, and $N(S)$ again denotes the neighborhood of S in Q_d. Let $g_k(d) = 2^{d-1} - k(d-k) - \binom{k}{2}$, as in Theorem 2.4.

By k-suitability, $Q_d - U_M$ has no deficient set of size less than k. For $S \subseteq X$ with $k \leq |S| \leq g_k(d)$, Theorem 2.4 guarantees $|N(S)| \geq |S| + k(d-k) + \binom{k-1}{2}$. Since $|M| \leq k(d-k) + \binom{k-1}{2}$, we have $|N(S) - U_M| \geq |S|$, and hence S is not deficient in $Q_d - U_M$.

Finally, suppose that $|S| > g_k(d)$. Let S' be a subset of S with size $g_k(d)$. As shown above, $|N(S')| \geq |S'| + k(d-k) + \binom{k-1}{2} = 2^{d-1} - (k-1)$. Since $S' \subseteq S$, we have $|N(S)| \geq 2^{d-1} - (k-1)$.

Let $T = Y - N(S) - U_M$. Since $T \subseteq Y - N(S)$, we have $|T| < k$. Since M is k-suitable, $|N(T) - U_M| \geq |T|$. Letting $Q' = Q_d - U_M$, within the bipartite graph Q' we have $|N_{Q'}(T)| - |T| \geq 0$. By Lemma 2.3, $|N_{Q'}(S)| - |S| \geq |N_{Q'}(T)| - |T| \geq 0$. This is precisely the needed inequality $|N(S) - U_M| \geq |S|$. □

Applying Theorem 3.1 when $k = 2$ and $d \geq 5$ strengthens the result of Limaye and Sarvate; they proved the same statement with d in place of $2d-4$.

Corollary 3.2. *For $d \geq 5$, any matching with at most $2d-4$ edges in which no uncovered vertex has a covered neighborhood extends to a perfect matching in Q_d.*

Our lemmas and theorem yield $f_k(d) \geq k(d-k) + \binom{k-1}{2}$ only when $k \leq d-3$. At the end of this section, we will show that $f_k(d) < k(d-k) + \binom{k-1}{2}$ when $k \in \{d-2, d-1\}$.

Nevertheless, $(2d-4)$-extendibility for 2-suitable matchings remains true in the hypercube Q_d for $d \leq 4$. Limaye and Sarvate [3] proved this as part of their result, and there is another direct proof in [9].

Our first step toward sharpness of the bound on $f_k(d)$ is a construction for $k = 2$.

Example 3.3. In 2-suitable matchings, deficient 1-sets are forbidden, but not deficient 2-sets. To prove that Corollary 3.2 is sharp, we construct a matching M of size $2d-3$ leaving a deficient set in $Q_d - U_M$ consisting of two vertices u and v at distance 2 in Q_d. Let x and y be their common neighbors. Let $R = N(u) - \{x, y\}$ and $S = N(v) - \{x, y\}$. We choose

$2d-3$ edges to cover $R \cup S \cup \{x\}$, all of which lies in one partite set. Eliminating this set from $N(u) \cup N(v)$ leaves only their common neighbor y, making $\{u, v\}$ deficient in $Q_d - U_M$.

For $S \subseteq \{1, \ldots, d\}$, let v_S denote the vertex of Q_d whose binary d-tuple name is the incidence vector of S. We may let $x = v_{\varnothing}$, $u = v_{\{1\}}$, and $v = v_{\{2\}}$. Now $R = \{v_{\{1,i\}} \colon 3 \leq i \leq d\}$ and $S = \{v_{\{2,i\}} \colon 3 \leq i \leq d\}$. We achieve all the requirements by letting $M = \{v_{\{1,i\}}v_{\{1,2,i\}} \colon i \geq 3\} \cup \{v_{\{2,i\}}v_{\{2,i,d\}} \colon 3 \leq i \leq d-1\} \cup \{v_{\emptyset}v_{\{3\}}, v_{\{d\}}v_{\{2,d\}}\}$. □

Example 3.3 applies for $d \geq 3$. With Corollary 3.2, it yields $f_2(d) = 2d - 4$ for $d \geq 5$. As noted earlier, this formula also holds for $2 \leq d \leq 4$. For $k > 2$, we need a somewhat different construction to show that the bound on $f_k(d)$ in Theorem 3.1 is sharp for $d \geq k + 3$.

Theorem 3.4. *If k and d are positive integers with $k < d \neq 2$, then Q_d contains a k-suitable matching of size $k(d-k) + \binom{k-1}{2} + 1$ that does not extend to a perfect matching. Thus $f_k(d) \leq k(d-k) + \binom{k-1}{2}$, with equality when $d \geq k + 3$.*

Proof. For $k = 1$, let $M = \{v_{\{i\}}v_{\{i,i+1\}} \colon 1 \leq i \leq d\}$, with subscript elements taken modulo d; note that $|M| = d$ and that $v_{\varnothing}$ is isolated in $Q_d - U_M$. For $k = 2$, the matching in Example 3.3 suffices. Thus we may assume $k \geq 3$.

Let $A = \{v_{\{1\}}, \ldots, v_{\{k\}}\}$. Note that A consists of k vertices of weight 1, neighbors of $v_{\varnothing}$. Every two vertices of A have another common neighbor of weight 2. Hence each vertex of A has $d - k$ neighbors adjacent to nothing else in A. Thus $|N(A)| = k(d-k) + \binom{k}{2} + 1$.

We construct a k-suitable matching M with $k(d-k) + \binom{k-1}{2} + 1$ edges that covers all of $N(A)$ except $k-1$ specified vertices. Thus A is a deficient k-set in $Q_d - U_M$, and M does not extend to a perfect matching.

Let $B = \{v_{\{i,k\}} \colon 1 \leq i \leq k-1\}$ and $M' = \{v_C v_{C \Delta \{k\}} \colon v_C \in N(A) - B\}$, where Δ denotes symmetric difference. That is, vertices of weight 2 in $N(A) - B$ are matched to vertices of weight 1 or 3 depending on whether they have a 1 in position k. Note that $|M'| = |N(A)| - |B| = k(d-k) + \binom{k}{2} + 1 - (k-1)$. We modify M' to create M.

Let $R = \{v_{\varnothing}v_{\{k\}}, v_{\{k,d\}}v_{\{d\}}, v_{\{1,d\}}v_{\{1,d,k\}}\}$; note that $R \subseteq M'$. Form M by replacing R with $\{v_{\varnothing}v_{\{d\}}, v_{\{k,d\}}v_{\{1,d,k\}}, v_{\{1,d\}}v_{\{1,2,d\}}\}$ in M', so $|M| = |M'| = k(d-k) + \binom{k-1}{2} + 1$. In Figure 1, R is dashed, M is bold, and V_r denotes the set of vertices of weight r.

The purpose of the change from M' to M is to leave A completely uncovered. Having $\{1, k, d\} \neq \{1, 2, d\}$ is the reason this construction requires $k \geq 3$. It remains to show that $Q_d - U_M$ has no deficient set of size less than k.

First consider $S \subseteq X - U_M$ with $|S| < k$, where X consists of the vertices with odd weight. For $v_C \in S$ with $C \neq \{k\}$, let $f(v_C) = v_{C \Delta \{k\}}$. If $v_{\{k\}} \in S$, then $|S| < k$ guarantees some $v_{\{i\}} \notin S$ with $1 \leq i \leq k-1$; let $f(v_{\{k\}}) = v_{\{i,k\}} \in B$. By construction, f is injective. To prove that $|N(S) - U_M| \geq |S|$, it suffices to show that $f(v_C) \notin U_M$ for all $v_C \in S$.

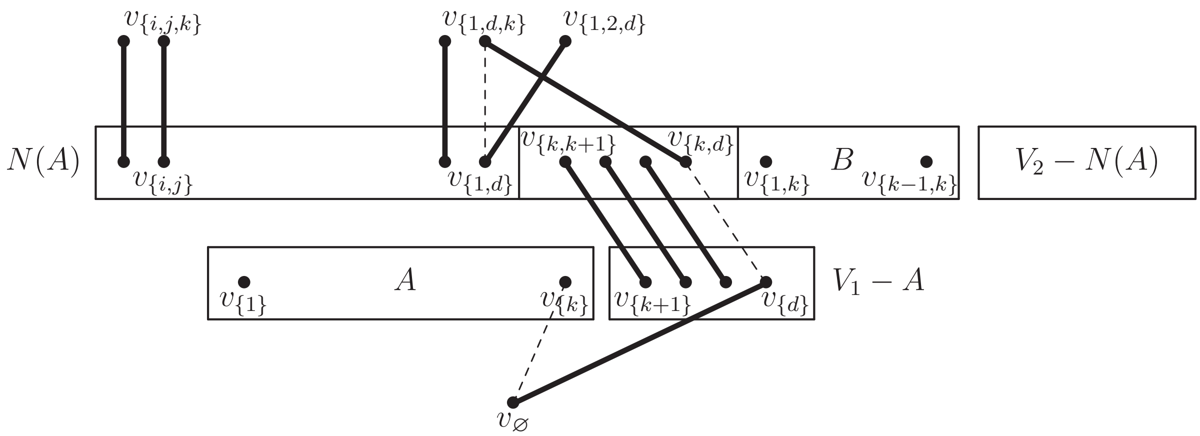

Figure 1: The matching M in Theorem 3.4

For $v_C \in V_1 \cap S$, we have $v_C \in A$, since $V_1 - A \subseteq U_M$; in this case $f(v_C) \in B$, and $B \cap U_M = \varnothing$. For $v_C \in V_3 \cap S$, a neighbor of v_C in U_M must be in V_2. If this is $f(v_C)$, then $k \in C$, and also $v_C \in U_M$. Vertices of larger weight have no neighbors in U_M.

Finally, consider $S \subseteq Y - U_M$ with $|S| < k$, where Y consists of the vertices with even weight. For $v_C \in S$, let $f(v_C) = v_{C \Delta \{k\}}$ (note that $v_\varnothing \notin S$). Again f is injective. If $v_C \in B$, then $f(v_C) \in A$. If $v_C \in V_2 - N(A)$, then $C \subseteq \{k+1, \ldots, d\}$, and $f(v_C) \notin U_M$.

If $v_C \in V_4$, then $f(v_C) \notin U_M$, unless $C = \{1, 2, k, d\}$. Here one of the $d-4$ neighbors of v_C in V_5 is available to substitute for $f(v_C)$ unless S contains all of $\{v_{\{1,2,i,d\}} \colon 3 \le i \le d-1\}$. Since $|S| < k \le d-3$, this cannot happen, and $v_{\{1,2,k,d\}}$ has an available neighbor. For v_C of larger weight, $f(v_C) \notin U_M$. Hence again we have shown that $|N(S) - U_M| \ge |S|$. □

Corollary 3.5. *If $d \ge k+3$, then $f_k(d) = k(d-k) + \binom{k-1}{2}$.* □

We close this section with an example showing that $f_k(d) < k(d-k) + \binom{k-1}{2}$ when $d \in \{k+1, k+2\}$. For $k \in \{d-1, d-2\}$, we have $k(d-k) + \binom{k-1}{2} = 1 + \binom{d-1}{2}$. For $d \ge 5$, we construct a matching M in Q_d of size $1 + \binom{d-1}{2}$ that is $(d-2)$-suitable (and hence $(d-1)$-suitable) but fails to extend to a perfect matching.

Example 3.6. Let $d \ge 5$. Recall that V_1 denotes the set of vertices of weight 1. Our matching M will cover all but $d-1$ neighbors of V_1. Since $|V_1| = d$, such a matching does not extend to a perfect matching. For $(d-2)$-suitability, we show that $G - U_M$ has no deficient set of size less than $d-2$. Since $|N(V_1)| = \binom{d}{2} + 1$, covering all but $d-1$ neighbors yields $|M| = 1 + \binom{d-1}{2}$.

We leave uncovered $v_\varnothing$ and $\{v_{\{i,d\}} \colon 2 \le i \le d-1\}$. Cover $v_{\{1,d\}}$ with $v_{\{1,d\}}v_{\{1,2,d\}}$. Cover each remaining vertex $v_{\{i,j\}}$ (with $i, j \le d-1$) by adding $v_{\{i,j\}}v_{\{i,j,d\}}$ to M, except that since $v_{\{1,2,d\}}$ is already used we add $v_{\{1,2\}}v_{\{1,2,3\}}$ to cover $v_{\{1,2\}}$.

It remains to prove that M is $(d-2)$-suitable; we use the same technique as in Theorem 3.4. Consider $S \subseteq X - U_M$, where X is the set of vertices of odd weight and $|S| < d-2$. For $v_C \in S - \{v_{\{1\}}, v_{\{d\}}\}$, let $f(v_C) = v_{C \triangle \{d\}}$. If $v_{\{1\}} \in S$, then let $f(v_{\{1\}}) = v_{\varnothing}$. If $v_{\{d\}} \in S$, then let $f(v_{\{d\}}) = v_{\{i,d\}}$, where $v_{\{i\}} \in (V_1 - \{v_{\{1\}}, v_{\{d\}}\}) - S$, which exists since $|S| < d-2$. Note that $v_{\{i,d\}}$ is uncovered. Since $f(S) \cap U_M = \varnothing$, we have $|N(S) - U_M| \geq |S|$.

For such a subset of the vertices of even weight, again let $f(v_C) = f(v_{C \triangle \{d\}})$ for $v_C \in S$. The only uncovered vertex of even weight whose image under f is covered is $v_{\{1,2,3,d\}}$. One of its neighbors of weight 5 is available to use instead of $v_{\{1,2,3\}}$ to complete the injection into $X - U_M$ unless $\{v_{\{1,2,3,i\}} \colon 4 \leq i \leq d\} \subseteq S$. Since $|S| < d-2$, equality holds. However, all vertices of weight 3 in U_M other than $v_{\{1,2,3\}}$ have the form $v_{\{i,j,d\}}$. Since $d \geq 5$, we can set $f(v_{\{1,2,3,4\}}) = v_{\{1,2,4\}}$ and $f(v_{\{1,2,3,d\}}) = v_{\{1,2,3,4,d\}}$ to complete the needed injection. □

4 Extendibility of induced matchings

Let G be a d-dimensional grid graph, in particular the cartesian product of d copies of the m-vertex path P_m. As mentioned in the Introduction, Tseng and Anstee [8] proved that every set of edges in G with pairwise distance at least 3 extends to a perfect matching in G, meaning that $G - U_M$ has a perfect matching.

We show that in the special case $m = 2$, where G is the hypercube, the distance threshold can be improved. For extendibility of matchings, pairwise distance 2 between edges of M suffices. This condition is equivalent to M being an *induced matching*, meaning that M is the subgraph of Q_d induced by U_M. The result is sharp in that deleting the vertices of an induced matching in a cartesian power of P_4 can isolate a corner vertex.

Theorem 4.1. *Every induced matching in Q_d extends to a perfect matching.*

Proof. Let M be an induced matching in Q_d. We explicitly construct a matching in $Q_d - U_M$. Let X and Y be the partite sets, consisting of the vertices of odd weight and even weight, respectively. For each vertex u, let u' denote the vertex obtained from v by changing the last coordinate in the d-tuple name of u. That is, uu' is an edge whose endpoints differ in the last coordinate. Note that if $x \in X$, then $x' \in Y$.

Since each edge of M has a vertex in both X and Y, it suffices to find pairwise disjoint edges in $Q_d - U_M$ that cover $X - U_M$. For $x \in X - U_M$, match x to x' unless $x' \in U_M$. If $x' \in U_M$, then there exists y' such that $x'y' \in M$. Let $y = (y')'$. Note that x is adjacent to y and that the distance condition guarantees that $y \notin U_M$. Hence we can match x to y.

We must show that no two vertices x and $\hat{x}$ get matched to the same vertex $z \in Y$. They cannot both request z in the first way, since $z = x' = \hat{x}'$ requires $x = \hat{x}$.

If z is requested in the second way by both x and $\hat{x}$, then $z = y = \hat{y}$, where $x'y', \hat{x}'\hat{y}' \in M$. However, $y = \hat{y}$ implies $y' = \hat{y}'$. Now $x = \hat{x}$, since M is a matching.

Finally, if $z = x'$ but z is requested in the second way by $\hat{x}$, then $z = (\hat{y}')'$, where $\hat{x}'\hat{y}' \in M$. However, $x' = z = (\hat{y}')'$ requires $x = \hat{y}'$, which contradicts $x \notin U_M$. □

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