Equitable Hypergraph Orientations

Douglas B. West

Department of Mathematics
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

preprint & slides at
http://www.math.uiuc.edu/~west/pubs/publink.html

Joint work with
Yair Caro and Raphael Yuster

A Classical Exercise

Thm. Every graph can be oriented so that the indegree and outdegree differ by at most 1 at each vertex.

A Classical Exercise

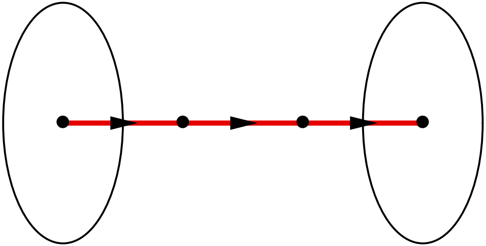
Thm. Every graph can be oriented so that the indegree and outdegree differ by at most 1 at each vertex.

Pf. If some vertices have too much outdegree, then some have excess indegree.

A Classical Exercise

Thm. Every graph can be oriented so that the indegree and outdegree differ by at most 1 at each vertex.

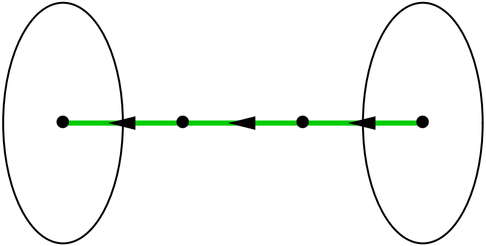
Pf. If some vertices have too much outdegree, then some have excess indegree.



A Classical Exercise

Thm. Every graph can be oriented so that the indegree and outdegree differ by at most 1 at each vertex.

Pf. If some vertices have too much outdegree, then some have excess indegree.

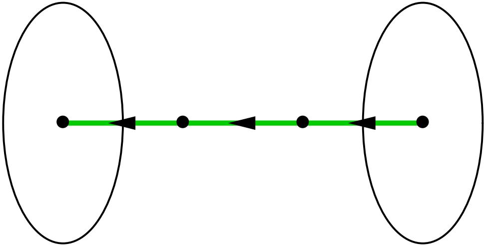


Reverse such a path to reduce the imbalance. Repeat. ■

A Classical Exercise

Thm. Every graph can be oriented so that the indegree and outdegree differ by at most 1 at each vertex.

Pf. If some vertices have too much outdegree, then some have excess indegree.



Reverse such a path to reduce the imbalance. Repeat. ■

Aim: Generalize this result to uniform hypergraphs.

Definitions

Consider r-uniform hypergraphs.

Definitions

Consider r-uniform hypergraphs.

Def. An orientation is a linear ordering of the vertices in each edge. (Each vertex is in one of r positions.)

Definitions

Consider r-uniform hypergraphs.

Def. An orientation is a linear ordering of the vertices in each edge. (Each vertex is in one of r positions.)

Def. (generalization of $d^-(v)$ and $d^+(v)$)
For $S \subset [r]$ and $U \subset V(H)$ with $|U| = |S|$,
$d_S(U)$ = #edges with U in the positions given by S.

Definitions

Consider r-uniform hypergraphs.

Def. An orientation is a linear ordering of the vertices in each edge. (Each vertex is in one of r positions.)

Def. (generalization of $d^-(v)$ and $d^+(v)$)
For $S \subset [r]$ and $U \subset V(H)$ with $|U| = |S|$,
$d_S(U)$ = #edges with U in the positions given by S.

Def. When H is r-uniform and $p \in [r]$, an orientation of H is p-equitable if $|d_S(U) - d_T(U)| \le 1$ whenever U is a p-set of vertices and S and T are p-sets of positions.

Definitions

Consider r-uniform hypergraphs.

Def. An orientation is a linear ordering of the vertices in each edge. (Each vertex is in one of r positions.)

Def. (generalization of $d^-(v)$ and $d^+(v)$)
For $S \subset [r]$ and $U \subset V(H)$ with $|U| = |S|$,
$d_S(U)$ = #edges with U in the positions given by S.

Def. When H is r-uniform and $p \in [r]$, an orientation of H is p-equitable if $|d_S(U) - d_T(U)| \leq 1$ whenever U is a p-set of vertices and S and T are p-sets of positions.

- I.e., each p-set of vertices occurs “equally often” in each p-set of positions.

Definitions

Consider r-uniform hypergraphs.

Def. An orientation is a linear ordering of the vertices in each edge. (Each vertex is in one of r positions.)

Def. (generalization of $d^-(v)$ and $d^+(v)$)
For $S \subset [r]$ and $U \subset V(H)$ with $|U| = |S|$,
$d_S(U)$ = #edges with U in the positions given by S.

Def. When H is r-uniform and $p \in [r]$, an orientation of H is p-equitable if $|d_S(U) - d_T(U)| \leq 1$ whenever U is a p-set of vertices and S and T are p-sets of positions.

- I.e., each p-set of vertices occurs “equally often” in each p-set of positions.

Ex. Equitable orientation of graphs
= 1-equitable orientations of 2-uniform hypergraphs.

Results

Results

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph has a p-equitable orientation.

Results

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph has a p-equitable orientation.

- The complete 4-uniform K_n^4 has no 2-equitable orientation when $n \equiv 6 \mod 12$.

Results

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph has a p-equitable orientation.

- The complete 4-uniform K_n^4 has no 2-equitable orientation when $n \equiv 6 \mod 12$.

Thm. If $1 < p < r-1$ and $\binom{r}{p}$ divides $\binom{n-p}{r-p}$, then to have a p-equitable orientation of K_n^r it is necessary that r divides $\binom{n-1}{r-1}$.

Results

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph has a p-equitable orientation.

- The complete 4-uniform K_n^4 has no 2-equitable orientation when $n \equiv 6 \mod 12$.

Thm. If $1 < p < r-1$ and $\binom{r}{p}$ divides $\binom{n-p}{r-p}$, then to have a p-equitable orientation of K_n^r it is necessary that r divides $\binom{n-1}{r-1}$.

Def. (generalization of line graph $L(G)$)
The p-set multigraph $L_p(H)$ of a hypergraph H has vertex set $E(H)$, and the multiplicity of the edge joining e and e' is the number of p-sets in $e \cap e'$.

Results

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph has a p-equitable orientation.

- The complete 4-uniform K_n^4 has no 2-equitable orientation when $n \equiv 6 \mod 12$.

Thm. If $1 < p < r-1$ and $\binom{r}{p}$ divides $\binom{n-p}{r-p}$, then to have a p-equitable orientation of K_n^r it is necessary that r divides $\binom{n-1}{r-1}$.

Def. (generalization of line graph $L(G)$)
The p-set multigraph $L_p(H)$ of a hypergraph H has vertex set $E(H)$, and the multiplicity of the edge joining e and e' is the number of p-sets in $e \cap e'$.

Thm. If H is r-uniform and $L_p(H)$ is $\binom{r}{p} - 1$-degenerate, then H has a p-equitable orientation.

1-Equitable Orientations

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph H has a p-equitable orientation.

1-Equitable Orientations

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph H has a p-equitable orientation.

Pf. Let $X = \binom{V(H)}{p}$ and $Y = E(H)$.

1-Equitable Orientations

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph H has a p-equitable orientation.

Pf. Let $X = \binom{V(H)}{p}$ and $Y = E(H)$.

Make an X, Y-bigraph B with $Ue \in E(B)$ iff $U \subseteq e$.

1-Equitable Orientations

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph H has a p-equitable orientation.

Pf. Let $X = \binom{V(H)}{p}$ and $Y = E(H)$.

Make an X, Y-bigraph B with $Ue \in E(B)$ iff $U \subseteq e$.

Since $p \in \{1, r-1\}$, each vertex of Y has degree r in B.

1-Equitable Orientations

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph H has a p-equitable orientation.

Pf. Let $X = \binom{V(H)}{p}$ and $Y = E(H)$.

Make an X, Y-bigraph B with $Ue \in E(B)$ iff $U \subseteq e$.

Since $p \in \{1, r-1\}$, each vertex of Y has degree r in B.

Form B' by splitting each $U \in X$ into $\lceil d_B(U)/r \rceil$ vertices; all copies of U except the last inherit r edges.

1-Equitable Orientations

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph H has a p-equitable orientation.

Pf. Let $X = \binom{V(H)}{p}$ and $Y = E(H)$.

Make an X, Y-bigraph B with $Ue \in E(B)$ iff $U \subseteq e$.

Since $p \in \{1, r-1\}$, each vertex of Y has degree r in B.

Form B' by splitting each $U \in X$ into $\lceil d_B(U)/r \rceil$ vertices; all copies of U except the last inherit r edges.

Since $\Delta(B') = r$, matchings $M_1, \ldots, M_r$ decompose B'.

1-Equitable Orientations

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph H has a p-equitable orientation.

Pf. Let $X = \binom{V(H)}{p}$ and $Y = E(H)$.

Make an X, Y-bigraph B with $Ue \in E(B)$ iff $U \subseteq e$.

Since $p \in \{1, r-1\}$, each vertex of Y has degree r in B.

Form B' by splitting each $U \in X$ into $\lceil d_B(U)/r \rceil$ vertices; all copies of U except the last inherit r edges.

Since $\Delta(B') = r$, matchings $M_1, \ldots, M_r$ decompose B'.

If $U_j e \in M_i$, then in position i of the ordering of edge e put U (if $p = 1$) [or $e - U$ if $p = r - 1$].

1-Equitable Orientations

Thm. For $p \in \{1, r-1\}$, every r-uniform hypergraph H has a p-equitable orientation.

Pf. Let $X = \binom{V(H)}{p}$ and $Y = E(H)$.

Make an X,Y-bigraph B with $Ue \in E(B)$ iff $U \subseteq e$.

Since $p \in \{1, r-1\}$, each vertex of Y has degree r in B.

Form B' by splitting each $U \in X$ into $\lceil d_B(U)/r \rceil$ vertices; all copies of U except the last inherit r edges.

Since $\Delta(B') = r$, matchings $M_1, \ldots, M_r$ decompose B'.

If $U_j e \in M_i$, then in position i of the ordering of edge e put U (if $p = 1$) [or $e - U$ if $p = r - 1$].

Each U is put in each of the r possible positions at most $\lceil d_B(U)/r \rceil$ and at least $\lceil d_B(U)/r \rceil - 1$ times. ■

A Necessary Condition

Thm. If $1 < p < r - 1$ and $\binom{r}{p}$ divides $\binom{n-p}{r-p}$, then a p-equitable orientation of K_n^r requires r to divide $\binom{n-1}{r-1}$.

A Necessary Condition

Thm. If $1 < p < r-1$ and $\binom{r}{p}$ divides $\binom{n-p}{r-p}$, then a p-equitable orientation of K_n^r requires r to divide $\binom{n-1}{r-1}$.

Pf. Write a p-equitable orientation of K_n^r as a matrix.

r columns

4	5	1	2	$\binom{n}{r}$ rows

A Necessary Condition

Thm. If $1 < p < r - 1$ and $\binom{r}{p}$ divides $\binom{n-p}{r-p}$, then a p-equitable orientation of K_n^r requires r to divide $\binom{n-1}{r-1}$.

Pf. Write a p-equitable orientation of K_n^r as a matrix.

r columns

4	5	1	2	$\binom{n}{r}$ rows

Each p-set of vertices occurs in $\binom{n-p}{r-p}$ rows.

A Necessary Condition

Thm. If $1 < p < r - 1$ and $\binom{r}{p}$ divides $\binom{n-p}{r-p}$, then a p-equitable orientation of K_n^r requires r to divide $\binom{n-1}{r-1}$.

Pf. Write a p-equitable orientation of K_n^r as a matrix.

r columns

4	5	1	2	$\binom{n}{r}$ rows

Each p-set of vertices occurs in $\binom{n-p}{r-p}$ rows.

Let $s = \binom{n-p}{r-p}/\binom{r}{p}$; each p-set of vertices occurs s times in each p-set of columns.

A Necessary Condition

Thm. If $1 < p < r-1$ and $\binom{r}{p}$ divides $\binom{n-p}{r-p}$, then a p-equitable orientation of K_n^r requires r to divide $\binom{n-1}{r-1}$.

Pf. Write a p-equitable orientation of K_n^r as a matrix.

r columns

4	5	1	2	$\binom{n}{r}$ rows

Each p-set of vertices occurs in $\binom{n-p}{r-p}$ rows.

Let $s = \binom{n-p}{r-p}/\binom{r}{p}$; each p-set of vertices occurs s times in each p-set of columns.

A vertex z occurs $s\binom{n-1}{p-1}$ times among p given columns.

A Necessary Condition

Thm. If $1 < p < r-1$ and $\binom{r}{p}$ divides $\binom{n-p}{r-p}$, then a p-equitable orientation of K_n^r requires r to divide $\binom{n-1}{r-1}$.

Pf. Write a p-equitable orientation of K_n^r as a matrix.

r columns

4	5	1	2	$\binom{n}{r}$ rows

Each p-set of vertices occurs in $\binom{n-p}{r-p}$ rows.

Let $s = \binom{n-p}{r-p}/\binom{r}{p}$; each p-set of vertices occurs s times in each p-set of columns.

A vertex z occurs $s\binom{n-1}{p-1}$ times among p given columns.

Since z occurs equally often in each p-set of columns, it occurs as many times in its p most frequent as in its p least frequent columns.

A Necessary Condition

Thm. If $1 < p < r - 1$ and $\binom{r}{p}$ divides $\binom{n-p}{r-p}$, then a p-equitable orientation of K_n^r requires r to divide $\binom{n-1}{r-1}$.

Pf. Write a p-equitable orientation of K_n^r as a matrix.

r columns

4	5	1	2	$\binom{n}{r}$ rows

Each p-set of vertices occurs in $\binom{n-p}{r-p}$ rows.

Let $s = \binom{n-p}{r-p} / \binom{r}{p}$; each p-set of vertices occurs s times in each p-set of columns.

A vertex z occurs $s\binom{n-1}{p-1}$ times among p given columns.

Since z occurs equally often in each p-set of columns, it occurs as many times in its p most frequent as in its p least frequent columns.

$\therefore$ z occurs equally in each column: r divides $\binom{n-1}{r-1}$. ∎

Exclusions

Thm. $\binom{r}{p}$ divides $\binom{n-p}{r-p}$ $\Rightarrow$ r must divide $\binom{n-1}{r-1}$.

Exclusions

Thm. $\binom{r}{p}$ divides $\binom{n-p}{r-p}$ $\Rightarrow$ r must divide $\binom{n-1}{r-1}$.

Cor. K_n^4 has no 2-equitbl orientation for $n \equiv 6 \bmod 12$.

Exclusions

Thm. $\binom{r}{p}$ divides $\binom{n-p}{r-p}$ $\Rightarrow$ r must divide $\binom{n-1}{r-1}$.

Cor. K_n^4 has no 2-equitbl orientation for $n \equiv 6 \bmod 12$.

Pf. $\binom{n-p}{r-p} = (n-2)(n-3)/2$ and $\binom{r}{p} = 6$. Divisibility holds when $n \equiv 6, 11 \pmod{12}$. Necessary condition that 4 divides $\binom{n-1}{3}$ fails when $n \equiv 6 \pmod{12}$. ■

Exclusions

Thm. $\binom{r}{p}$ divides $\binom{n-p}{r-p}$ $\Rightarrow$ r must divide $\binom{n-1}{r-1}$.

Cor. K_n^4 has no 2-equitbl orientation for $n \equiv 6 \bmod 12$.

Pf. $\binom{n-p}{r-p} = (n-2)(n-3)/2$ and $\binom{r}{p} = 6$. Divisibility holds when $n \equiv 6, 11 \mod 12$. Necessary condition that 4 divides $\binom{n-1}{3}$ fails when $n \equiv 6 \mod 12$. ■

Cor. If p is a prime power ($p = q^s$) and $m \in \mathbf{N}$, then $K_{(m+1)p}^{mp}$ has no p-equitable orientation.

Exclusions

Thm. $\binom{r}{p}$ divides $\binom{n-p}{r-p}$ $\Rightarrow$ r must divide $\binom{n-1}{r-1}$.

Cor. K_n^4 has no 2-equitbl orientation for $n \equiv 6 \bmod 12$.

Pf. $\binom{n-p}{r-p} = (n-2)(n-3)/2$ and $\binom{r}{p} = 6$. Divisibility holds when $n \equiv 6, 11 \mod 12$. Necessary condition that 4 divides $\binom{n-1}{3}$ fails when $n \equiv 6 \mod 12$. ■

Cor. If p is a prime power ($p = q^s$) and $m \in \mathbf{N}$, then $K_{(m+1)p}^{mp}$ has no p-equitable orientation.

Pf. $\binom{n-p}{r-p} = \binom{r}{r-p} = \binom{r}{p}$, and $\binom{n-1}{r-1} = \binom{n-1}{n-r} = \binom{n-1}{p}$. The exponent on q in the product of any q^s consecutive integers is $(q^s-1)/(q-1)$, so q does not divide $\binom{n-1}{p}$.

Hence mp does not divide $\binom{n-1}{p}$. ■

Sparse p-set Multigraphs

Thm. If H is r-uniform and $L_p(H)$ is $\binom{r}{p} - 1$-degenerate, then H has a p-equitable orientation.

Sparse p-set Multigraphs

Thm. If H is r-uniform and $L_p(H)$ is $\binom{r}{p} - 1$-degenerate, then H has a p-equitable orientation.

Pf. Order $E(H)$ as $e_1, \ldots, e_m$ so that $\forall\, i$, fewer than $\binom{r}{p}$ pairs (P, j) exist with $j < i$ and $P \subseteq e_i \cap e_j$ and $|P| = p$.

Sparse p-set Multigraphs

Thm. If H is r-uniform and $L_p(H)$ is $\binom{r}{p} - 1$-degenerate, then H has a p-equitable orientation.

Pf. Order $E(H)$ as $e_1, \ldots, e_m$ so that $\forall\, i$, fewer than $\binom{r}{p}$ pairs (P, j) exist with $j < i$ and $P \subseteq e_i \cap e_j$ and $|P| = p$.

We will orient each of $e_1, \ldots, e_m$ so no p-set occupies the same set of positions more than once. (The result is then a p-equitable orientation.) Orient e_1 arbitrarily.

Sparse p-set Multigraphs

Thm. If H is r-uniform and $L_p(H)$ is $\binom{r}{p}-1$-degenerate, then H has a p-equitable orientation.

Pf. Order $E(H)$ as $e_1, \ldots, e_m$ so that $\forall\, i$, fewer than $\binom{r}{p}$ pairs (P, j) exist with $j < i$ and $P \subseteq e_i \cap e_j$ and $|P| = p$.

We will orient each of $e_1, \ldots, e_m$ so no p-set occupies the same set of positions more than once. (The result is then a p-equitable orientation.) Orient e_1 arbitrarily.

For each of the pairs (P, j), the occurrence of P in the ordering of e_j forbids $p!(r-p)!$ orderings for e_i (to avoid repeating the set of positions for P).

Sparse p-set Multigraphs

Thm. If H is r-uniform and $L_p(H)$ is $\binom{r}{p} - 1$-degenerate, then H has a p-equitable orientation.

Pf. Order $E(H)$ as $e_1, \ldots, e_m$ so that $\forall\, i$, fewer than $\binom{r}{p}$ pairs (P, j) exist with $j < i$ and $P \subseteq e_i \cap e_j$ and $|P| = p$.

We will orient each of $e_1, \ldots, e_m$ so no p-set occupies the same set of positions more than once. (The result is then a p-equitable orientation.) Orient e_1 arbitrarily.

For each of the pairs (P, j), the occurrence of P in the ordering of e_j forbids $p!(r-p)!$ orderings for e_i (to avoid repeating the set of positions for P).

Since there are fewer than $\binom{r}{p}$ such pairs, some ordering among the $r!$ orderings remains available. ■

An Open Problem

- Very large edges yield many possible orientations.

An Open Problem

- Very large edges yield many possible orientations.

Def. The degree $d(U)$ of a p-set U in H is the number of edges containing U. Let $\Delta_p(H) = \max\{d(U) : |U| = p\}$.

An Open Problem

- Very large edges yield many possible orientations.

Def. The degree $d(U)$ of a p-set U in H is the number of edges containing U. Let $\Delta_p(H) = \max\{d(U)\colon\ |U| = p\}$.

- Since $d_S(U)$ is the number of times U occupies the set S of positions in a given orientation, $d(U) = \sum_S d_S(U)$.

An Open Problem

- Very large edges yield many possible orientations.

Def. The degree $d(U)$ of a p-set U in H is the number of edges containing U. Let $\Delta_p(H) = \max\{d(U)\colon\ |U| = p\}$.

- Since $d_S(U)$ is the number of times U occupies the set S of positions in a given orientation, $d(U) = \sum_S d_S(U)$.

Conj. Given $p, k \in \mathbf{N}$, there exists r_0 such that when $r > r_0$, every r-uniform hypergraph H with $\Delta_p(H) \le k$ has a p-equitable orientation.

An Open Problem

- Very large edges yield many possible orientations.

Def. The degree $d(U)$ of a p-set U in H is the number of edges containing U. Let $\Delta_p(H) = \max\{d(U): |U| = p\}$.

- Since $d_S(U)$ is the number of times U occupies the set S of positions in a given orientation, $d(U) = \sum_S d_S(U)$.

Conj. Given $p, k \in \mathbf{N}$, there exists r_0 such that when $r > r_0$, every r-uniform hypergraph H with $\Delta_p(H) \le k$ has a p-equitable orientation.

- We prove a relaxed version.

An Open Problem

- Very large edges yield many possible orientations.

Def. The degree $d(U)$ of a p-set U in H is the number of edges containing U. Let $\Delta_p(H) = \max\{d(U) \colon |U| = p\}$.

- Since $d_S(U)$ is the number of times U occupies the set S of positions in a given orientation, $d(U) = \sum_S d_S(U)$.

Conj. Given $p, k \in \mathbb{N}$, there exists r_0 such that when $r > r_0$, every r-uniform hypergraph H with $\Delta_p(H) \le k$ has a p-equitable orientation.

- We prove a relaxed version.

Def. An orientation of an r-uniform hypergraph H is nearly p-equitable if $|d_S(U) - d_T(U)| \le 2$ whenever U is a p-set of vertices and S and T are p-sets of positions.

Nearly p-Equitable

Thm. For $p, k, r \in \mathbf{N}$ with $\binom{r}{p} > 9\binom{k-1}{2}$, every r-uniform H with $\Delta_p(H) \leq k$ has a nearly p-equitable orientation.

Nearly p-Equitable

Thm. For $p, k, r \in \mathbf{N}$ with $\binom{r}{p} > 9\binom{k-1}{2}$, every r-uniform H with $\Delta_p(H) \leq k$ has a nearly p-equitable orientation.

Pf. Order each edge randomly. We want each p-set of vertices to be in each p-set of positions at most twice.

Nearly p-Equitable

Thm. For $p, k, r \in \mathbf{N}$ with $\binom{r}{p} > 9\binom{k-1}{2}$, every r-uniform H with $\Delta_p(H) \le k$ has a nearly p-equitable orientation.

Pf. Order each edge randomly. We want each p-set of vertices to be in each p-set of positions at most twice.

Events: For a p-set U in edges e, f, g, let $A_{U,\{e,f,g\}}$ mean U occupies the same p positions in all three.

Nearly p-Equitable

Thm. For $p, k, r \in \mathbf{N}$ with $\binom{r}{p} > 9\binom{k-1}{2}$, every r-uniform H with $\Delta_p(H) \le k$ has a nearly p-equitable orientation.

Pf. Order each edge randomly. We want each p-set of vertices to be in each p-set of positions at most twice.

Events: For a p-set U in edges e, f, g, let $A_{U,\{e,f,g\}}$ mean U occupies the same p positions in all three.

$\Pr(A_{U,\{e,f,g\}}) = \binom{r}{p}^{-2}$.

Nearly p-Equitable

Thm. For $p, k, r \in \mathbf{N}$ with $\binom{r}{p} > 9\binom{k-1}{2}$, every r-uniform H with $\Delta_p(H) \le k$ has a nearly p-equitable orientation.

Pf. Order each edge randomly. We want each p-set of vertices to be in each p-set of positions at most twice.

Events: For a p-set U in edges e, f, g, let $A_{U,\{e,f,g\}}$ mean U occupies the same p positions in all three.

$\Pr(A_{U,\{e,f,g\}}) = \binom{r}{p}^{-2}$. Also, $A_{U,\{e,f,g\}}$ is mutually independent of the family F of events whose sets of three edges contain none of $\{e, f, g\}$.

Nearly p-Equitable

Thm. For $p, k, r \in \mathbf{N}$ with $\binom{r}{p} > 9\binom{k-1}{2}$, every r-uniform H with $\Delta_p(H) \le k$ has a nearly p-equitable orientation.

Pf. Order each edge randomly. We want each p-set of vertices to be in each p-set of positions at most twice.

Events: For a p-set U in edges e, f, g, let $A_{U,\{e,f,g\}}$ mean U occupies the same p positions in all three.

$\Pr(A_{U,\{e,f,g\}}) = \binom{r}{p}^{-2}$. Also, $A_{U,\{e,f,g\}}$ is mutually independent of the family F of events whose sets of three edges contain none of $\{e, f, g\}$.

For a p-set $W \subset e$, at most $k-1$ other edges contain W, so at most $\binom{k-1}{2}$ events have the form $A_{W,\{e,x,y\}}$, and there are $\binom{r}{p}$ choices for W. Considering also f and g, fewer than $3\binom{k-1}{2}\binom{r}{p}$ other events are excluded from F.

Finishing the Proof

e $\Pr\left[A_{U,\{e,f,g\}}\right] \cdot 3\binom{k-1}{2}\binom{r}{p} < 3\frac{1}{\binom{r}{p}^2} \cdot 3\binom{k-1}{2}\binom{r}{p} < 1$

Finishing the Proof

$$e \Pr\left[A_{U,\{e,f,g\}}\right] \cdot 3\binom{k-1}{2}\binom{r}{p} < 3\frac{1}{\binom{r}{p}^2} \cdot 3\binom{k-1}{2}\binom{r}{p} < 1$$

By the Lovász Local Lemma, with positive probability no bad event occurs. ∎

Finishing the Proof

$$\mathbf{e}\,\Pr\left[A_{U,\{e,f,g\}}\right] \cdot 3\binom{k-1}{2}\binom{r}{p} < 3\frac{1}{\binom{r}{p}^2} \cdot 3\binom{k-1}{2}\binom{r}{p} < 1$$

By the Lovász Local Lemma, with positive probability no bad event occurs. ■

Another Open Problem

Finishing the Proof

$\mathbf{e}\,\Pr\left[A_{U,\{e,f,g\}}\right] \cdot 3\binom{k-1}{2}\binom{r}{p} < 3\frac{1}{\binom{r}{p}^2} \cdot 3\binom{k-1}{2}\binom{r}{p} < 1$

By the Lovász Local Lemma, with positive probability no bad event occurs. ∎

Another Open Problem

Def. imbalance of orientation $= \max_{U,S,T} |d_S(U) - d_T(U)|$.
$f_p(H)$ = least imbalance over orientations of H.
$f(p,r,n)$ = maximum of $f_p(H)$ over n-vertex r-uniform hypergraphs.

Finishing the Proof

e $\Pr\left[A_{U,\{e,f,g\}}\right] \cdot 3\binom{k-1}{2}\binom{r}{p} < 3\frac{1}{\binom{r}{p}^2} \cdot 3\binom{k-1}{2}\binom{r}{p} < 1$

By the Lovász Local Lemma, with positive probability no bad event occurs. ∎

Another Open Problem

Def. imbalance of orientation $= \max_{U,S,T} |d_S(U) - d_T(U)|$.
$f_p(H)$ = least imbalance over orientations of H.
$f(p,r,n)$ = maximum of $f_p(H)$ over n-vertex r-uniform hypergraphs.

- $f(1,r,n) = f(r-1,r,n) = 1$.

Finishing the Proof

$\mathbf{e}\,\Pr\left[A_{U,\{e,f,g\}}\right]\cdot 3\binom{k-1}{2}\binom{r}{p} < 3\frac{1}{\binom{r}{p}^2}\cdot 3\binom{k-1}{2}\binom{r}{p} < 1$

By the Lovász Local Lemma, with positive probability no bad event occurs. ■

Another Open Problem

Def. imbalance of orientation $= \max_{U,S,T}|d_S(U) - d_T(U)|$.
$f_p(H)$ = least imbalance over orientations of H.
$f(p,r,n)$ = maximum of $f_p(H)$ over n-vertex r-uniform hypergraphs.

- $f(1,r,n) = f(r-1,r,n) = 1$.

Ques. For fixed $p, r \in \mathbf{N}$, is $f(p,r,n)$ bounded by a value independent of n?