A FIBONACCI TILING OF THE PLANE

Charles W. Huegy
Polygonics, 2 Mann St., Irvine, CA 92612-2709

Douglas B. West
University of Illinois, Urbana, IL 61801-2975, west@math.uiuc.edu

Abstract. We describe a tiling of the plane, motivated by architectural constructions of domes, in which the Fibonacci series appears in many ways.

1. INTRODUCTION

Many large sports arenas and convention centers are built with domed roofs. Because the area to be covered is so large, a framework for the roof is needed that will provide a lot of support without contributing much weight. Here we describe a way of designing such a framework that gives rise naturally to the Fibonacci recurrence in various aspects of a tiling of the plane.

Roughly speaking, one can view the framework as a portion of a polytope with vertices on a sphere, near the highest point of the sphere. For strength, all regions are triangles. Viewed from above, the framework is a tiling of a disc. This extends to a tiling of the plane. For stability against stress caused by gravitational forces, we rely on segments that are chords of circles of longitude and latitude when the dome is viewed as a cap at the top of a globe. In the projection onto a horizontal plane that maps the top to the origin, the longitudinal segments lie on lines from the origin; we call these *ribs*. The latitudinal segments we call *struts*; these appear as concentric polygons in the tiling. The remaining segments we call *trusses*. This terminology is motivated by the engineering usage.

Various parameters of the design can be adjusted for architectural or engineering considerations. Nevertheless, once the number of vertices on the central polygon is chosen, the resulting graphs formed by the framework are isomorphic.

Our focus on ribs and struts is motivated by the work of R. Buckminster Fuller [1]. He argued that domes are strongest when the edges lie along great circles; hence the name *geodesic dome*. Our ribs lie along great circles. Our struts meet the endpoints of ribs and lie along lesser circles. These lesser circles provide tension/compression rings adding to the strength of the structure, as suggested in [5]. Such tension/compression rings don't arise in Fuller-type tilings, but with a tiling based on concentric circles they arise naturally.

Running head: FIBONACCI TILING
AMS codes: 11B39
Keywords: Fibonacci number, tiling, recurrence
Completed December, 1998.

Diagonal trusses provide lateral stability and complete the triangulation. Fuller held that this was essential for maximum strength; adding trusses to obtain a triangulation increases rigidity.

Fuller's original dome partitioned the icosahedron into many smaller triangles. The truncated icosahedron has pentagons and hexagons as faces but uses edges of a single length [8]. By placing a central vertex in each face, one can regain the structural strength of triangles using only two types of triangles. Since the time of Fuller, a considerable attention has been given to minimizing the number of types of triangles needed to construct ever larger domes [7]. Engineers have also studied Fuller's concept of *tensegrity forces* holding domes together [6].

Our tiling is isomorphic to a vertical projection of the design for a dome. Tiling the plane with polygons is a well-studied area [2,4]. Our tiling is motivated by engineering considerations and exhibits Fibonacci growth as it expands from the origin. Fibonacci numbers also occur in hyperbolic tilings of the plane by triangles or heptagons that are regular with degree 7 or 3, respectively [3].

2. THE TILING

We describe an infinite graph embedded in the plane. The edges are ribs, struts, and trusses as described earlier. The struts form polygons (called *rings*) whose vertices lie on concentric circles centered at the origin. The ribs and trusses join vertices on consecutive rings, with the ribs lying along geometric rays from the origin.

We begin with a regular polygon around the origin. The number of sides is s_0, which is the fundamental parameter of the design. The edges of this polygon are struts forming ring 0. In general, let s_k be the number of struts in the kth ring, and let r_k and t_k be the number of ribs and trusses, respectively, reaching out to the kth ring. Figure 1 shows the central portion of the tiling, with ring 0 through ring 6, in the case $s_0 = 4$.

The graph is generated by two rules that produce the ribs, struts, and trusses in successive rings. The key structural aspect of the design is the first rule, which introduces struts to support the weight of earlier ribs in a symmetric fashion. The application of the rules can be seen clearly in Figure 2, where the ribs are drawn as dashed segments for clarity.

Rule 1: Each rib R that reaches ring k at a point x produces two trusses from x that reach ring $k+1$ at the endpoints of a strut on ring $k+1$ that is perpendicularly bisected by the radial extension of R. Such a vertex x is a *bifurcation point* of ring k. Letting p_k be the number of bifurcation points on ring k, we have $p_k = r_k$.

Rule 2: Each strut on ring $k-1$ produces two trusses reaching ring k at a common point x that initiates a rib to ring $k+1$. Since x also is the common endpoint of two struts on ring k, two trusses emanate from x to ring $k+1$ by the same rule. Thus we call x a *trifurcation point* of ring k. Letting q_k be the number of trifurcation points on ring k, we have $r_{k+1} = q_k = s_{k-1}$.

To get started from ring 0, we use a degenerate form of Rule 2, declaring that all vertices in ring 0 are trifurcation points from which ribs extend to ring 1.

Since ring k is a polygon, the number of vertices on ring k is the same as the number of struts on ring k. The vertices on ring k are generated from struts and bifurcation points on ring $k-1$. There is one point on ring k for each strut on ring $k-1$, and there are two points on ring k for each bifurcation point on ring $k-1$. Thus $s_k = s_{k-1} + 2r_{k-1}$.

Alternatively, the points are bifurcation or trifurcation points, so $s_k = p_k + q_k$. We can also count directly the struts generated at ring k; we have one from each bifurcation point and two from each trifurcation point at ring $k-1$. Thus $s_k = p_{k-1} + 2q_{k-1}$. From these two equations, we obtain

$$s_k = p_{k-1} + 2q_{k-1} = s_{k-1} + q_{k-1} = s_{k-1} + s_{k-2}.$$

Since $p_{k+1} = r_{k+1} = q_k = s_{k-1}$, the other sequences use the same values, except that values appear in $\langle q\rangle$ delayed by one step and in $\langle p\rangle, \langle r\rangle$ delayed by two steps. To generate the sequences, we need the initial values. Our parameter is s_0. All the points in ring 0 are trifurcation points. Thus we have $p_0 = r_0 = 0$ and $q_0 = s_0$. Thus $s_1 = 2s_0$, and our sequence is s_0 times the Fibonacci sequence. Specifically, with the convention $F_0 = F_1 = 1$, we have $r_k = s_0 F_{k-1}$.

In light of the tiling in [4], we note that our tiling is a refinement of a tiling formed from irregular hexagons. The bifurcation point on level k at the outer endpoint of each rib is incident to five edges: the rib, the two neighboring struts, and the two trusses that ascend to the next level. Adding the other triangle sharing the strut between those two trusses completes a hexagonal region consisting of two triangles between levels $k-1$ and k, three between levels k and $k+1$, and one between levels $k+1$ and $k+2$. Each such hexagonal region contains one rib, and each rib generates one such hexagon. We call the hexagonal region containing a rib the *fundamental hexagon* of the rib (see Figure 3).

The fundamental hexagons are pairwise disjoint. Except for the central s_0-gon and the triangles neighboring it, the plane is tiled by these hexagons. When $s_0 = 3$, added the central hexagonal region completes a tiling of the plane by hexagons.

One can also observe that each rib begins an infinite subgraph of the tiling. These subgraphs are pairwise isomorphic in their adjacency structure.

3. FURTHER REMARKS

To fix the remainder of the planar representation, we must choose the radii of the rings and locate the trifurcation points on each ring.

Between successive bifurcation points there are one or two trifurcation points; we call the latter "double" trifurcation points. Around a single trifurcation point, we obtain two struts of the same length; let s'_k be the number of struts of this type on the kth ring. When there are double trifurcation points, we place them so that the three resulting struts have equal length; let s''_k be the number of struts of this type.

Thus each ring has struts of at most two lengths. We next determine the number of each type. As shown in Figure 2, each single trifurcation point is generated by a bifurcation point (end of a rib) two rings earlier. On the other hand, double trifurcation points are generated by bifurcation points in the preceding ring. Thus $s'_k = 2r_{k-2}$ and $s''_k = 3r_{k-1}$.

Since $\langle r\rangle$ is a multiple of the Fibonacci sequence, so are $\langle s'\rangle$ and $\langle s''\rangle$. Because the ratio of successive Fibonacci numbers is not constant, it is not possible to choose the radii of the rings to ensure that only two lengths of struts are used in the tiling. However, since those ratios oscillate around (and approach) the golden ratio ϕ, and since circumference is proportional to radius, choosing the radius of ring k to be ϕ^k will keep the lengths of struts "asymptotically" to two values.

We have chosen this approach in our illustrations. Once the trifurcation points are placed on one ring and the radius for the next ring has been chosen, the new ribs are extended out from these trifurcation points to reach the new ring, and the new trifurcation points are computed by trisection or bisection according to whether there are one or two of them between the neighboring ribs. Thus the tiling has been completely determined by choosing the radii to be powers of ϕ.

We remark on one more occurrence of the Fibonacci recurrence. We have expressed our sequence by $r_k = s_0 F_{k-1}$. The expression $s_k = s'_k + s''_k = 2r_{k-2} + 3r_{k-1}$ relating three multiples of Fibonacci sequences is an instance of a famous identity involving the Fibonacci numbers. With $r_k = F_{k-1}$ and $s_k = F_{k+1}$ in our indexing ($F_0 = F_1 = 1$), our expression for s_k becomes $F_{k+1} = F_2 F_{k-3} + F_3 F_{k-2}$. The generalization of this is $F_{n+m} = F_{n-1}F_{m-1} + F_n F_m$, which has a simple combinatorial proof under the model that the Fibonacci number F_n counts the lists of 1's and 2's that sum to n.

Alternatively, if one has computed $\langle r \rangle$ but not $\langle s \rangle$, this identity involving the two types of struts yields it by iteration of the Fibonacci recurrence:

$$s_k = 2F_{k-3} + 3F_{k-2} = 2F_{k-1} + F_{k-2} = F_{k-1} + F_k = F_{k+1}.$$

We close with several remarks about engineering aspects. The design presented here will be used as a semester-long example in an architecture course at the University of Illinois. The course will develop specifications for the structure that will then be tested by civil engineers. We expect that this testing will reveal that the design has greater strength per unit weight of materials as a result of using tension/compression rings corresponding to the concentric circles of the tiling. The ribs also strengthen the structure by spreading load evenly.

Another engineering issue is related to the fundamental hexagons discussed in Section 2. Deleting the struts leaves a decomposition of each fundamental hexagonal into three quadrangular regions. Each such quadrangle straddles a strut, and thus the clockwise-facing boundaries of these quadrangles decompose the edge set into paths of length two. This allows us to minimize the number of structural components by forming a quadrangle as two triangular plates, with the clockwise-facing boundary expressed as upward flanges and the opposite boundary as downward flanges.

Our tiling is proposed as the vertical projection into ground level from a three-dimensional design. An important aspect of such an implementation, which we have not considered here, is the choice of relative elevations of the rings in the dome. This adds another dimension to the testing. Further parameters are the number of edges in the central polygon and the radii of the concentric circles.

Finally, we note that other variations of the central structure lead to Lucas numbers instead of Fibonacci numbers.

References

[1] R.B. Fuller, U.S. Patent 2,682,235: Geodesic Dome, June 29, 1954.

[2] B. Grünbaum and G.C. Shephard, *Tilings and Patterns.* W.H. Freeman, 1987.

[3] H. Harborth, Concentric cycles in mosaic graphs, in *Applications of Fibonacci numbers, Vol 3* (G.E. Bergum et al, eds.), (Kluwer Acad. Publ., 1990), 123–128.

[4] H. Herda, Tiling the plane with incongruent regular polygons, *Fibonacci Quarterly* **19.5** (1981), 437–440.

[5] C. Huegy, U.S. Patent 4,901,483: Spiral Helix Tensegrity Dome, September 22, 1988.

[6] H. Kenner, *Geodesic Math and How to Use It.* Univ. of California Press, 1976.

[7] T. Tarnai, Geodesic dome with three different bar lengths, *Structural Topology* **8** (1983), 23–24.

[8] M.J. Wenninger, *Spherical Models.* Cambridge University Press, 1979.