# Implications among linkage properties in graphs

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**Abstract**

Given a graph $H$ with vertices $w_1, \ldots, w_m$, a graph $G$ with at least $m$ vertices is *$H$-linked* if for every choice of vertices $v_1, \ldots, v_m$ in $G$, there is a subdivision of $H$ in $G$ such that $v_i$ is the branch vertex representing $w_i$ (for all $i$). This concept generalizes the notions of $k$-linked, $k$-connected, and $k$-ordered graphs. For graphs $H_1$ and $H_2$ with the same order that are not contained in stars, the property of being $H_1$-linked implies that of being $H_2$-linked if and only if $H_2 \subseteq H_1$. The implication also holds when $H_1$ is obtained from $H_2$ by replacing an edge $xy$ with an edge from $y$ to a new vertex $x'$. Other instances of non-implication are obtained, using a lemma that the number of vertices appearing in minimum vertex covers of a graph $G$ is at most the vertex cover number plus the size of a maximum matching.

## 1 Introduction

Many applications require measures of the "connectedness" of a graph. A graph is *$k$-connected* if it has more than $k$ vertices and deletion of any $k-1$ vertices leaves a connected subgraph. By a fundamental result (Menger's Theorem [10]), this is equivalent to the existence of $k$ pairwise internally disjoint paths joining any pair of vertices.

More restrictive conditions have been studied. A graph is *$k$-linked* if for every list $(v_1, \ldots, v_{2k})$ of vertices, there are pairwise internally disjoint paths $P_1, \ldots, P_k$ such that each $P_i$ is a path joining $v_i$ and $v_{i+k}$. A graph is *$k$-ordered* if for every list of $k$ distinct vertices, there is a cycle that visits those vertices in the given order; this again specifies $k$ pairwise internally disjoint paths.

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An *H-subdivision* of a graph $H$ is obtained from $H$ by replacing each edge with a path whose internal vertices (if any) are new vertices of degree 2. The *branch vertices* in the $H$-subdivision are the original vertices of $H$. An *H-subdivision* in a graph $G$ is a subgraph of $G$ that is an $H$-subdivision. The concept of $H$-subdivision leads to a natural generalization of $k$-linked graphs and $k$-ordered graphs.

With $\{w_1, \ldots, w_m\}$ denoting the vertices of $H$, a graph $G$ with at least $m$ vertices is *H-linked* if every injective mapping $f\colon V(H) \to V(G)$ extends to an $H$-subdivision in $G$; that is, for each choice of distinct vertices $v_1, \ldots, v_m$ in $G$, there is an $H$-subdivision in which $v_i$ is the branch vertex representing $w_i$, for all $i$. A graph $G$ has the *H-linkage property* if $G$ is $H$-linked. The notion of $H$-linked graphs was introduced independently in [6] and [3].

In the definition of $k$-linked graphs, it is sufficient to study only lists of $2k$ distinct vertices. Thus $G$ is $k$-linked if and only if $G$ is $M_k$-linked, where $M_k$ is a matching of size $k$. Similarly, $G$ is $k$-ordered if and only if $G$ is $C_k$-linked, where $C_k$ is a cycle of order $k$.

Several papers have been devoted to $H$-linkage properties, such as [3, 4, 6, 7, 8, 9]. These all focus on sufficient degree conditions (Dirac-type or Ore-type) for a graph to be $H$-linked. In contrast, here we compare $H$-linkage properties for distinct $H$. For example, $C_k$-linked implies $C_{k-1}$-linked, since $(k-1)$-ordered is a weaker condition than $k$-ordered. Also, various such properties are equivalent to $k$-connectedness, using a trivial observation.

**Fact A** *If $H_2 \subseteq H_1$, then every $H_1$-linked graph is $H_2$-linked.*

Let $S_k$ be the star with $k$ edges and $I_k$ be the graph with $k+1$ vertices and one edge.

**Proposition 1** *If $H$ is a graph such that $I_k \subseteq H \subseteq S_k$, then a graph $G$ is $H$-linked if and only if it is $k$-connected.*

**Proof.** The definitions of $k$-connected and $I_k$-linked are essentially the same. The $S_k$-linkage property is that for any set $U$ of $k$ vertices and a vertex $v \notin U$, there are $k$ paths from $v$ to $U$ that pairwise share only $v$. This is the “Fan Condition” that Dirac [2] proved equivalent to being $k$-connected. Since $I_k \subseteq H \subseteq S_k$, Fact A now yields

$$k\text{-connected} \Leftrightarrow S_k\text{-linked} \Rightarrow H\text{-linked} \Rightarrow I_k\text{-linked} \Leftrightarrow k\text{-connected}. \qquad \square$$

The equivalence of $I_k$-linked with $k$-connected motivates the requirement that $G$ have at least as many vertices as $H$ to be $H$-linked.

To study the relation between $H_1$-linked and $H_2$-linked for general $H_1$ and $H_2$, let $\mathcal{G}_H$ denote the family of $H$-linked graphs. We write $H_2 \le H_1$ when $\mathcal{G}_{H_1} \subseteq \mathcal{G}_{H_2}$. This means that every $H_1$-linked graph is $H_2$-linked; equivalently, the $H_1$-linkage property implies the $H_2$-linkage property. We have defined a partial order on the isomorphism classes of graphs. Fact A states that $H_2 \subseteq H_1$ implies $H_2 \le H_1$.

The resulting poset $\mathcal{P}$ is the containment poset among the families $\mathcal{G}_H$ for all $H$. At the 2005 conference in Denver celebrating Joan Hutchinson’s 60th birthday, Ron Gould described the subposet of $\mathcal{P}$ formed by the graphs with three vertices. Here we determine the subposets corresponding to the $k$-vertex graphs, for all $k$.

We also study whether $H_2 \le H_1$ when $|V(H_1)| \neq |V(H_2)|$. Fact A applies when $H_1$ or $H_2$ contains the other, but otherwise the problem is more difficult. Fact B below holds because $K_{|V(H_1)|}$ is $H_1$-linked but not $H_2$-linked, or similarly for the graph obtained by deleting one edge from $K_{|V(H_2)|}$.

**Fact B** *If $H_2$ has more vertices than $H_1$, then $H_2 \not\le H_1$.*

On the other hand, in Section 2 we prove that $H_2 \le H_1$ in some cases where $H_1$ has more vertices than $H_2$ but does not contain $H_2$. A *splinter* operation on $H$ deletes an edge $xy$ and replaces it with an edge $x'y$, where $x'$ is a new vertex of degree 1. Theorem 5 states that if $H'$ is obtained from $H$ by a splinter operation, then $H \le H'$.

Applying successive splinter operations thus yields a spectrum of successively stronger properties between $H$-linked and $k$-linked, where $H$ has $k$ edges and no isolated vertices. In particular, $M_k$ is the unique maximal element of the subposet of $\mathcal{P}$ on the graphs with $k$ edges. Most of this subposet is unknown, but not for $k = 4$.

**Example 2** Among graphs with four edges, the results we prove in this paper determine all of the relations except for five pairs of graphs. Four of them are the pairs of *consecutive* graphs in the following list (where "+" means disjoint union and $K'_{1,3}$ arises from a claw by subdividing one edge):

$$K'_{1,3}, \quad C_4, \quad K_2 + K_{1,3}, \quad K_2 + K_3, \quad 2P_3.$$

We also do not know whether $C_4 \le K_2 + K_3$ is true. This example is interesting partly because we do know that $C_4 \not\le 2K_1 + K_3$. To see this, consider a 5-connected plane graph with a face of length 4. It is well known that such a graph is not 2-linked, and hence it is not $C_4$-linked, since $M_2 \subset C_4$. On the other hand, every 5-connected graph is $(2K_1 + K_3)$-linked, since deleting any two vertices leaves a 3-connected graph, and Dirac [2] proved that every $k$-connected graph has a cycle through any $k$ vertices, which for $k = 3$ is equivalent to 3-ordered. Fact B yields $2K_1 + K_3 \not\le C_4$.

Subposets of $\mathcal{P}$ formed by fixing the number of vertices are easier to analyze than those where the number of edges is fixed. The job is completed by Theorem 7 in Section 3: when $H_1$ and $H_2$ have the same number of vertices and $H_2$ is not contained in a star, $H_2 \le H_2$ if and only if $H_2 \subseteq H_1$. Theorem 7 and Fact B yield the corollary that if $H$ is not contained in a star, then the $H$-linkage and $H'$-linkage properties are equivalent only if $H \cong H'$.

Theorem 10 in Section 4 generalizes the negative part of Theorem 7 to yield instances where $H_1$-linked does not imply $H_2$-linked even though $H_1$ may have more vertices and edges than $H_2$. This and Theorem 5 provide the first steps toward analyzing the rest of the poset $\mathcal{P}$. A lemma of interest in its own right (Lemma 9) states that in any graph $G$, the number of vertices belonging to minimum-sized vertex covers is at most the vertex cover number plus the maximum size of a matching.

Finally, the special role of $M_k$ leads us to introduce a problem that generalizes the well-known problem of finding the minimum value $f(k)$ such that every $f(k)$-connected graph is $k$-linked [1, 5, 11] (it is known that $2k \le f(k) \le 10k$). Since $I_r$-linked is equivalent to $r$-connected, $f(k)$ is one more than the number of isolated vertices that must be added to $K_2$ to obtain a graph $H$ such that $H$-linked implies $M_k$-linked. Since $H_2 \le M_k$ for every graph $H$ with $k$ edges and no isolated vertices, and $I_r \le H_1$ for every nontrivial graph $H_1$ with $r+1$ vertices, we may define $f(H_1, H_2)$ to be the least $p$ such that $H_2 \le H_1 + pK_1$. We offer this problem as a subject for future study but provide no results on it.

## 2 Establishing $H_2 \le H_1$

We begin with easy observations. We use $G + v$ to denote the graph obtained from $G$ by adding an isolated vertex $v$, and we use $G \vee v$ to denote the graph obtained from $G$ by adding a vertex $v$ with neighborhood $V(G)$.

**Lemma 3** *A graph $G$ is $H$-linked if and only if $G \vee u$ is $(H + v)$-linked.*

**Proof.** *Necessity.* Assume that $G$ is $H$-linked and consider any injective $f\colon V(H + v) \to V(G \vee u)$. Let $w = f(v)$. Since $(G \vee u) - w$ contains a copy of $G$, this subgraph contains an $H$-subdivision with the desired branch vertices to complete the $(H+v)$-subdivision in $G \vee u$.

*Sufficiency.* Assume that $G \vee u$ is $(H + v)$-linked and consider any injective $f\colon V(H) \to V(G)$. Extend $f$ to $V(H + v)$ by setting $f(v) = u$. Since $G \vee u$ is $(H + v)$-linked, $f$ extend to an $(H + v)$-subdivision in $G \vee u$, and deleting $u$ yields the desired extension of $f$ on $V(H)$ to an $H$-subdivision in $G$. □

**Lemma 4** *Graphs $H_1$ and $H_2$ satisfy $H_2 \le H_1$ if and only if $H_2 + v \le H_1 + v$.*

**Proof.** *Necessity.* Suppose that $H_2 \le H_1$. If $G$ is $(H_1 + v)$-linked, then Lemma 3 implies that $G - u$ is $H_1$-linked, for any $u \in V(G)$. Hence $G - u$ is also $H_2$-linked. Given $f\colon V(H_2+v) \to V(G)$, let $u = f(v)$. Since $G-u$ is $H_2$-linked, the desired subdivision exists.

*Sufficiency.* Suppose that $H_2+v \le H_1+v$. If $G$ is $H_1$-linked, then $G\vee u$ is $(H_1+v)$-linked, by Lemma 3, and hence $G\vee u$ is $(H_2+v)$-linked. Now Lemma 3 implies that $G$ is $H_2$-linked. □

A *splinter* operation at a vertex $x$ in a graph $H$ forms a new graph $H'$ by deleting an edge $xy$ and introducing a new vertex $x'$ adjacent only to $y$. In $H'$, the new vertex $x'$ has degree 1, and $x$ has degree $d_H(x) - 1$. The new graph has the same number of edges as $H$.

**Theorem 5** *If $H'$ is obtained from a graph $H$ by a splinter operation, then $H \leq H'$.*

**Proof.** Let $x'$ be the new vertex in $H'$, and let $x$ be the vertex from which its edge was splintered. Let $h$ and $h'$ be the numbers of vertices in $H$ and $H'$; note that $h' = h + 1$.

Let $G$ be an $H'$-linked graph. We need to show that $G$ is also $H$-linked. Since $H'$ has an edge and has $h+1$ vertices, $I_h \subseteq H'$. Hence by Fact 0 and Proposition 1, $G$ is $I_h$-linked and $h$-connected.

Consider an injective mapping $f\colon V(H) \to V(G)$ with $u = f(x)$. Since $G$ is $h$-connected, $d_G(u) \geq h$. Choose $v \in N_G(u) - f(V(H))$. Define $f'\colon V(H') \to V(G)$ by $f'(x') = v$ and $f'(y) = f(y)$ for $y \neq x'$. From the extension of $f'$ to an $H'$-subdivision in $G$, we obtain the desired $H$-subdivision in $G$ by adding the edge $vu$ to the path with endpoint $v$. □

**Corollary 6** *A graph is $k$-linked if and only if it is $H$-linked for every graph $H$ with $k$ edges and no isolated vertices.*

**Proof.** From a graph $H$ with $k$ edges and no isolated vertices, we can obtain $M_k$ by a succession of splinter operations. Hence Theorem 5 implies that every $k$-linked graph is $H$-linked for every such $H$. The converse is immediate, since $M_k$ is such a graph. □

# 3 Pairs of Graphs with Equal Order

For graphs with $n$ vertices, we show that the corresponding subposet of $\mathcal{P}$ is given by graph containment except for the subgraphs of $S_{n-1}$. Let $\beta(H)$ denote the vertex cover number of a graph $H$, the minimum size of a vertex subset incident to every edge.

Our next theorem is a special case of Theorem 10, but its proof is much less complicated. We give it here to motivate the proof for Theorem 10.

**Theorem 7** *If $H_2$ and $H_1$ both have $n$ vertices, and $\beta(H_2) \geq 2$, then $H_2 \leq H_1$ if and only if $H_2 \subseteq H_1$.*

**Proof.** Sufficiency is by Fact A. For the converse, if $H_2$ is contained neither in $H_1$ nor in a star, then we construct an $H_1$-linked graph that is not $H_2$-linked. Let $m = |E(H_2)|$.

Let $G = K_{n+m-1} - E(H_2)$; that is, $G$ is the complement of the disjoint union of $H_2$ with $m-1$ isolated vertices. When we map $V(H_2)$ into $V(G)$ by sending the vertices to their natural images with respect to the missing edges, extension to an $H_2$-subdivision requires an added vertex on the path representing each edge of $H_2$, but there are only $m-1$ added vertices available. Hence $G$ is not $H_2$-linked.

It remains to prove that $G$ is $H_1$-linked. Consider an injective mapping $f\colon V(H_1) \to V(G)$. Let $x_iy_i$ denote the $i$th edge of $H_1$. Let $u_i = f(x_i)$ and $v_i = f(y_i)$ for each $i$. Let $s$ be the number of edges of $H$ whose endpoints are mapped into nonadjacent pairs in $G$ by $f$. We may index the edges so that $u_iv_i \notin E(G)$ for $1 \leq i \leq s$ and $u_iv_i \in E(G)$ for $i > s$.

To complete the extension to an $H_1$-subdivision, we must find pairwise internally disjoint paths to represent the $s$ missing edges. There are also $m - s$ other nonadjacent pairs in $G$. Since $H_2 \not\subseteq H_1$, the nonadjacent pairs in $G$ cannot all be edges desired for the $H_1$-subdivision, and hence $s \leq m - 1$. Let $S = \bigcup_{i=1}^{s} \{u_i, v_i\}$.

Let $W = f(V(H_1))$, so $S \subseteq W$. Let $T = V(G) - W$, so $|T| = m - 1$. Let $T_2 = \{v \in T\colon\ S \subseteq N_G(v)\}$, and let $T_1 = T - T_2$ (see Fig. 1). Let $t_1 = |T_1|$ and $t_2 = |T_2|$, so $t_1 + t_2 = m - 1$. There are exactly $m$ nonadjacent pairs of vertices in $V(G)$. Since each vertex of $T_1$ forms such a pair with some vertex of $S$, we have $s + t_1 \leq m$, and hence $t_2 \geq s - 1$.

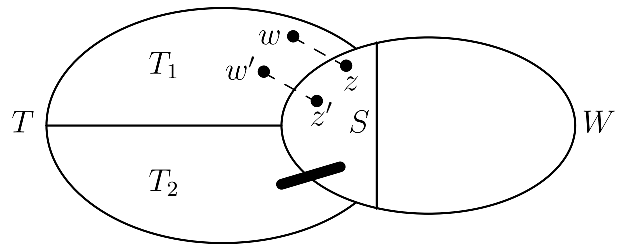


Fig. 1. Vertex partition of $G$

If $t_2 \geq s$, then $s$ distinct vertices of $T_2$ can be used to complete the $s$ needed paths, all with length 2. Thus we may assume that $t_2 = s - 1$. Now $s + t_1 = m$ (and also $t_1 \geq 1$). This means that every vertex of $T_1$ is nonadjacent to exactly one vertex in $S$, and together with the $s$ pairs corresponding to $E(H_2)$ these are all the nonadjacent pairs in $G$.

Let $w$ be a vertex of $T_1$, with unique nonneighbor $z$ in $S$. If there exists $i \leq s$ such that $z \notin \{u_i, v_i\}$, then we use the path $\langle u_i, w, v_i \rangle$ to link $u_i$ and $v_i$. Otherwise, $\beta(H_2) > 1$ requires another vertex $w'$ in $T$ with unique nonneighbor $z'$ in $S$ such that $z' \neq z$. If $z'$ avoids some pair $\{u_i, v_i\}$, then we proceed as above; otherwise, $z$ and $z'$ are both contained in all such pairs, so $s = 1$ and we use the path $\langle z, w', w, z' \rangle$ to link $u_1$ and $v_1$.

In each case, the remaining $s - 1$ needed paths each use one vertex of $T_2$. □

**Corollary 8** *If $H_1$ is not contained in a star, then the $H_1$-linkage and $H_2$-linkage properties are equivalent if and only if $H_1 \cong H_2$.*

**Proof.** By Fact B, we may assume that $|V(H_1)| = |V(H_2)|$. If $H_2$ is not contained in a star, then the conclusion is stated by Theorem 7. If $H_2$ is contained in a star but $H_1$ is not, then Theorem 7 implies that $H_1 \not\leq H_2$. □

We add two remarks about the proof of Theorem 7. First, the proof gives an algorithm for extending $f$ into an $H_1$-subdivision in $G$. Second, the construction of a graph that is $H_1$-linked but not $H_2$-linked extends easily to provide an infinite family of such examples. We can successively add any number of vertices joined to all vertices not incident to the deleted copy of $H_2$.

# 4 Forbidding $H_1 \leq H_2$

In this section, we compare $H_1$-linkage and $H_2$-linkage when $H_1$ and $H_2$ do not have the same number of vertices. Note that if $H_2$ has more vertices, then always $H_2 \not\leq H_1$, since deleting an edge from a complete graph with $|V(H_2)|$ vertices then yields a graph that is $H_1$-linked but not $H_2$-linked.

We will generalize the construction of Theorem 7 to apply to pairs where $H_1$ has $n+k$ vertices, and $H_2$ has $n$ vertices and $m$ edges. If the number of common edges is not too big, and the vertex cover number is not too small, then again $K_{n+m-1} - E(H_2)$ will be a graph that is $H_1$-linked but not $H_2$-linked. The precise statement is in Theorem 10.

We will need a lemma about vertex covers; it is of independent interest. Let $\alpha'(G)$ denote the maximum size of a matching in $G$. The edges of a matching require distinct vertices in a cover, so $\alpha'(G) \leq \beta(G)$. Hence the lemma implies that the total number of vertices in minimum vertex covers of $G$ is at most $2\beta(G)$.

An $X,Y$-*bigraph* is a bipartite graph with partite sets $X$ and $Y$. For a graph $G$ and $S,T \subseteq V(G)$, let $G[S,T]$ denote the maximal $S,T$-bigraph contained in $G$.

**Lemma 9** *In a graph $G$, the total number of vertices that belong to minimum vertex covers of $G$ is at most $\beta(G) + \alpha'(G)$.*

**Proof.** Let $S$ be the set of all vertices belonging to minimum vertex covers; suppose that $|S| > \beta(G) + \alpha'(G)$. Hence there are at least two minimum vertex covers, $C_1$ and $C_2$. Let $A = C_1 \cap C_2$, $B_1 = C_1 - C_2$, $B_2 = C_2 - C_1$, and $D = S - (C_1 \cup C_2)$ (see Fig. 2). Since $|C_1| = |C_2|$, also $|B_1| = |B_2|$. Let $b = |B_1| = |B_2|$, so $|A| = \beta(G) - b$.

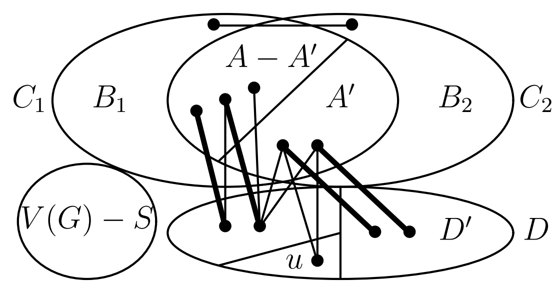


Fig. 2. Vertex covers in $G$

Let $G_B = G[B_1, B_2]$ and $G_A = G[A, D]$. Since deleting a vertex cover leaves an independent set, the only edges in $G$ not incident to $A$ are those of $G_B$. Hence $\beta(G) \leq |A| + \beta(G_B)$. Thus $\beta(G_B) \geq b$, which yields $\alpha'(G_B) \geq b$, by the König–Egerváry Theorem. Since $G_B$ and $G_A$ are disjoint, $\alpha'(G_B) + \alpha'(G_A) \leq \alpha'(G)$, and hence $\alpha'(G_A) \leq \alpha'(G) - b$.

In $G_A$, let $Q$ be a minimum vertex cover and $M$ be a maximum matching (edges drawn in Fig. 2 show $G_A$, with $M$ bold). Since $|Q| = |M|$ (by the König–Egerváry Theorem), each edge of $M$ is incident to exactly one vertex of $Q$. Let $A' = Q \cap A$, and let $D'$ be the subset of $D$ matched to $A'$ by $M$; we have $Q \cap D' = \varnothing$. Thus no edges join $D'$ and $A - A'$.

Since $M$ covers $\alpha'(G_A)$ vertices in $D$, and $|D| = |S - (C_1 \cup C_2)| > \alpha'(G) - b$, some vertex $u$ in $D$ is not covered by $M$. Nevertheless, $N_G(u) \subseteq A'$. Let $R$ be a minimum vertex cover of $G$ containing $u$. Let $R' = A' \cup (R - D')$. Since all edges incident to $D'$ are covered by $A$, $R'$ is a vertex cover of $G$. Since $M$ yields $|A'| = |D'|$, we have $|R'| = |R| = \beta$. However, since $N_G(u) \subseteq A'$, also $R' - \{u\}$ is a vertex cover, which contradicts the minimality of $R$. □

Our general construction of a graph that is $H_1$-linked but not $H_2$-linked fails when $H_2$ lies in a special family. A *double-star* is a tree with exactly two non-leaf vertices. Let $\mathcal{H}_{r,k}$ be the class of graphs with $k+1$ components consisting of a double-star with $r-k$ edges plus $k$ isolated edges. Such graphs have vertex cover number $k+2$. It should be noted that the main part of the proof suffices when $\beta(H_2) \geq k+3$ and ignores the special family. The last page of the proof is needed only for the case $\beta(H_2) = k+2$.

**Theorem 10** *Let $H_1$ and $H_2$ be simple graphs, where $H_1$ has $n+k$ vertices and $H_2$ has $n$ vertices and $m$ edges. If $|E(H_1 \cap H_2)| \leq m-k-1$ and*
*(1) $\beta(H_2) \geq k+3$, or*
*(2) $\beta(H_2) = k+2$ and ($H_2 \notin \mathcal{H}_{m,k}$ or $m > 2k+2$),*
*then $K_{n+m-1} - E(H_2)$ is $H_1$-linked but not $H_2$-linked.*

**Proof.** Let $G = K_{n+m-1} - E(H_2)$. The construction is the same as in Theorem 7, not dependent on $H_1$, so the argument that $G$ is not $H_2$-linked is the same.

It remains to prove that $G$ is $H_1$-linked. Again, consider $f\colon V(H_1) \to V(G)$, and let $x_iy_i$ denote the $i$th edge of $H_1$, with $u_i = f(x_i)$ and $v_i = f(y_i)$ for each $i$. Again let $s$ be the number of edges of $H_1$ mapped into nonadjacent pairs, and index them so that $u_iv_i \notin E(G)$ for $1 \leq i \leq s$ and $u_iv_i \in E(G)$ for $i > s$.

Again we need paths for the $s$ missing edges and $G$ has $m-s$ other nonadjacent pairs. The restriction on $|E(H_1) \cap E(H_2)|$ means that $f$ cannot map more than $m-k-1$ edges of $H_1$ onto missing edges. Thus $s \leq m-k-1$. Define $S, W, T, T_1, T_2, t_1, t_2$ as in Theorem 7 (see Fig. 1). Since $H_1$ has $n+k$ vertices, $t_1 + t_2 = m-k-1$.

Let $U = \{1, \ldots, s\}$. Let $B$ be an auxiliary $U, T_1$-bigraph defined by putting $i$ adjacent to $w$ if and only if $w$ is a common neighbor of $u_i$ and $v_i$ in $G$. If $\alpha'(B) \geq s - t_2$, then the paths of length 2 (in $G$) that correspond to a matching of size $s - t_2$ combine with paths of length 2 through vertices of $T_2$ to complete the desired $H_1$-subdivision.

Hence we may assume that $\alpha'(B) < s - t_2$, and hence also $\beta(B) < s - t_2$. Let $X_1 \cup Y_1$ be a minimum vertex cover of $B$, where $X_1 \subseteq U$ and $Y_1 \subseteq T_1$ (see Fig. 3). Let $x_1 = |X_1|$ and $y_1 = |Y_1|$, so $x_1 + y_1 \leq s - t_2 - 1$. Let $X_2 = U - X_1$ and $Y_2 = T_1 - Y_1$, so every $u \in Y_2$ is adjacent to at most one vertex of each pair in $S$ corresponding to a vertex of $X_2$.

We have $\beta(B) < s - t_2$ under the assumption that the desired $H_1$-subdivision with branch vertices specified in $W$ does not exist in $G$. Will we obtain a contradiction to this assumption in most cases by using the vertex cover of $B$ to build a cover of $E(\overline{G})$ with fewer than $\beta(H_2)$ vertices.

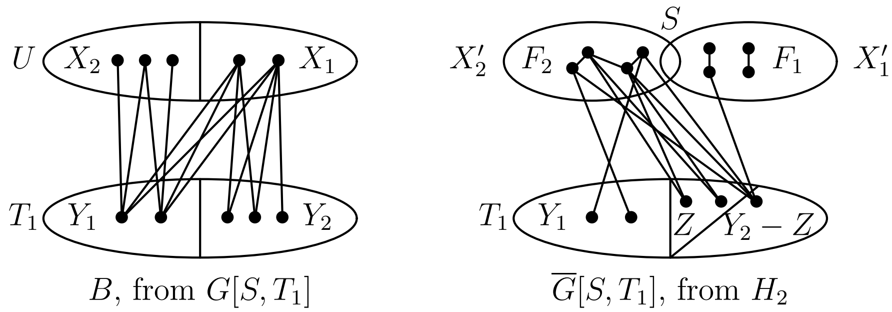


Fig. 3. Auxiliary bigraph in $G$; structure for vertex cover in $\overline{G}$.

Let $l$ be the number of edges in $\overline{G}[S,T_1]$. Since $S$ covers all edges of $\overline{G}$ incident to $S$, at most $m-s-l$ vertices are needed to cover the rest. For $j \in \{1,2\}$, let $X_j' = \bigcup_{i\in X_j}\{u_i, v_i\}$ (see Fig. 3), and let $F_j$ be the minimal subgraph of $\overline{G}$ whose edges set is $\{u_iv_i\colon\ i \in X_i\}$ To complete the vertex cover, we will choose vertices to cover the edge sets of $F_1$, $F_2$, $\overline{G}[S,Y_1]$, and $\overline{G}[S,Y_2]$.

We cover $\overline{G}[S,Y_1]$ using the $y_1$ vertices of $Y_1$. For the edge $u_iv_i$ in $F_1$, we choose $u_i$ or $v_i$, thus covering $F_1$ using at most $x_1$ vertices of $X_1'$.

For the remaining edges, let $F = F_2 \cup \overline{G}[S,Y_2]$. Let $c = \beta(F_2)$. For $u \in Y_2$, let $N_u = N_{\bar{G}}(u) \cap S$. By the choice of the cover $X_1 \cup Y_1$ in $B$, the vertex $u$ is adjacent in $G$ to at most one vertex of edge in $F_2$. Hence $N_u \cap X_2'$ is a vertex cover of $F_2$, which yields $|N_u| \geq c$. Let $Z = \{u \in Y_2\colon\ |N_u| = c\}$ and $z = |Z|$. If $u \in Z$, then $N_u \cap X_2'$ is a minimum vertex cover of $F_2$. By Lemma 9, $|\bigcup_{u\in Z} N_u| \leq 2c$. Thus $(Y_2 - Z) \cup (\sum_{u\in Z} N_u)$ is a vertex cover for $F$ with size at most $(t_1 - y_1 - z) + 2c$.

We have constructed a vertex cover showing $\beta(\overline{G}) \leq (m-s-l)+y_1+x_1+(t_1-y_1-z)+2c$. We hope to bound this quantity by $k+2$.

Let $r = |E(\overline{G}[S,Y_2])|$. Since $\overline{G}[S,T_1]$ has $l$ edges, with each vertex of $T_1$ incident (by definition of $T_1$) to at least one, $r \leq l - y_1$. However, $r \geq \sum_{u\in Y_2}|N_u| \geq c|Y_2| + |Y_2 - Z|$. Together, $c(t_1 - y_1) + (t_1 - y_1 - z) \leq l - y_1$, so $t_1 - y_1 - z \leq (l - y_1) - c(t_1 - y_1)$. Also, the expressions $m - k - 1 = t_1 + t_2$ and $x_1 + y_1 \leq s - t_2 - 1$ yield $x_1 + y_1 \leq s - m + k + t_1$. Thus

$$\begin{aligned}\beta(\overline{G}) &\leq x_1 + y_1 + m - s - l + (t_1 - y_1 - z) + 2c\\ &\leq k + t_1 - y_1 - c(t_1 - y_1) + 2c\\ &= k + 2 - (c-1)(t_1 - y_1 - 2).\end{aligned} \tag{1}$$

Let $q = (c-1)(t_1 - y_1 - 2)$. If $q$ is positive, then already we contradict the hypothesis that $\beta(\overline{G}) \geq k+2$. Consider the ways that $q$ can be nonpositive.

If $c = 0$ or $y_1 = t_1$ (by definition $y_1 \leq t_1$), then $X_1 = U$ or $Y_1 = T$, respectively. The former contradicts $\beta(B) < s - t_2$, while the latter yields $s - 1 \geq t_1 + t_2 = m - k - 1 \geq s$, a contradiction. Hence we may assume that $c \geq 1$ and $t_1 > y_1$.

Whenever $z \leq c$, we can improve the upper bound on $\beta(F)$ from $(t_1 - y_1 - z) + 2c$ to $t_1 - y_1 + c$ by using all of $Y_2$ together with a minimum vertex cover of $F_2$. If also $z > 1$, then we can improve it by one more, by using $N_u$ as that cover for some $u \in Z$, since then we can drop $u$. Thus we reduce the upper bound by $c$ if $z = 0$ and by $c - z + 1$ if $z > 0$; call this *replacement reduction*. We call reducing the upper bound on $\beta(\overline{G})$ below $k + 2$ a *win*.

If $q < 0$, then $c \geq 2$ and $t_1 = y_1 + 1$, and we obtain $q = -(c - 1)$. In this case, $|Y_2| = 1$ and $z \leq 1$. By replacement reduction, we reduce the upper bound on $\beta(\overline{G})$ by $c$, a win.

Hence we may assume that $q = 0$, so $c = 1$ or $t_1 = y_1 + 2$. We still have the contradiction if $\beta(H_2) \geq k+3$, so we may assume that $\beta(H_2) = k+2$. (That is, the proof is now complete for $\beta(H_2) \geq k + 3$; the remainder is needed only when $\beta(\overline{G}) = k + 2$.)

If $t_1 = y_1+2$ and $c \geq 2$, then $|Y_2| = 2$ and $z \leq 2$. Hence $z \leq c$, and replacement reduction reduces the upper bound by $c$ if $z = c = 2$ and by $c - z + 1$ otherwise. This is a win.

Hence we may assume that $c = 1$. This implies that $F_2$ is a star with at least one edge. If $|X_2| > 1$, then $F_2$ has a minimum cover $\{w\}$ that covers all of $\overline{G}[X_2, Z]$. In that situation, $\{w\} \cup (Y_2 - Z)$ covers $E(F)$, so $\beta(F) \leq 1 + t_1 - y_1 - z$. Since (1) used $\beta(F) \leq t_1 - y_1 - z + 2c$, this is a win. Hence we conclude that $x_2 = 1$. Furthermore, since we win if $\bigcup_{u \in Z} N_u$ is a single vertex, we have $\bigcup_{u \in Z} N_u = X_2'$.

We win by improving the bounds $\beta(F_1) \leq x_1$ or $\beta(\overline{G}[S, Y_1]) \leq y_1$ unless $F_1$ is a matching and $x_1 + y_1 = s - t_2 - 1$. With $x_2 = 1$, we have $x_1 = s - 1$, and hence $y_1 = -t_2 \leq 0$. We conclude that $y_1 = t_2 = 0$ and $Y_2 = T_1$.

We claim further that $Y_2 = Z$. We win on $|E(\overline{G}[S, Y_2])|$ unless vertices of $Z$ and $Y_2 - Z$ have degree exactly 1 and 2 in $\overline{G}[S, Y_2]$, and we win on $\beta(\overline{G}[S, Y_2]$ unless $(Y_2 - Z) \cup (\bigcup_{u \in Z} N_u)$ is a minimum vertex cover for $\bar{G}[S, Y_2]$. Since already $\bigcup_{u \in Z} N_u = X_2'$, each vertex $w \in Y_2 - Z$ must have a neighbor $w' \in X_1'$. However, $F_1$ is a matching, so we can choose the vertex covering the edge containing $w'$ in $F_1$ to be $w'$. Now $w$ is not needed in the cover; we win.

Hence $Z = T_1$, and each vertex of $T_1$ is adjacent to one vertex in $X_2'$. Since our minimum vertex cover contains all of $X_2'$, we win on $\beta(F_1)$ if $X_2' \cap X_1' \neq \varnothing$. Finally, we win unless $m - s - l$ vertices beyond $\beta(F)$ are needed to cover the edges of $\overline{G}$ not incident to $S$. This requires that those edges form a matching, and that also no edges join $X_1'$ and $X_2'$.

We have proved that $\overline{G}$ consists of a double-star (with center $X_2'$ and leaves $Z$) plus isolated edges and vertices. The double-star has $t_1$ leaves, and $t_1 = m - k - 1$ (since $t_2 = 0$). Hence the double-star has $m - k$ edges, and there are $k$ other isolated edges in $\overline{G}$, so $H_2 \in \mathcal{H}_{m,k}$.

Since $l = t_1$, the cover uses $k + 1 - s$ vertices outside $S$ to cover those edges. Thus $s \leq k + 1$, and from the beginning we have $s \leq m - k - 1$.

If $s < m-k-1$, then let $X_2' = \{u, v\}$ and choose $w \in N_{\overline{G}}(u) \cap Z$ and $w' \in N_{\overline{G}}(v) \cap Z$. We have $ww' \in E(G)$. Hence we can use the path $\langle u, w', w, v \rangle$ to link $u$ and $v$. The maximum matching of size $s - 1$ in $B$ provides paths for the other desired missing edges, so we obtain the desired $H_1$-subdivision.

Hence we may assume that $s = m - k - 1$. Now $m = k + 1 + s \leq 2k + 2$, which completes the proof. □

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