Spanning Cycles through Specified Edges in Bipartite Graphs

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Joint work with
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Thm. Suffic. for F-Hamilt. when F is perfect matching:
(Häggvist [1979]) $\sigma_2(G) \ge n + 1$.
(Las Vergnas [1972]) G is bipartite and $\sigma(G) \ge n/2+2$.

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Unsharp: Las Vergnas: $\sigma(G) \ge n/2 + 2$ when $k = n/2$.

Sharpness Examples

- F has t components: t_1 odd length, t_2 even length.

$\sigma(G) \geq n/2 + \lceil k/2 \rceil + \epsilon$ suffices; let $\tau(F) = \lceil k/2 \rceil + \epsilon$.

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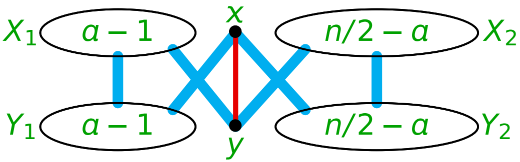
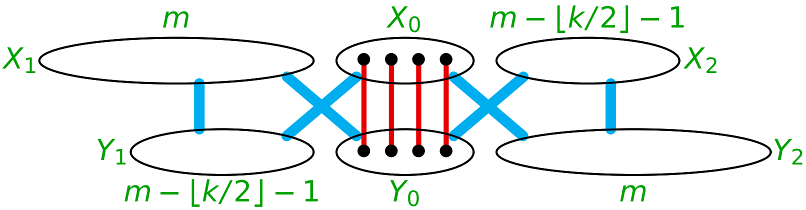
Ex. For $k \in \{1, 2\}$, add to $K_{a-1,a-1} + K_{n/2-a,n/2-a}$ new vertices x and y with $xy \in F$ as below; $\sigma(G) = n/2 + 1$, but G has no cycle through xy.



Main Example

Ex. For $k \geq 3$ and $m \geq \lfloor k/2 \rfloor + 1$, let $n = 4m + 2\lceil k/2 \rceil - 2$.
Start with $K_{m,m-\lfloor k/2 \rfloor -1} + K_{m-\lfloor k/2 \rfloor -1,m}$.
Add k vertices to each partite set, inducing the set F.
Add all other edges, as shown below.



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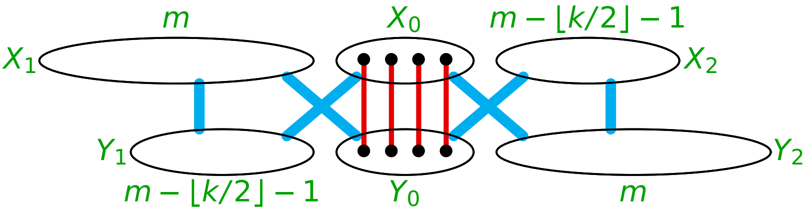

For $x \in X_1$ and $y \in Y_2$,
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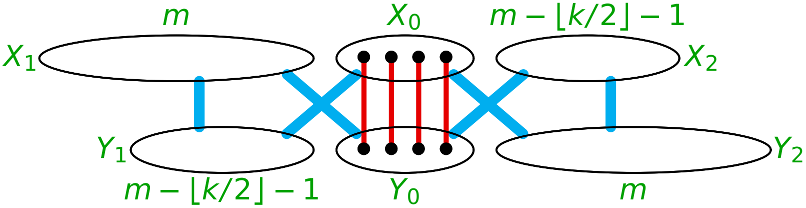

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Deleting $X_0 \cup Y_0$ from a spanning cycle through F leaves k paths. Hence $X_1 \cup Y_1$ or $X_2 \cup Y_2$ is covered by $\leq \lfloor k/2 \rfloor$ paths. Since the sizes differ by $\lfloor k/2 \rfloor + 1$, impossible.

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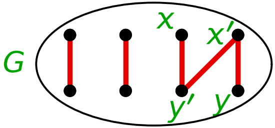

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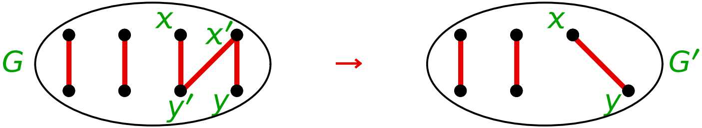

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Replace xy in C' with $\{xy', y'x', x'y\}$ to form C.

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Establish conditions allowing use of a spanning path through F to obtain a spanning cycle through F.

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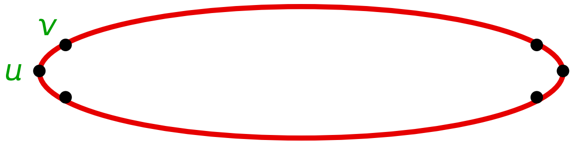

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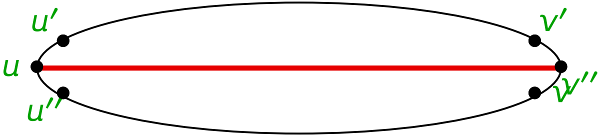

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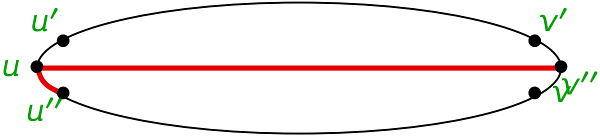

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Pathlengths at most 2 in F $\Rightarrow$ F contains at most one edge of C incident to u or v, say uu''.

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Path $C[u', v] \cup vu \cup C[u, v']$ is a spanning path through F in G. **Lemma** applies! ∎

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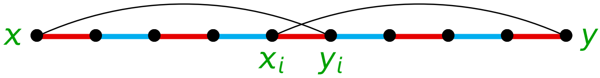

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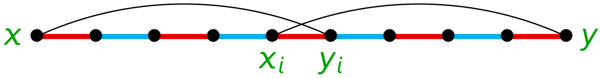

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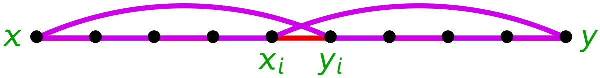

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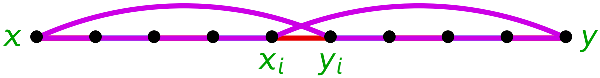

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Obs. An unselected full odd edge turns a spanning path through F into a spanning cycle though F.

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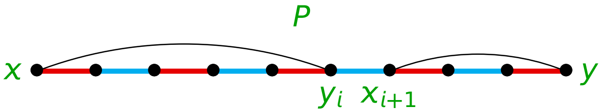

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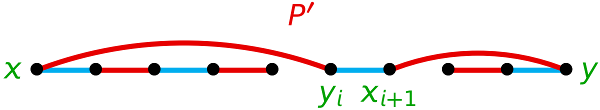

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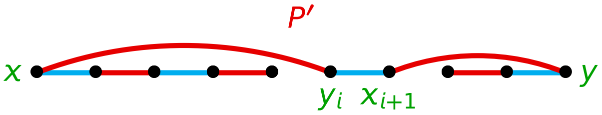

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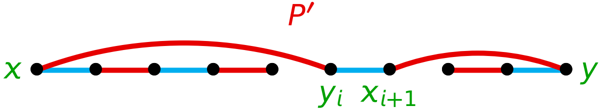

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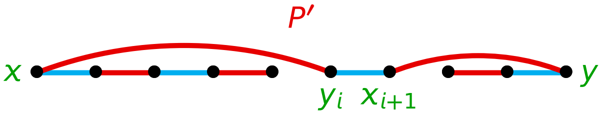

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Edge $y_i x_{i+1}$ is odd in neither P nor P'; other selected edges are odd in exactly one.

Choose $Q \in \{P, P'\}$ with fewest selected odd edges: $|E_{odd}(Q) \cap F| \le \lfloor (k-1)/2 \rfloor < \lceil k/2 \rceil$. ■

Obs. $\sigma(G) \ge n/2 + \lceil k/2 \rceil$ and $|E_{odd}(Q) \cap F| < \lceil k/2 \rceil$
$\Rightarrow$ $\exists$ an unselected full odd edge along P
$\Rightarrow$ $\exists$ spanning cycle through F.

The Remaining Steps

Suffic. Conditions for a spanning path P through F to give spanning cycle through F (when $\sigma(G) \geq n/2 + \tau(F)$).

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- Some full even edge on P is in F_1.

(Fewer than $\lfloor k/2 \rfloor$ selected odd edges.)

The Remaining Steps

Suffic. Conditions for a spanning path P through F to give spanning cycle through F (when $\sigma(G) \geq n/2 + \tau(F)$).

- Some full odd edge on P is unselected (not in F).
- Some full even edge on P is in F_1.

(Fewer than $\lfloor k/2 \rfloor$ selected odd edges.)

Each remaining step: 2-3 pages of alternative paths through F (etc.) to reduce to earlier suffic. conditions.

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- Some full even edge on P is in F_1.
(Fewer than $\lfloor k/2 \rfloor$ selected odd edges.)

Each remaining step: 2-3 pages of alternative paths through F (etc.) to reduce to earlier suffic. conditions.

- Selected edges split half odd, half even along P.

The Remaining Steps

Suffic. Conditions for a spanning path P through F to give spanning cycle through F (when $\sigma(G) \geq n/2 + \tau(F)$).

- Some full odd edge on P is unselected (not in F).
- Some full even edge on P is in F_1.

(Fewer than $\lfloor k/2 \rfloor$ selected odd edges.)

Each remaining step: 2-3 pages of alternative paths through F (etc.) to reduce to earlier suffic. conditions.

- Selected edges split half odd, half even along P.
- Both end edges of P are unselected.

The Remaining Steps

Suffic. Conditions for a spanning path P through F to give spanning cycle through F (when $\sigma(G) \geq n/2 + \tau(F)$).

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- Selected edges split half odd, half even along P.
- Both end edges of P are unselected.
- One end edge of P is unselected.
- Neither end edge of P is unselected.

The Remaining Steps

Suffic. Conditions for a spanning path P through F to give spanning cycle through F (when $\sigma(G) \geq n/2+\tau(F)$).

- Some full odd edge on P is unselected (not in F).

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(Fewer than $\lfloor k/2 \rfloor$ selected odd edges.)

Each remaining step: 2-3 pages of alternative paths through F (etc.) to reduce to earlier suffic. conditions.

- Selected edges split half odd, half even along P.
- Both end edges of P are unselected.
- One end edge of P is unselected.
- Neither end edge of P is unselected.

Any spanning path through F suffices! ■