Spanning Cycles through Specified Edges in Bipartite Graphs

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Abstract

Pósa proved that if G is an n-vertex graph in which any two nonadjacent vertices have degree sum at least $n+k$, then G has a spanning cycle containing any specified family of disjoint paths with a total of k edges. We consider the analogous problem for a bipartite graph G with n vertices and parts of equal size. Let F be a subgraph of G whose components are nontrivial paths. Let k be the number of edges in F, and let t_1 and t_2 be the numbers of components of F having odd and even length, respectively. We prove that G has a spanning cycle containing F if any two nonadjacent vertices in opposite partite sets have degree-sum at least $n/2+\tau(F)$, where $\tau(F)=\lceil k/2\rceil+\epsilon$ (here $\epsilon=1$ if $t_1=0$ or if $(t_1,t_2)\in\{(1,0),(2,0)\}$, and $\epsilon=0$ otherwise). We show also that this threshold on the degree-sum is sharp when $n>3k$.

1 Introduction

In a graph, a cycle through all the vertices is a *spanning cycle* or *Hamiltonian cycle*, and a graph with such a cycle is a *Hamiltonian graph*. The study of sufficient conditions for Hamiltonian cycles is a classical topic in graph theory. Dirac's Theorem [1] states that every n-vertex graph with minimum degree at least $n/2$ is Hamiltonian. Ore [2] strengthened this: it suffices to have $\sigma_2(G)\ge n$, where $\sigma_2(G)=\min\{d(x)+d(y)\colon\ xy\notin E(G)\}$. Further refinements have studied sufficient conditions on degrees for spanning cycles through specified edges (loops and multiple edges are forbidden).

We consider analogues of these results for bipartite graphs. An X,Y*-bigraph* is a bipartite graph with partite sets X and Y. It is *balanced* if $|X|=|Y|$. For an X,Y-bigraph G, let $\sigma(G)=\min\{d(x)+d(y)\colon\ x\in X, y\in Y, xy\notin E(G)\}$. Gould [5] used $\sigma_{1,1}(G)$ for this quantity to distinguish it from $\sigma_2(G)$. Since we study only balanced bipartite graphs in this paper, we use the simplified notation $\sigma(G)$. Always n denotes $|V(G)|$.

The analogue of Ore's Theorem for balanced bipartite graphs was proved by Moon and Moser [4]: $\sigma(G)\ge n/2+1$ implies that G is Hamiltonian. The disjoint union of the complete bipartite graphs $K_{a,a}$ and $K_{n/2-a,n/2-a}$ shows that the result is sharp (see Figure 1(a)).

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Researchers also studied degree thresholds for the existence of spanning cycles through a specified set F of edges, calling a graph F-*Hamiltonian* when such a cycle exists. Of course, F must be a *linear forest*, meaning that every component of F is a path. We require all the paths to be nontrivial (positive length). When F is a perfect matching in a graph G, Häggkvist [6] proved that $\sigma_2(G) \geq n+1$ is sufficient for G to be F-Hamiltonian. Las Vergnas [7] proved the bipartite analogue, showing that $\sigma(G) \geq n/2+2$ suffices when F is a perfect matching. Again the threshold is sharp.

More generally, we seek a spanning cycle through a linear forest with k edges. For general graphs, $\sigma_2(G) \geq n+k$ suffices (Pósa [8]). Faudree, Gould, and Jacobson [10] proved that when F has t components and k edges, with $2 \leq k+t \leq n$, the condition $\sigma_2(G) \geq n+k$ guarantees that G has a cycle of length r containing F for all r such that $2t+k \leq r \leq n$.

We seek the threshold on $\sigma(G)$ to guarantee that G is F-Hamiltonian whenever G is an n-vertex balanced bipartite graph and F is a linear forest in G having k edges. When F is a matching, the requirement on $\sigma(G)$ as a function only of n was studied by Amar, Flandrin, Gancarzewicz, and Wojda [9]. They proved that if $\sigma(G) > 2n/3$, then every matching in G lies in some Hamiltonian cycle, and this threshold on $\sigma(G)$ is sharp. Our problem adds the parameter k, and we seek the sufficiency threshold for $\sigma(G)$ in terms of n and k.

Usually the answer is $\sigma(G) \geq n/2 + \lceil k/2 \rceil$, but the threshold is larger by 1 for some arrangements of k edges. Suppose that the k edges of F form t_1 components of odd length and t_2 components of positive even length. Let

$$\epsilon(t_1,t_2) = \begin{cases} 1 & t_1 = 0 \\ 1 & (t_1,t_2) \in \{(1,0),(2,0)\} \\ 0 & \text{otherwise} \end{cases},$$

and let $\tau(F) = \lceil k/2 \rceil + \epsilon(t_1,t_2)$. Our main result is that if $\sigma(G) \geq n/2 + \tau(F)$, then G is F-Hamiltonian. Furthermore, this threshold on $\sigma(G)$ is sharp when $n > 3k$. Note that when $n = 2k$, the result of Las Vergnas yields $n/2+2$ as the threshold. When $n < 3k$ and F is a matching, the result of Amar et al. [9] yields $2n/3$ as the threshold, but the sharpness example for their result requires $n > 3k$, like ours.

Pósa's result for linear forests in general graphs does not depend on the number of components in the forest. His general result follows easily from the case of matchings. In the bipartite analogue, the general case reduces analogously to the case where each specified path has length 1 or 2. Paths of odd and even lengths behave differently in the bipartite setting because traversing them does or does not switch partite sets.

In Section 2 we present sharpness constructions for all cases with $n > 3k$. In Section 3 we reduce the sufficiency argument to the case where all components of the linear forest have length at most 2, and we outline the steps needed to complete the proof. The remainder of the paper proves the remaining needed structural statement that if $\sigma(G) \geq n/2 + \tau(F)$ and G has a spanning path through F (where paths in F have length at most 2), then G also has a spanning cycle through F.

2 Sharpness Constructions

In this section we introduce needed terminology and provide constructions showing that the results are sharp. We use $V(G)$ and $E(G)$ for the vertex and edge sets of a graph G. Let $G + H$ denote the disjoint union of graphs G and H, let $G[A]$ denote the subgraph of G induced by vertex set A, and let $N(v)$ denote the set of neighbors of v.

We begin with sharpness constructions when all paths in the linear forest F have length 1 or 2. This will be the main case in the sufficiency proof, so we introduce special terminology.

Definition 2.1. A *short forest* is a linear forest whose components have length 1 or 2. When there are t_1 components of length 1 and t_2 of length 2, we also call this a (t_1, t_2)-*short forest*.

We will abuse notation slightly by often viewing F as a specified set of edges rather than a subgraph, but the usage will be clear from context. For example, when P is a path (or a cycle) in G, we say that P *passes through* F if $F \subseteq E(P)$.

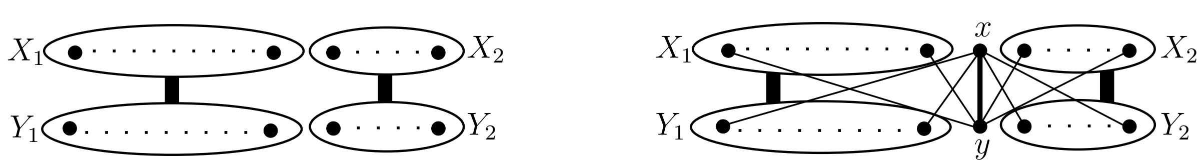
Since k always denotes the number of edges in F, we have $k = t_1 + 2t_2$ when F is a short forest, which includes all cases with $k \le 2$. We first consider the special case $\epsilon(t_1, t_2) = 1$. The construction in Figure 1(a) for $k = 0$ proves sharpness for the Moon–Moser result [4]. Since the graph in Figure 1(b) has a perfect matching containing xy, that construction also proves sharpness of Las Vergnas's result. Note that the short forests for which $\epsilon(t_1, t_2) = 1$ are those with $(t_1, t_2) \in \{(0, t), (1, 0), (2, 0)\}$, where t is any nonnegative integer.

Lemma 2.2. Let n be even and greater than $2(t_1 + 2t_2 + 1)$. If $\epsilon(t_1, t_2) = 1$, then there is an n-vertex balanced X, Y-bigraph G and a (t_1, t_2)-short forest F in G such that $\sigma(G) = n/2 + \tau(F) - 1$ and G has no spanning cycle through F.

Proof. Since $\epsilon(t_1, t_2) = 1$ and $k = t_1 + 2t_2$, we have $\tau(F) - 1 = \lceil t_1/2 \rceil + t_2$.

For $t_1 = t_2 = 0$, the graph G in Figure 1(a) is $K_{a,a} + K_{n/2-a,n/2-a}$. It is disconnected and hence has no spanning cycle, but $\sigma(G) = n/2$.

For $t_2 = 0$ and $t_1 \in \{1, 2\}$, where $\tau(F) - 1 = 1$, we construct G in Figure 1(b) from $K_{a-1,a-1} + K_{n/2-a,n/2-a}$ by adding x to X and y to Y with $N(x) = Y \cup \{y\}$, $N(y) = X \cup \{x\}$, and $xy \in F$. Although $\sigma(G) = n/2 + 1$, there is no spanning cycle through xy. If $t_1 = 2$, then F has another edge not incident to x or y, but still there is no cycle through xy.



(a) construction for $k = 0$. (b) construction for $k \in \{1, 2\}$.

Figure 1: Sharpness constructions for (t_1, t_2)-short forests with $t_2 = 0$ and $t_1 \le 2$.

The remaining case is $t_1 = 0$ and $t_2 > 0$. Let G have partite sets $X_1 \cup X_0 \cup X_2$ and $Y_1 \cup Y_0 \cup Y_2$, with $|X_1| = |Y_1| = |X_2| = |Y_2| = m$ and $|X_0| = |Y_0| = t_2$, where $m \ge t_2$. Let

$E(G)$ consist of all edges joining the partite sets except those from X_1 to Y_2 and from X_2 to Y_1; see Figure 2. Let F consist of a perfect matching in $G[X_0 \cup Y_0]$ plus a matching of size t_2 in $G[X_2 \cup Y_0]$; note that F is $(0, t_2)$-short. If G has a spanning cycle C through F, then deleting $V(F)$ cuts C into t_2 paths. Since covering $G - V(F)$ requires at least $t_2 + 1$ paths (one for $G[X_1 \cup Y_1]$ and at least t_2 for $G[X_2 \cup Y_2]$), no such cycle exists. □

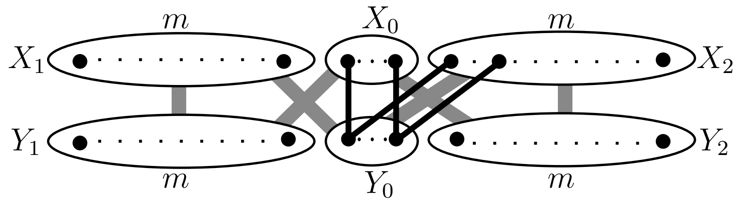


Figure 2: Sharpness when $t_1 = 0$.

The pairs (t_1, t_2) with $\epsilon(t_1, t_2) = 0$ are those such that $t_1 \geq 3$ or $t_1 t_2 > 0$. The next construction differs from those above because $|X_1| \neq |Y_1|$. Note that $n > 3k$ is required.

Lemma 2.3. Fix t_1 and t_2 with $\epsilon(t_1, t_2) = 0$ and let $k = t_1 + 2t_2$. For $n \in \mathbb{N}$ with $n \geq 2\lceil \frac{k+1}{2} \rceil + 2k$ and $n \equiv 2\lceil \frac{k}{2} \rceil - 2 \pmod 4$, there is an n-vertex balanced bipartite graph G and a (t_1, t_2)-short forest F in G such that $\sigma(G) = n/2 + \tau(F) - 1$ and G has no spanning cycle through F.

Proof. Since $\epsilon(t_1, t_2) = 0$, we have $\tau(F) = \lceil k/2 \rceil$. Fix $m \in \mathbb{N}$ with $m \geq \lfloor t_1/2 \rfloor + t_2 + 1$. Let G have partite sets $X_0 \cup X_1 \cup X_2$ and $Y_0 \cup Y_1 \cup Y_2$ with $|X_0| = |Y_0| = t_1 + t_2$, $|X_2| = |Y_1| = m - \lfloor t_1/2 \rfloor - 1$, and $|X_1| = |Y_2| = m$. Let $E(G)$ consist of all edges joining the partite sets except those from X_1 to Y_2 and from X_2 to Y_1; see Figure 3. Let F consist of a perfect matching in $G[X_0 \cup Y_0]$ plus a matching of size t_2 in $G[X_2 \cup Y_0]$; note that F is (t_1, t_2)-short. For $x \in X_1$ and $y \in Y_2$, we have $d_G(x) + d_G(y) = 2(m - \lfloor t_1/2 \rfloor - 1 + t_1 + t_2) = n/2 + \lceil k/2 \rceil - 1$, and such a pair has the smallest degree-sum. The construction exists for $m \geq \lfloor t_1/2 \rfloor + t_2 + 1$, yielding all values of n specified in the hypothesis.

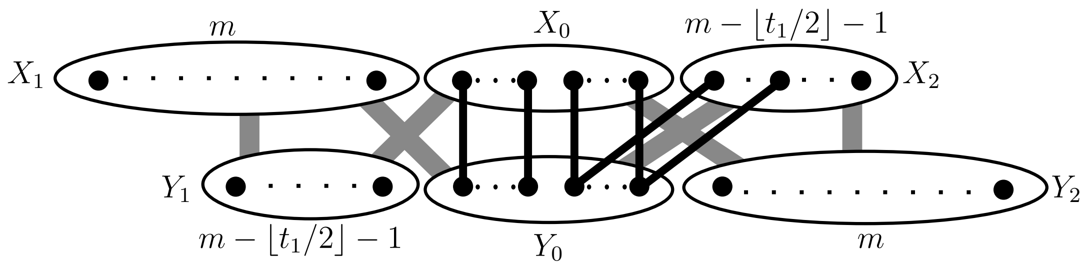


Figure 3: Sharpness when $\epsilon(t_1, t_2) = 0$.

Assume a spanning cycle C through F. Since F consists of $t_1 + t_2$ paths, deleting $V(F)$ cuts C into at most $t_1 + t_2$ paths. Since $|X_1| - |Y_1| = \lfloor t_1/2 \rfloor + 1$, covering $G[X_1 \cup Y_1]$ needs at least $\lfloor t_1/2 \rfloor + 1$ paths; similarly, covering $G[(X_2 - V(F)) \cup Y_2]$ needs at least $\lfloor t_1/2 \rfloor + t_2 + 1$ paths. Since covering $G - V(F)$ needs more than $t_1 + t_2$ paths, no such cycle exists. □

Lemmas 2.2 and 2.3 provide sharpness constructions whenever $k = t_1 + 2t_2$. From the sharpness constructions for (t_1, t_2)-short forests, we obtain sharpness for linear forests with longer paths.

Lemma 2.4. Let F be a k-edge linear forest in an n-vertex bipartite graph G with $\sigma(G) = n/2 + \tau(F) - 1$. If G is not F-Hamiltonian, then there is an $(n+2)$-vertex bipartite graph G' containing a $(k+2)$-edge linear forest F' with the same number of components of each parity as F, such that $\sigma(G') = (n+2)/2 + \tau(F') - 1$ and G' is not F'-Hamiltonian.

Proof. Let xy be an edge in F with $x \in X$. Form G' from G by adding two new vertices x' and y' and setting $N(y') = X$ and $N(x') = Y$. Note that $\sigma(G') = \sigma(G) + 2$. Form F' by adding to $F - \{xy\}$ the edges $\{xy', y'x', x'y\}$. This does not change the parity of the length of any path, so $\tau(F') = \tau(F) + 1$. Hence $\sigma(G') = |V(G')|/2 + \tau(F') - 1$.

Any spanning cycle through F' in G' can be converted to a spanning cycle through F in G by replacing the path through x, y', x', y with the edge xy. Thus G' is not F'-Hamiltonian. □

Repeating this construction yields examples for any desired list of path-lengths showing that $\sigma(G) = n/2 + \tau(F) - 1$ is not sufficient, given that such an example exists with the same number of odd and even components when the lengths of the paths are at most 2. We have exhibited such examples when $n > 3k$.

3 Outline of the Sufficiency Proof

Our first step is to reduce proving sufficiency to the case of short forests, by in essence reversing the construction in Lemma 2.4.

Lemma 3.1. Let G be an n-vertex balanced X, Y-bigraph. If $\sigma(G) \geq n/2 + \tau(F)$ guarantees a spanning cycle through F whenever F is a short linear forest in G, then it also suffices without the length restriction.

Proof. Let F consist of k edges forming t_1 paths of odd length and t_2 paths of even (positive) length. When $k = t_1 + 2t_2$, the forest F is short and there is nothing to prove. We proceed by induction on k with t_1 and t_2 fixed. For $k > t_1 + 2t_2$, some path in F has length at least 3; let x, y', x', y be consecutive vertices along it. Form G' from $G - \{x', y'\}$ by adding the edge xy (if not already present). Let F' in G' be the same as F except for replacing the specified path through x, y', x', y with the edge xy.

Since t_1 and t_2 do not change, $\tau(F') = \tau(F) - 1$. Since each vertex of G' loses at most one neighbor in $\{x', y'\}$, we have $\sigma(G') \geq \sigma(G) - 2 = |V(G')|/2 + \tau(F')$. Hence the induction hypothesis yields a spanning cycle C' through F' in G'. Obtain the desired cycle C in G by replacing xy in C' with the path through x, y', x', y. □

Our main task, which takes the bulk of the paper, will be to prove that G is F-Hamiltonian when the following conditions hold: F is a short forest, $\sigma(G) \geq n/2 + \tau(F)$, and G has a spanning path through F. A relatively easy induction on k then completes the sufficiency proof. To clarify the structure of the proof, we present this induction first. The basis step, for $k = 0$, is the Moon-Moser result. We prove it here to make our result self-contained and to motivate some notions that we will use frequently later. When $k = 0$, we only need a spanning cycle. Note also that $\epsilon(t_1, t_2) = 1$ when $k = 0$.

Proposition 3.2. [4] If G is an n-vertex balanced X, Y-bigraph and $\sigma(G) \geq n/2 + 1$, then G has a spanning cycle.

Proof. Adding edges preserves the condition $\sigma(G) \geq n/2 + 1$, so a maximal counterexample has a spanning path P with nonadjacent endpoints x and y. Since $d_G(x) + d_G(y) \geq \sigma(G) \geq n/2 + 1$ and there are $n/2$ odd-indexed edges along P, some odd-indexed edge $x'y'$ contains neighbors of both x and y. Now $(P - x'y') \cup \{xy', x'y\}$ is a spanning cycle (Figure 4). $\square$

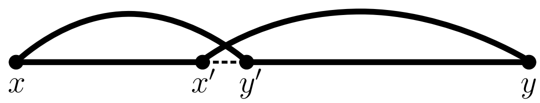


Figure 4: Substituting xy' and $x'y$ for $x'y'$.

The cycle produced in this proof is a concatenation of subpaths of P with adjacent endpoints. To express it in this way, we need appropriate notation for paths and subpaths.

Definition 3.3. An *x, y-path* is a path with endpoints x and y. Given vertices u and v on a path P, we write $P(u, v)$ for the ordered list of vertices along P from u to v. Given a cycle C and an edge uv on C, we write $C(u, v)$ for the list of vertices along the path $C - uv$ from u to v. When L is a list of consecutive vertices on a path, $\langle L \rangle$ denotes the path through the vertices of L in the specified order; this designates only the path, not the subgraph induced by L, which we write as $G[L]$. Analogously, when L is the full list of vertices along a cycle in order, $[L]$ denotes that cycle; the square brackets suggest "closing" the path.

Now we can write the cycle in Figure 4 as $[P(x, x'), P(y, y')]$. Next we present the overall induction argument that uses the structural claim.

Lemma 3.4. If $\sigma(G) \geq n/2 + \tau(F)$ implies that G is F-Hamiltonian whenever F is a short forest and G has a spanning path through F, then $\sigma(G) \geq n/2 + \tau(F)$ implies that G is F-Hamiltonian for every linear forest F in G.

Proof. By Lemma 3.1, we may restrict our attention to short forests. For these we use induction on k, the number of edges. The case $k = 0$ is the Moon–Moser result proved in Proposition 3.2, since $\epsilon(t_1, t_2) = 1$ when $k = 0$.

For $k > 0$, let uv be an edge of F, and let $F' = F - uv$ and $k' = k - 1$. Note that F' is a (t'_1, t'_2)-short forest in G for some t'_1 and t'_2 with $k' = t'_1 + 2t'_2$. Since $\tau(F) = \lceil k/2 \rceil + \epsilon(t_1, t_2)$ and $\tau(F') = \lceil k'/2 \rceil + \epsilon(t'_1, t'_2)$, we have $\tau(F) \geq \tau(F')$ unless k is even, $\epsilon(t_1, t_2) = 0$, and $\epsilon(t'_1, t'_2) = 1$. This requires $t_1 = 2$, and then no choice for t_2 is possible.

We conclude that $\sigma(G) \geq n/2 + \tau(F')$. Now the induction hypothesis implies that G has a spanning cycle C through $F - uv$. If $uv \in E(C)$, then C is a spanning cycle through F, as desired. Otherwise, let u' and u'' be the neighbors of u on C, and let v' and v'' be the neighbors of v on C, with v' and u' on different sides of the chord uv as in Figure 5.

Since paths in F have length at most 2, at most one edge in $\{uu', uu'', vv', vv''\}$ is in F. If such an edge exists, then by symmetry we may assume it is uu''. Let Q be the path $C - uu'$. The path $\langle Q(u', v), Q(u, v')\rangle$, is a spanning path through F in G. Now the structural hypothesis guarantees that G has a spanning cycle through F. □

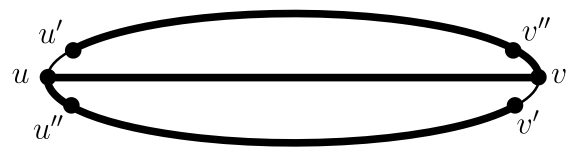


Figure 5: Cycle C.

Thus our task is to prove that the hypothesis in Lemma 3.4 is a true statement. We begin by formalizing two important concepts from Proposition 3.2: parity of edges along a spanning path and having both endpoints of the path as neighbors.

Definition 3.5. Let G be an X, Y-bigraph containing an x, y-path P of odd length. An edge of P is an *odd edge* or *even edge* (with respect to P) when it has *odd position* or *even position* in a listing of the edges in order from one end of P. We write $E_{\mathsf{odd}}(P)$ for the set of all odd edges on P and $E_{\mathsf{even}}(P)$ for the set of all even edges on P. An edge on an x, y-path P is *full* (with respect to P) if one endpoint is adjacent to x and the other is adjacent to y. The edge is *half-full* (with respect to P) if exactly one of these edges exists.

In this language, we generalize the idea used in the Moon-Moser result; we will use this remark frequently. We write $u \leftrightarrow v$ when u and v are adjacent in G; otherwise, $u \not\leftrightarrow v$.

Remark 3.6. Let G be an n-vertex balanced X, Y-bigraph, and let P be a spanning x, y-path in G. Since each endpoint of an edge along P has at most one neighbor in $\{x, y\}$, the pigeonhole principle implies that if $x \not\leftrightarrow y$ and $d_G(x)+d_G(y) \geq n/2+p$, then there are at least p full odd edges and at least $p+1$ full even edges along P. Moreover, if $d_G(x)+d_G(y) = n/2+p$ and there are exactly p full odd edges on P, then all other odd edges on P are half-full.

Having reduced our task to proving the hypothesis of Lemma 3.4, we henceforth adopt the setting of that statement as a uniform restriction on G and F. We will not continue to repeat these hypotheses, so we gather them here as a definition.

Definition 3.7. *The Scenario.* Throughout the rest of the paper, G denotes a fixed n-vertex balanced X, Y-bigraph, F is a short forest in G consisting of k edges, with t_1 single-edge components and t_2 double-edge components, and $\sigma(G) \geq n/2 + \tau(F)$. All uses of x, x', x_i indicate vertices in X, and all uses of y, y', y_i indicate vertices in Y. We call the edges of F the *selected* edges. Let F_1 denote the set of isolated edges in F, and let F_2 denote the set of edges of F in paths of length 2. Always P denotes a given spanning path through F with nonadjacent endpoints $x \in X$ and $y \in Y$; hence $d_G(x) + d_G(y) \geq \sigma(G)$ and the end-edges of P are not full.

Our task, given the scenario of Definition 3.7, is to produce a spanning cycle through F. We show successively that various conditions suffice to ensure such a cycle. We already observed in proving the Moon-Moser result that having an unselected full odd edge suffices. The subsequent sufficient conditions are:

On P there are fewer than $\tau(F)$ selected odd edges (Lemma 3.8).
Some full even edge on P is in F_1 (Lemma 3.9).
Along P, half of the selected edges are odd and half are even (Section 4).
Both end-edges of P are unselected (Section 5).
One end-edge of P is unselected (Section 6).
Both end-edges of P are selected (Section 7).

The last three steps together include all cases for P and hence imply that the specified conditions guarantee a spanning cycle through F. This will complete the proof. We do not start with those cases because their proofs use the earlier, easier cases. The first two conditions are easy to show sufficient, and we close this section with that.

Lemma 3.8. If fewer than $\tau(F)$ odd edges of P are selected, then G is F-Hamiltonian.

Proof. Since $\sigma(G) \geq n/2 + \tau(F)$, at least $\tau(F)$ odd edges are full. Since fewer than $\tau(F)$ are selected, some full odd edge is unselected, which we have observed is sufficient. □

Lemma 3.9. If F_1 contains a full even edge of P, then G is F-Hamiltonian.

Proof. Let $y'x'$ be such an edge. Consecutive vertices x'', y', x', y'' exist along P. Let $Q = \langle P(x'', x), y', x', P(y, y'')\rangle$ (see Figure 6). Since $y'x' \in F_1$, we have $x''y', x'y'' \notin F$, and hence Q passes through F. We may therefore assume $x'' \not\leftrightarrow y''$, which yields $d_G(x'') + d_G(y'') \geq n/2 + \tau(F)$. Since every edge other than $y'x'$ has different parity on P and Q, one of P and Q has fewer than $\lceil k/2 \rceil$ selected odd edges. Since $\tau(F) \geq \lceil k/2 \rceil$, Lemma 3.8 applies. □

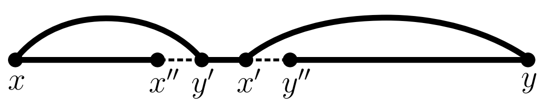


Figure 6: The path Q.

Henceforth, the phrase "Lemma A.B applies" means the hypotheses of that lemma (often Lemma 3.8) have been satisfied and hence its conclusion (always existence of a spanning cycle through F) holds, thereby completing the proof of that case.

4 Paths Splitting F by Parity

Given spanning paths P and Q through F such that every selected edge has opposite parity in P and Q, one of $\{P, Q\}$ has at most $\lfloor k/2 \rfloor$ selected odd edges. Since $\tau(F) \geq \lceil k/2 \rceil$, Lemma 3.8 thus suffices when k is odd (or $\epsilon(t_1, t_2) = 1$) and such P and Q exist. When k is even, this observation is not sufficient, and we need an additional structural lemma.

Definition 4.1. The spanning path P through F *splits* F if $|F \cap E_{\mathsf{odd}}(P)| = |F \cap E_{\mathsf{even}}(P)|$. When $x'y'$ is a full odd edge on P (hence not an end edge), preceded by y'' and followed by x'' on P, we define $P^{x'y'}$ to be the path $\langle P(x'', y), x', y', P(x, y'')\rangle$ (see Figure 7).

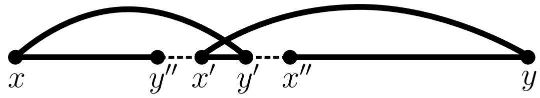


Figure 7: The path $P^{x'y'}$

Every edge in both P and $P^{x'y'}$ has the same parity on both paths, because movement from the "X-end" to the "Y-end" of $P^{x'y'}$ traverses common edges of P and $P^{x'y'}$ in the same direction (contrast this with Figure 6, where all edges except $y'x'$ change parity).

Lemma 4.2. If k is even and P splits F, then G is F-Hamiltonian.

Proof. Suppose G is not F-Hamiltonian. Lemma 3.8 applies unless at least $\tau(F)$ full odd edges are selected. Since P splits F, there are exactly $k/2$ selected odd edges. Hence $\tau(F) = k/2$, which requires $\epsilon(t_1, t_2) = 0$ and hence $t_1 \geq 2$ (t_1 is even when k is even). Every selected odd edge is full, and the other odd edges are half-full. Since $x \not\leftrightarrow y$, the end-edges of P are not full. Thus the end-edges of any path through F that splits F are unselected.

Since every path through F splits F_2 by parity, P also must split F_1. Let $x'y'$ be an odd edge in F_1. Since all selected odd edges are full, $P^{x'y'}$ exists. Since $x'y' \in F_1$, this path also contains F, and it splits F since all edges of F have the same parity in P and $P^{x'y'}$. If the odd edge nearest to $x'y'$ in either direction is selected, then an end-edge of $P^{x'y'}$ is selected. The preceding paragraph forbids this when G is not F-Hamiltonian. Hence we may assume that any two selected odd edges of P incident to a common even edge are both in F_2.

Let $r = t_1/2$. Let x_1y_1 be the odd edge in F_1 closest to x on P; similarly choose x_2y_2 closest to y. These edges are distinct when $r \geq 2$. We consider three cases for r.

Case 1: $r \geq 3$. Let x_3y_3 be a third selected odd edge in F_1. Let $\langle u_1, v_1, x_3, y_3, u_2, v_2\rangle$ be the 6-vertex portion of P centered at x_3y_3 (see Figure 8). Since $x_3y_3 \in F_1$ and successive selected odd edges lie in F_2, none of $u_1v_1, v_1x_3, y_3u_2, u_2v_2$ is selected.

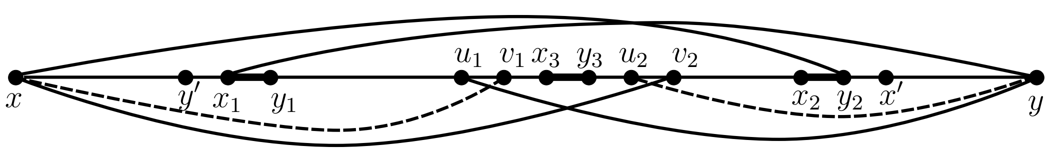


Figure 8: Three selected odd edges.

If $u_2 \leftrightarrow y$, then let $Q = \langle P(x', y), P(u_2, y_2), P(x, y_3)\rangle$, where x' follows y_2 on P. Edges in $E(Q) \cap E(P)$ have the same parity in both paths, so Q splits F but has a selected end-edge. This is forbidden, so $u_2 \not\leftrightarrow y$. By symmetry, $v_1 \not\leftrightarrow x$. Since all unselected odd edges on P are half-full, $v_2 \leftrightarrow x$ and $u_1 \leftrightarrow y$.

Consider paths $\langle P(x, u_1), P(y, v_1)\rangle$ and $\langle P(u_2, x), P(v_2, y)\rangle$ through F. Every edge of F_1 except x_3y_3 has different parity on these paths. Since x_3y_3 is even on both and $|F| = 2r+2t_2$, one of the two has fewer than $r + t_2$ selected odd edges, and Lemma 3.8 applies.

Case 2: $r = 2$. Let $\langle y', x_1, y_1, u_1, v_1\rangle$ and $\langle u_2, v_2, x_2, y_2, x'\rangle$ be the 5-vertex portions of P centered at y_1 and x_2 (see Figure 9). Again $x_1y_1, x_2y_2 \in F_1$ implies that the other edges of these two subpaths are unselected. If $u_1 \leftrightarrow y$, then $\langle P(x', y), P(u_1, y_2), P(x, y_1)\rangle$ has a selected end-edge, so $u_1 \not\leftrightarrow y$. Similarly, $x \not\leftrightarrow v_2$. Now, since unselected odd edges are half-full, $x \leftrightarrow v_1$ and $y \leftrightarrow u_2$. Hence $u_1v_1 \neq u_2v_2$.

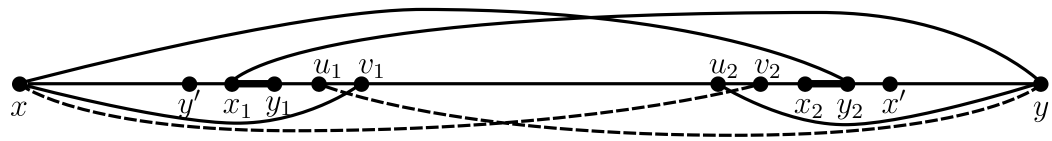


Figure 9: Exactly two selected odd edges.

The two edges of $F_1 - \{x_1y_1, x_2y_2\}$ are both even edges of P. If either lies in $\langle P(v_1, u_2)\rangle$, then $\langle P(u_1, x), P(v_1, u_2), P(y, v_2)\rangle$ has at most $1 + t_2$ selected edges in odd position, and Lemma 3.8 applies. If each even edge of F_1 lies in $\langle P(x, u_1)\rangle$ or $\langle P(v_2, y)\rangle$, then by symmetry we may assume that $\langle P(x, u_1)\rangle$ contains such an edge e. Now the edges y_1x_1, y_2x_2, and e all have even position in $\langle P(u_1, x_1), P(y, v_1), P(x, y')\rangle$. Hence this path through F has at most $1 + t_2$ selected odd edges, and Lemma 3.8 applies.

Case 3: $r = 1$. Let x_1y_1 be the odd edge of F_1, and let e be the even edge. By symmetry in X and Y, we may assume $e \in \langle P(x, x_1)\rangle$. Let $\langle x_0, y_0, x_1, y_1\rangle$ be the 4-vertex portion of P ending with x_1y_1. Since $x_1y_1 \in F_1$, both y_0x_1 and x_0y_0 are unselected.

If $x_0 \leftrightarrow y$, then $\langle P(x, x_0), P(y, y_0)\rangle$ has only t_2 selected odd edges, and Lemma 3.8 applies. Hence $x_0 \not\leftrightarrow y$. Since unselected odd edges are half-full, $x \leftrightarrow y_0$. Since $\epsilon(t_1, t_2) = 0$, we have $t_2 > 0$, and F_2 is nonempty. Let b be the center of a component $\langle P(a, b, c)\rangle$ of F_2.

Compare P with the path obtained from $P^{x_1y_1}$ by interchanging X and Y; in both the end-edges are unselected, F is split, and x_1y_1 is full. Both paths have b and e on the same side of x_1y_1, or both have them on opposite sides. Hence we may assume $b \in Y$ in the first case (Figure 10) and $b \in X$ in the second case (Figure 11). Let d be the vertex before a on P, and let x' be the vertex before y.

In the first case, with $a \in X$, the edge ab is selected and odd, hence full, hence $a \leftrightarrow y$. Now $\langle P(x', x_1), y, P(a, y_0), P(x, d)\rangle$ has only t_2 selected odd edges, and Lemma 3.8 applies.

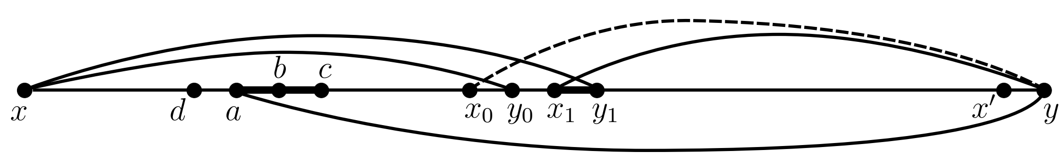


Figure 10: The even edge of F_1 and $\langle d, a, b, c\rangle$ on the same side of x_1y_1.

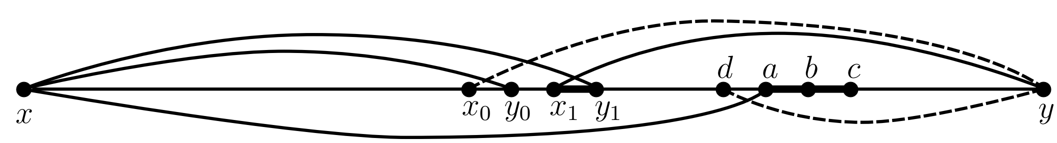


Figure 11: The even edge of F_1 and $\langle d, a, b, c\rangle$ on opposite sides of x_1y_1.

In the second case, if $d \leftrightarrow y$, then $\langle P(a,y), P(d,x)\rangle$ splits F and has a selected end-edge, so we may assume $d \not\leftrightarrow y$. Unselected odd edges are half-full, so $x \leftrightarrow a$. Now $\langle P(d,x_1), P(y,a), P(x,y_0)\rangle$ has only t_2 selected odd edges, and Lemma 3.8 applies. □

5 Paths with Both End-edges Unselected

In this section we complete the proof for the case of a spanning path whose end-edges are unselected. In the previous section our focus was on such a path, with the additional hypothesis that it splits F. Having eliminated that case, we may now assume that the numbers of even and odd selected edges along P differ. The first two lemmas are tools.

Lemma 5.1. If there are at most $\lfloor k/2\rfloor$ selected odd edges along P, then G is F-Hamiltonian.

Proof. Lemma 3.8 applies when fewer than $\lceil k/2\rceil$ odd edges are selected, and Lemma 4.2 applies when equality holds. □

Lemma 5.2. Given that P has unselected full even edges $y_i x_i$ and $y_j x_j$, let Q be the portion of P between them. If the end-edges of P are unselected, and the inequality below holds, then G is F-Hamiltonian.

$$|E_{\mathsf{odd}}(Q)\cap F_1| - |E_{\mathsf{even}}(Q)\cap F_1| \le 2\lfloor t_1/2\rfloor - t_1 + 1$$

Proof. Name the vertices so that $y_i x_i$ is later than $y_j x_j$ along P, so $Q = \langle P(x_j, y_i)\rangle$. With y' and x' neighboring x and y, let $R = \langle P(x', x_i), y, P(x_j, y_i), x, P(y_j, y')\rangle$ (see Figure 12). Edges in both P and R have different parity in R and P, except for those in Q. Thus

$$|E_{\mathsf{odd}}(R)\cap F_1| + |E_{\mathsf{odd}}(P)\cap F_1| = t_1 + |E_{\mathsf{odd}}(Q)\cap F_1| - |E_{\mathsf{even}}(Q)\cap F_1| \le 2\lfloor t_1/2\rfloor + 1.$$

We conclude that P or R has at most $\lfloor t_1/2\rfloor$ edges of F_1 in odd position. Since $y_i x_i$, $y_j x_j$, and the end-edges of P are unselected, R and P both pass through F. One of them has at most $\lfloor t_1/2\rfloor + t_2$ selected odd edges, so Lemma 5.1 applies. □

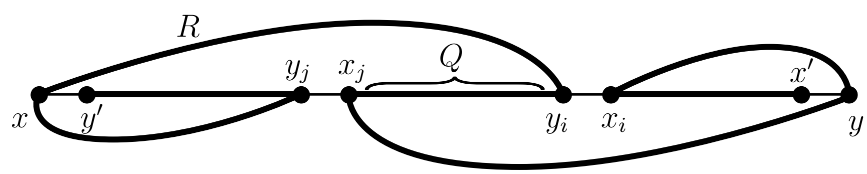


Figure 12: Horizontal path P and modified path R.

Lemma 5.3. If $t_1 \le 2$, or if both end-edges of P are unselected, then G is F-Hamiltonian.

Proof. There are at least $\tau(F)+1$ full even edges along P. Let S be the set of full even edges outside F_2; note that $|S| \ge \lceil t_1/2\rceil + \epsilon(t_1,t_2) + 1$. If any edge of S is in F_1, then G is F-Hamiltonian by Lemma 3.9, so we may assume $S \cap F = \varnothing$. Index S as $y_1x_1,\dots,y_sx_s$ in order along P from x to y. Let $Q_j = \langle P(x_j, y_{j+1})\rangle$ for $1 \le j < s$. Note that always $s \ge 2$.

Case 1: $t_1 \le 2$. Note first that if $t_1 = 0$, then there are exactly $k/2$ selected odd edges along P, and Lemma 5.1 applies. If $t_1 \in \{1, 2\}$, then $\lfloor t_1/2 \rfloor = 2\lfloor t_1/2 \rfloor - t_1 + 1$. If $s \ge 3$, then paths Q_1 and Q_2 exist; one of them contains at most $\lfloor t_1/2 \rfloor$ edges of F_1, so in this case Lemma 5.2 applies. If $t_2 = 0$, then $\epsilon(t_1, t_2) = 1$ and $s \ge 3$, as desired. If $t_2 > 0$, then $\epsilon(t_1, t_2) = 0$, but still $s \ge 3$ if some even edge in F_2 is not full. Hence we may assume that $t_2 > 0$ and that all even edges of F_2 are full.

Since $|F_1| \le 2$, Lemma 5.1 applies unless every edge of F_1 is odd. Since $\tau(F) = 1 + t_2$, there are at least $t_2 + 1$ full odd edges, with at most t_2 in F_2. Since an unselected full odd edge yields a spanning cycle through F, we may assume that some odd edge $\hat{x}\hat{y}$ in F_1 is full.

Since $t_2 > 0$, by symmetry we may assume F_2 has an edge in $\langle P(\hat{y}, y)\rangle$. Let d, a, b, c be four vertices in order along $\langle P(\hat{y}, y)\rangle$ such that $ab, bc \in F_2$ (see Figure 11, with $\hat{x}\hat{y}$ replacing x_1, y_1 in the figure). If $a \notin Y$, then consider $P^{\hat{x}\hat{y}}$ instead of P and interchange X and Y; hence we may assume $a \in Y$. Now ab is a full even edge in F_2, so $x \leftrightarrow a$ (as in Figure 11). The path $\langle P(d, x), P(a, y)\rangle$ has $\hat{y}\hat{x}$ in even position, so it has at most $\lfloor k/2 \rfloor$ selected odd edges, and Lemma 5.1 applies.

Case 2: $t_1 \ge 3$. In this case $\epsilon(t_1, t_2) = 0$. Let y' and x' be the neighbors of x and y on P. For $1 \le j < s$, let $R_j = \langle P(x', x_{j+1}), y, P(x_j, y_{j+1}), x, P(y_j, y')\rangle$; this is just the path R in Figure 12 with $i = j + 1$. If $|E_{\mathsf{odd}}(Q_j) \cap F_1| - |E_{\mathsf{even}}(Q_j) \cap F_1| \le 2\lfloor t_1/2 \rfloor - t_1 + 1$ for some j, then Lemma 5.2 applies. For t_1 even, we may thus assume that $|E_{\mathsf{odd}}(Q_j) \cap F_1| \ge 2$ for all j. Hence $|E_{\mathsf{odd}}(P) \cap F_1| \ge 2s - 2 \ge t_1 = |F_1|$. We conclude that all edges of F_1 have odd position in P, and that every Q_j contains exactly two of them. Hence exactly two members of F_1 (those in Q_j) have odd position in R_j; since $t_1 \ge 4$, Lemma 5.1 applies.

The remaining case is t_1 odd. Let $p = \lceil t_1/2 \rceil$; note that $s > p$. Lemma 5.2 applies unless

$$|E_{\mathsf{odd}}(Q_j) \cap F_1| - |E_{\mathsf{even}}(Q_j) \cap F_1| \ge 1 \qquad \text{for } 1 \le j \le p. \qquad (*)$$

Since $|F_1| = t_1 < 2p$, we have $|E_{\mathsf{odd}}(Q_j) \cap F_1| = 1$ and $|E_{\mathsf{even}}(Q_j) \cap F_1| = 0$ for some j. Since selected edges outside Q_j have opposite parity in P and R_j, for this j we have $|E_{\mathsf{odd}}(R_j) \cap F_1| + |E_{\mathsf{odd}}(P) \cap F_1| = t_1 + 1$. If R_j or P has at most $\lfloor t_1/2 \rfloor$ odd edges in F_1, then Lemma 5.1 applies. Hence each has exactly p, meaning that F_1 has exactly p odd edges and $p - 1$ even edges on P. Now $(*)$ requires $|E_{\mathsf{odd}}(Q_j) \cap F_1| = 1$ and $|E_{\mathsf{even}}(Q_j) \cap F_1| = 0$ for $1 \le j \le p$. Hence each even edge of F_1 is in $\langle P(x, y_1)\rangle$ or $\langle P(x_{p+1}, y)\rangle$, and all odd edges are in $\langle P(x_1, y_{p+1})\rangle$. By symmetry, we may assume that $P(x_{p+1}, y)$ has at most $\lfloor (p-1)/2 \rfloor$ even edges of F_1. Now $\langle P(x', x_2), y, P(x_1, y_2), P(x, y_1)\rangle$ has at most $\lfloor (p-1)/2 \rfloor + 1 + t_2$ selected odd edges (counting one in Q_1). Since $p + 1 \le t_1$ for $t_1 \ge 3$, Lemma 5.1 applies. □

6 Paths with One End-edge Selected

Several types of alternate paths will be useful in this section. We assume throughout this section that on P the initial edge xy' is selected and the final edge $x'y$ is unselected.

Lemma 6.1. If P has a full unselected even edge $\bar{y}\bar{x}$ preceded somewhere by an unselected odd edge $\hat{x}\hat{y}$ whose X-endpoint is adjacent to y, then G is F-Hamiltonian.

Proof. Let $Q = \langle P(x', \bar{x}), y, P(\hat{x}, x), P(\bar{y}, \hat{y})\rangle$ (see Figure 13); note that Q passes through F. Since Q travels backward along P from x', every edge of F has opposite parity on P and Q, so Lemma 5.1 applies. □

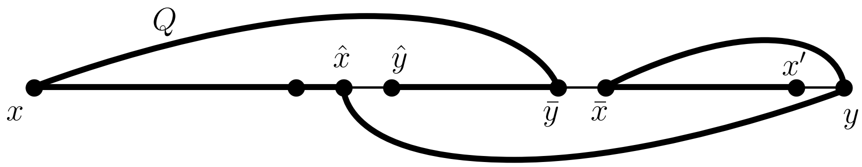


Figure 13: The path Q, toggling parity.

Lemma 6.2. If P has a full unselected even edge $\bar{y}\bar{x}$ followed somewhere by an unselected odd edge $\hat{x}\hat{y}$ whose Y-endpoint is adjacent to x, and $\langle P(\bar{x}, y)\rangle$ contains at least $\lceil t_1/2\rceil$ odd edges of F_1, then G is F-Hamiltonian.

Proof. Let $Q' = \langle P(\hat{x}, \bar{x}), P(y, \hat{y}), P(x, \bar{y})\rangle$ (see Figure 14). All selected edges of $\langle P(\bar{x}, y)\rangle$ appear with opposite parity on P and Q', including at least $\lceil t_1/2\rceil$ edges of F_1 in odd position on P. Hence $|E_{\mathsf{odd}}(Q') \cap F| \leq t_1 - \lceil t_1/2\rceil + t_2 \leq \lfloor k/2\rfloor$, and Lemma 5.1 applies. □

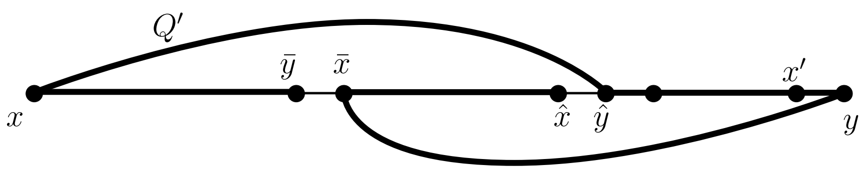


Figure 14: The path Q', mostly toggling parity.

Lemma 6.3. If one end-edge of P is unselected, then G is F-Hamiltonian.

Proof. By Lemma 5.3, we may assume $t_1 \geq 3$ and $\epsilon(t_1, t_2) = 0$. Since $\sigma(G) \geq n/2 + \lceil t_1/2\rceil + t_2$, there are at least $\lceil t_1/2\rceil + t_2 + 1$ full even edges along P.

The components of F_2 are paths of length 2. Let S be the set of full even edges on P that do not lie in F_2 and are not incident to a component of F_2 whose even edge is not full. Index the edges of S as $y_1x_1, \ldots, y_sx_s$ along P from x to y. Since S is obtained by discarding from the set of all full even edges at most one for each component of F_2, we have $s > \lceil t_1/2\rceil \geq 2$. If any edge of S is in F_1, then Lemma 3.9 applies; hence we may assume $S \cap F = \varnothing$.

For $1 \leq j < s$, let y'_j be the neighbor of x_j on P other than y_j. If $x_jy'_j \notin F$ for some such j, then Lemma 6.1 applies, using $x_jy'_j$ as $\hat{x}\hat{y}$ and y_sx_s as $\bar{y}\bar{x}$. Hence we may assume $x_jy'_j \in F$. Introducing the next two vertices, let $y_j, x_j, y'_j, x'_j, y''_j$ be consecutive along P. If $x_jy'_j \in F_2$, then $\langle x_j, y'_j, x'_j\rangle$ is a component of F_2, since $y_jx_j \notin F$. Since $y_jx_j \in S$, the next even edge $y'_jx'_j$ must also be full (by the definition of S). We conclude that $x'_j \leftrightarrow y$ and $x'_jy''_j \notin F$. Again Lemma 6.1 applies, with $x'_jy''_j$ as $\hat{x}\hat{y}$ and y_sx_s as $\bar{y}\bar{x}$.

Therefore, we may assume $x_j y'_j \in F_1$ for $1 \le j < s$. For such j, let x^-_{j+1} be the vertex before y_{j+1} on P. If $x^-_{j+1}y_{j+1} \notin F$, then Lemma 6.2 applies with $x^-_{j+1}y_{j+1}$ as $\hat{x}\hat{y}$ and y_1x_1 as $\bar{y}\bar{x}$, since $s-1 \ge \lceil t_1/2 \rceil$. Hence we may assume $x^-_{j+1}y_{j+1} \in F$. Introducing the two preceding vertices, let $x^{=}_{j+1}, y^-_{j+1}, x^-_{j+1}, y_{j+1}, x_{j+1}$ be consecutive along P. If $x^-_{j+1}y_{j+1} \in F_2$, then $\langle y^-_{j+1}, x^-_{j+1}, y_{j+1}\rangle$ is a component of F_2, since $y_{j+1}x_{j+1} \notin F$. Since $y_{j+1}x_{j+1} \in S$, the preceding even edge $y^-_{j+1}x^-_{j+1}$ must also be full (by the definition of S), and hence $y^-_{j+1} \leftrightarrow x$. Again Lemma 6.2 applies, with $x^{=}_{j+1}y^-_{j+1}$ as $\hat{x}\hat{y}$ and y_1x_1 as $\bar{y}\bar{x}$.

Therefore, we may assume for $1 \le j < s$ that $x^-_{j+1}y_{j+1} \in F_1$, along with our previous conclusion that $x_jy'_j \in F_1$. Let $Q'' = \langle P(x', x_2), y, P(x_1, y_2), P(x, y_1)\rangle$ (see Figure 15). All edges of $\langle P(x_2, x')\rangle$ have opposite parity in P and Q'', including $x_jy'_j$ and $x^-_{j+1}y_{j+1}$ for $2 \le j < s$. If for any such j the edges $x_jy'_j$ and $x^-_{j+1}y_{j+1}$ are not the same, then $|E_{\mathsf{odd}}(Q'') \cap F| \le t_1 + t_2 - (s-1) \le \lfloor k/2 \rfloor$ (again using $s - 1 \ge \lceil t_1/2 \rceil$), and Lemma 5.1 applies.

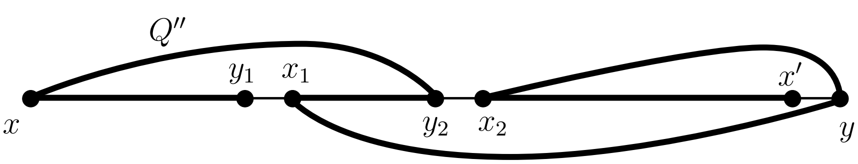


Figure 15: The path Q''.

Hence we may assume that $x^-_2, y_2, x_2, \ldots, x_{s-1}, y_s, x_s$ are consecutive on P, forming edges that alternate between F_1 and S. Let T denote the set of these edges in F_1. (Note $s \ge 3$.)

Now let $R = \langle P(x_s, y), P(x_1, y_s), P(x, y_1)\rangle$ (see Figure 16). Since R passes through F, we may assume its endpoints are non-adjacent, so $d_G(x_s) + d_G(y_1) \ge \sigma(G) \ge n/2 + \lceil k/2 \rceil$, and at least $\lceil k/2 \rceil$ odd edges of R are full. Lemma 3.8 applies unless at least $\lceil k/2 \rceil$ of them are in F. Exactly t_2 are in F_2, so at least $\lceil t_1/2 \rceil$ full odd edges of R are in F_1; call this set D.

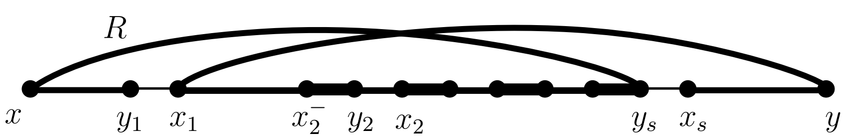


Figure 16: The path R.

All edges of F have the same parity on P and R, so the edges of D are in odd position also on P. We claim that $D \cap T = \varnothing$. If $x^-_j y_j \in D$ for some j with $2 \le j < s$, then fullness on R yields $y_j \leftrightarrow x_s$, and $[P(x_s, y), P(x_j, y_s), P(x, y_j)]$ is a spanning cycle through F. Also, if $x_{s-1}y_s \in D$, then fullness on R yields $x_{s-1} \leftrightarrow y_1$, and $[P(x_{s-1}, y), P(x_1, y_{s-1}), P(x, y_1)]$ is a spanning cycle through F.

Hence $D \cap T = \varnothing$. We have therefore found $\lceil t_1/2 \rceil + s - 1$ edges of F_1 in odd position on P. Since $s > \lceil t_1/2 \rceil$ and $|F_1| = t_1$, we conclude that $F_1 \subseteq E_{\mathsf{odd}}(P)$ and $s = t_1/2 + 1$.

Consider three consecutive vertices x^-_0, y_0, x_0 in $P(x, x_1)$, with y_0x_0 in even position (possibly $y_0x_0 = y_1x_1$). We prove Claim $(*)$: *If y_0x_0 is full and $x^-_0y_0 \notin F$, then G is F-Hamiltonian.* Since we may assume by Lemma 5.3 that the edge of P incident to x

is unselected, we may assume $x_0^- \neq x$. Introducing the two preceding vertices, we have $x_0^=, y_0^-, x_0^-, y_0, x_0$ consecutive on P. If $y_0^- x_0^- \notin F$, then $\langle P(x_0^-, x), P(y_0, y)\rangle$ is a spanning path through F with both end-edges unselected, and Lemma 5.3 applies.

Hence we may assume $y_0^- x_0^- \in F$. Since $F_1 \subseteq E_{\mathsf{odd}}(P)$, we have $y_0^- x_0^- \in F_2$. Since $x_0^- y_0 \notin F$, the component C of F_2 containing $y_0^- x_0^-$ is $\langle x_0^=, y_0^-, x_0^-\rangle$. Since P has at least $t_1/2 + t_2 + 1$ full even edges and $s = t_1/2 + 1$, every component of F_2 has a full even edge or is incident to a full even edge (by the definition of S). If the even edge of C is full, then $x_0^- \leftrightarrow y$, and $[P(x, x_0^-), P(y, y_0)]$ is a spanning cycle through F. Otherwise, $y_0^= x_0^=$ exists before $x_0^=$ and is full. Now $\langle x_0^-, y_0^-, x_0^=, P(y, y_0), P(x, y_0^=)\rangle$ is a spanning path through F; call it R' (see Figure 17). All selected edges after $y_0^=$ on P have opposite parity in P and R', including all $t_1/2$ edges of T. Hence at most $k/2$ selected edges have odd position on R', and Lemma 5.1 applies. This proves Claim $(*)$.

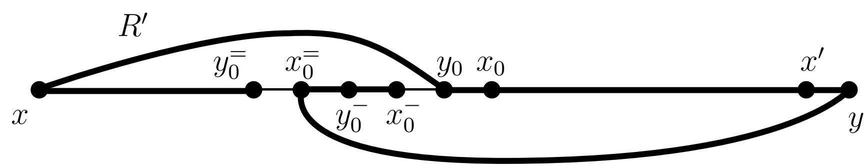


Figure 17: The path R' through F.

Now consider y_1x_1, the first edge of S. Let x_1^- be the other neighbor of y_1. By $(*)$, we may assume $x_1^- y_1 \in F$. If $x_1^- y_1 \in F_2$, then $x_1^- \neq x$, and we have $x_1^=, y_1^-, x_1^-, y_1, x_1$ consecutive on P with $y_1^- x_1^- \in F_2$ and $x_1^= y_1^- \notin F$. Since $y_1x_1 \in S$, we conclude that $x_1^- y_1^-$ is full (by the definition of S). Now $(*)$ applies with $x_1^=, y_1^-, x_1^-$ playing the role of x_0^-, y_0, x_0.

Hence we may assume that $x_1^- y_1 \in F_1$. Since the $t_1/2$ edges of T are in $\langle P(x_1, y_s)\rangle$, we have $x_1^- y_1 \in D$. Hence $x_1^- y_1$ is full with respect to R, so $y_1 \leftrightarrow x_s$, which contradicts the assumption that the endpoints of R are not adjacent. This completes the proof. □

7 Paths with Both End-edges Selected

The final case is when both end-edges of P lie in F. First we prove that this is sufficient under a threshold on n.

Lemma 7.1. If both end-edges of P are selected and $n > 2t_1 + 3t_2$, then G is F-Hamiltonian.

Proof. Since $n > 2t_1 + 3t_2 = |V(F)|$, some vertex of G is not incident to F. By symmetry in X and Y, we may assume it is in X and name it x_1. By Lemma 5.3, we may assume $t_1 \geq 3$ and $\epsilon(t_1, t_2) = 0$. Again let $p = \lceil t_1/2 \rceil$. Since $\sigma(G) \geq n/2 + p + t_2$, at least $p + t_2 + 1$ even edges of P are full. At least $p+1$ are in $E_{\mathsf{even}}(P) - F_2$; let y_0x_0 be one of them. If $y_0x_0 \in F_1$, then Lemma 3.9 applies, so we may assume $y_0x_0 \notin F$.

Since y_0x_0 is full, $[P(x, y_0)]$ and $[P(x_0, y)]$ are disjoint cycles that together cover $V(G)$ and all edges of F. Among these two cycles, let C be the one containing x_1 and C' be the other. Let y_1 and y_2 be the neighbors of x_1 on C; the choice of x_1 yields $x_1y_1, x_1y_2 \notin F$. Let

$P_1 = \langle C(x_1, y_1)\rangle$ and $P_2 = \langle C(x_1, y_2)\rangle$. Let $m = |V(C)|$ and $m' = |V(C')|$, so $m + m' = n$. Let $s = |F_2 \cap E(C)|$ and $s' = |F_2 \cap E(C')|$, so $s + s' = 2t_2$.

If y_1 has a neighbor v on C' such that an edge uv of C' is not in F, then $\langle C(x_1, y_1), C'(v, u)\rangle$ is a spanning path through F with x_1y_2 as an unselected end-edge, and Lemma 6.3 applies. Hence we may assume that both edges on C' incident to any neighbor of y_1 on C' are in F_2. Thus the only neighbors of y_1 in $V(C')$ are centers of components of F_2 contained in C', which yields $d_C(y_1) \geq d_G(y_1) - s'/2$.

Since F is short, we can choose an edge $x'y'$ of C' not in F (see Figure 18). Since $x' \not\leftrightarrow y_1$, we have $d_G(x') + d_G(y_1) \geq n/2 + p + t_2$. Since $d_C(y_1) \geq d_G(y_1) - s'/2$ and $d_C(x') \geq d_G(x') - m'/2$, we have $d_C(x') + d_C(y_1) \geq m/2 + p + s/2$. We conclude that among the $m/2$ edges in odd position on P_1, at least $p + s/2$ have neighbors of both y_1 and x'. Let $x''y''$ be one such edge. If $x''y'' \notin F$, then $\langle P_1(x_1, x''), P_1(y_1, y''), C'(x', y')\rangle$ is a spanning path through F having x_1y_2 as an unselected end-edge, and Lemma 6.3 applies.

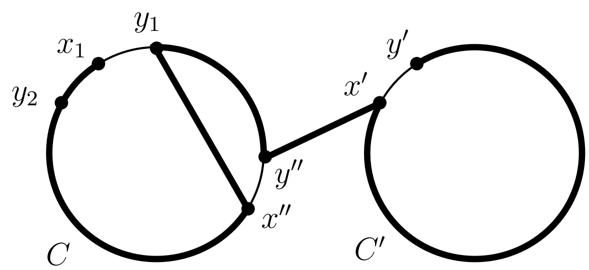


Figure 18: Two cycles.

Hence we may assume that $|E_{\mathsf{odd}}(P_1) \cap F| \geq p + s/2$. By applying these arguments using y_2 and P_2 in place of y_1 and P_1, also $|E_{\mathsf{odd}}(P_2) \cap F| \geq p + s/2$. Edges have opposite parity on P_1 and P_2, so $2p + s \leq |E(C) \cap F| \leq t_1 + s$. Since $p = \lceil t_1/2 \rceil$, equality must hold, and $F_1 \subseteq E(C)$, with half of F_1 in each of $E_{\mathsf{odd}}(P_1)$ and $E_{\mathsf{odd}}(P_2)$. Since P_1 and P_2 move in opposite directions from x_1 on $\langle P(x, y_0)\rangle$, the edges of $E_{\mathsf{odd}}(P_1)$ and $E_{\mathsf{odd}}(P_2)$ appear with opposite parity on the original path P. Therefore P splits F, and Lemma 4.2 applies. □

Lemma 7.2. Under the scenario of Definition 3.7, G is F-Hamiltonian.

Proof. We are left with the case where F is a spanning forest, both end-edges of P are selected, and $t_1 \geq 3$ (hence $\tau(F) = \lceil k/2 \rceil \geq 2$). If $t_2 = 0$, then F is a perfect matching in G; since $\sigma(G) \geq n/2 + 2$, the result of Las Vergnas [7] applies. Hence we may assume $t_2 \geq 1$.

We may name X and Y so that F_2 has a component with center in X; call it $\langle y_1, x_1, y_2\rangle$. Let $G' = G - x_1 - y_1$, $F' = F - \{x_1y_1, x_1y_2\}$, and $n' = n - 2$. Now F' is a short forest in G'. We have $\sigma(G') \geq \sigma(G) - 2$ and $\tau(F') = \tau(F) - 1$; hence $\sigma(G') \geq n'/2 + \tau(F')$.

Since y_2 is not incident to any edge of F', we have $n' > 2t'_1 + 3t'_2$, and Lemma 7.1 yields a spanning cycle C through F' in G'. Let x'_1 and x'_2 be neighbors of y_2 on C, so $x'_1y_2, x'_2y_2 \notin F$. Let $P_1 = C - x'_1y_2$ and $P_2 = C - x'_2y_2$. Let $Q_1 = \langle P_1(x'_1, y_2), x_1, y_1\rangle$ and $Q_2 = \langle P_2(x'_2, y_2), x_1, y_1\rangle$. Now $|E_{\mathsf{odd}}(Q_1) \cap F| + |E_{\mathsf{odd}}(Q_2) \cap F| = |F| = t_1 + 2t_2$, so Lemma 5.1 applies to Q_1 or Q_2. □

Theorem 7.3. Let G be an n-vertex balanced X, Y-bigraph, and let F be a linear forest in G with k edges forming t_1 paths of odd length and t_2 paths of positive even length. If $\sigma(G) \geq n/2 + \tau(F)$, then G has a spanning cycle through F.

Proof. Lemma 7.2 and Lemma 3.4. □

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