Note

Maximum Antichains of Rectangular Arrays

G. W. Peck

Bell Laboratories, Murray Hill, New Jersey, 07974

Introduction

Given a finite partially ordered set $\mathscr{P}$, it is often of interest to describe the largest possible subsets $S \subseteq \mathscr{P}$ which consist of *mutually incomparable* elements of $\mathscr{P}$. Such subsets are usually called *maximum antichains* or *maximum Sperner families* of $\mathscr{P}$ after Sperner [15], who first characterized these maximum antichains in the case that $\mathscr{P}$ is a finite Boolean lattice, i.e., the set of all subsets of some finite set partially ordered by inclusion. Recently, great advances have been made for questions of this type by Canfield, Greene, Harper, Katona, Kleitman, Lubell, Milne and others (e.g., see [1, 5–9, 11, 12]) In particular, Canfield [1–3], has recently settled in the negative the old question of Rota [13, 14]:

> For the set $\mathscr{P}$ of partitions of a finite set X partially ordered by refinement, is it true that the maximum antichain consists of all partitions of X having exactly k nonempty blocks, for a suitable value of k?

In this note we determine the maximum antichains and "semi-antichains" of certain geometrical partially ordered sets which we now describe.

Antichains of n-Dimensional Arrays

For a fixed positive integer n, we denote the n-tuples of positive integers $(a_1, ..., a_n)$, $(b_1, ..., b_n)$,..., by $\bar{a}, \bar{b}$,..., respectively. For a fixed n-tuple $\bar{m}$, let $\mathscr{R}(\bar{m})$ denote the "rectangular parallelepiped" of integer points $\{\bar{x} = (x_1, ..., x_n): 1 \leqslant x_i \leqslant m_i, 1 \leqslant i \leqslant n\}$. Using shorthand notation, we can write $\mathscr{R}(\bar{m}) = \{\bar{x}: \bar{1} \leqslant \bar{x} \leqslant \bar{m}\}$. For $1 \leqslant a \leqslant b \leqslant m$, the subset $R(\bar{a}; \bar{b})$ of $\mathscr{R}(\bar{m})$ is defined to be the set $\{\bar{x}: \bar{a} \leqslant \bar{x} \leqslant \bar{b}\}$. The partially ordered set $\mathscr{P}(\bar{m})$

397

is defined to be the set of all $R(\bar{a}; \bar{b})$ for $\bar{1} \leqslant \bar{a} \leqslant \bar{b} \leqslant \bar{m}$ partially ordered by inclusion.

In order to describe the maximum antichains of $\mathscr{P}(\bar{m})$ we need one more definition. By the *rank* of $R = R(\bar{a}; \bar{b})$, denoted by $\rho(R)$, we mean the integer $\sum_{i=1}^{n} (b_i - a_i)$. The geometric meaning of $\rho(R)$ is clear; it is just the sum of the edge lengths of R.

Let $\mathscr{R}_\rho = \mathscr{R}_\rho(\bar{m})$ denote the set of all $R(\bar{a}; \bar{b})$ in $\mathscr{P}(\bar{m})$ having rank ρ. Clearly the elements of $\mathscr{R}_\rho$ are mutually incomparable. The first result of this note is:

THEOREM 1. *Every maximum antichain of $\mathscr{P}(\bar{m})$ is of the form $\mathscr{R}_\rho$ for a suitable ρ.*

Proof. This result is a consequence of the following remarks.

1. The set $\mathscr{I}(m)$ of intervals $[a, b]$, $1 \leqslant a \leqslant b \leqslant m$, partially ordered by inclusion is a ranked, partially ordered set where the rank of $[a, b]$ is defined to be $b - a$. Since $\mathscr{I}(m)$ has $\rho_r = m - r$ elements of rank r, then the rank cardinalities are logarithmically convex, i.e., $\rho_r^2 \geqslant \rho_{r-1}\rho_{r+1}$. Also, it is easily checked that adjacent ranks enjoy the normalized matching (or LYM) property (see [9]).

2. By Harper's theorem [7], the direct product of $\mathscr{I}(m_1),..., \mathscr{I}(m_n)$, which is just the partially ordered set $\mathscr{P}(\bar{m})$, also has logarithmic convexity and normalized matching.

3. Standard techniques (cf. [9]) now immediately imply that the maximum antichains always consist of elements in those $\mathscr{R}_\rho$ with $|\mathscr{R}_\rho|$ maximum. An easy argument shows that even when there are several maximum-sized ranks in $\mathscr{P}(\bar{m})$, a maximum antichain can contain only elements having the same rank.

This completes the proof of Theorem 1. ∎

We note here that the stronger "Erdös" property (cf. [6]) actually holds for $\mathscr{P}(\bar{m})$, namely, that the size of the largest collection of $R(\bar{a}; \bar{b})$'s having no linearly ordered subset of size $k + 1$ cannot exceed the sum of the sizes of the k largest ranks.

SEMI-ANTICHAINS IN $R(\bar{m})$

Let δ_k denote the point having a single 1 as its kth component and all other components 0. We say that $\bar{x} \in R(\bar{a}; \bar{b})$ is a *corner* of $R(\bar{a}; \bar{b})$ if, for all k, at least one of $\bar{x} \pm \delta_k$ is not in $R(\bar{a}; \bar{b})$. For $\bar{x} \in R(\bar{a}; \bar{b})$, the *antipodal point* of $\bar{x}$ is defined to be $\bar{a} + \bar{b} - \bar{x}$. Finally, let us call a subset X of $\mathscr{R}(\bar{m})$ a *semi-*

antichain if for any $R(\bar{a}; \bar{b})$, $R(\bar{a}'; \bar{b}') \in X$ which have a common corner $\bar{x}$, the corresponding antipodal corners $\bar{a} + \bar{b} - \bar{x}$ and $\bar{a}' + \bar{b}' - \bar{x}$ have no common coordinate values.

An example of a semi-antichain in $\mathscr{R}(\bar{m})$ is $\mathscr{Q}(\bar{m})$, the set of all "cubes" in $\mathscr{R}(\bar{m})$, i.e., all $R(\bar{a}; \bar{b}) \in \mathscr{R}(\bar{m})$ with $\bar{b} - \bar{a}$ having all components equal. The next result shows that this is the (unique) maximum semi-antichain in $\mathscr{R}(\bar{m})$.

THEOREM 2. *The unique maximum semi-antichain in $\mathscr{R}(\bar{m})$ is $\mathscr{Q}(\bar{m})$.*

Proof. Let X be a maximum semi-antichain of $\mathscr{R}(\bar{m})$. Thus, $|X| \geqslant |\mathscr{Q}(\bar{m})|$. For each $\bar{a} \leqslant \bar{m}$ consider the set

$$C(\bar{a}) = \{R(\bar{a}; \bar{b}): \bar{a} \leqslant \bar{b} \leqslant \bar{m}\}.$$

Since X is a semi-antichain then

$$|X \cap C(\bar{a})| \leqslant 1 + \min(\bar{m} - \bar{a}), \tag{1}$$

where for $\bar{x} = (x_1, ..., x_n)$, $\min \bar{x}$ denotes $\min\{x_1, ..., x_n\}$. However, $\mathscr{Q}(\bar{m})$ satisfies (1) (in place of X) with equality for all $\bar{a}$. Thus, X must also satisfy (1) with equality for all $\bar{a} \leqslant \bar{m}$.

For $\bar{a} = \bar{m}$, $C(\bar{a})$ consists of the single set $R(\bar{m}; \bar{m}) = \{\bar{m}\}$. Since

$$|X \cap C(\bar{m})| = 1 \quad \text{then} \quad \{\bar{m}\} \in X.$$

Suppose for some $\bar{a} < \bar{m}$ (i.e., $a_i \leqslant m_i$ for all i and $\bar{a} \neq \bar{m}$), $R(\bar{x}; \bar{y}) \in X$ with $\bar{x} > \bar{a}$ if and only if $R(\bar{x}; \bar{y}) \in \mathscr{Q}(\bar{m})$. Assume that $R(\bar{a}; \bar{b}) \in X$ and $R(\bar{a}; \bar{b}) \notin \mathscr{Q}(\bar{m})$. Thus,

$$\min(\bar{b} - \bar{a}) < \max(\bar{b} - \bar{a}).$$

But $R(\bar{a}; \bar{b})$ clearly contains a cube Q of side $\min(\bar{b} - \bar{a})$ with which it shares a common corner $\bar{x} > \bar{a}$. Moreover, the respective antipodal corners share a common coordinate, namely, the ith coordinate where $b_i - a_i = \min(\bar{b} - \bar{a})$. This contradicts the assumption that X is a semi-antichain. Hence we conclude that $R(\bar{a}; \bar{b}) \in \mathscr{Q}(\bar{m})$. The proof of the theorem now follows by induction. ∎

CONCLUDING REMARKS

The special case of Theorem 1 with $m_1 = \cdots = m_n = 2$ was previously conjectured by Knuth (private communication) and proved by Chandra and Markowsky [4] and by Metropolis and Rota [10]. The special case of Theorem 2 with $n = 2$ was proved by West and Kleitman [16].

An interesting variation which has not yet been investigated is the characterization of the maximum antichains for the *convex* subsets of $\mathscr{R}(m, n)$, i.e., sets whose enclosing polygon is convex. A natural conjecture is that in this case one should choose all convex subsets containing exactly k points for a suitable value of k.

REFERENCES

1. E. R. CANFIELD, On a problem of Rota, to appear.
2. E. R. CANFIELD, On the location of the maximum Stirling number(s) of the second kind, to appear.
3. E. R. CANFIELD, Application of the Berry–Esséen inequality to combinatorial estimates, to appear.
4. A. K. CHANDRA AND G. MARKOWSKY, IBM Research Report RC6108 (No. 26343), 1976.
5. R. L. GRAHAM AND L. H. HARPER, Some results on matching in bipartite graphs, *SIAM J. Appl. Math.* 17 (1969), 1017–1022.
6. C. GREENE, Sperner families and partitions of a partially ordered set, *in* "Proc. Adv. Study Inst. on Combinatorics, Nijenrode in 1974," pp. 91–106, Math. Centrum, Amsterdam, 1974.
7. L. H. HARPER, The morphology of partially ordered sets, *J. Combinatorial Theory Ser. A* 17 (1974), 44–58.
8. D. J. KLEITMAN, M. EDELBERG, AND D. LUBELL, Maximal sized antichains in partial orders, *Discrete Math.* (1971), 47–53.
9. D. J. KLEITMAN, On an extremal property of antichains in partial orders. The LYM property and some of its implications and applications, "Proc. Adv. Study Inst. on Combinatorics, Nijenrode, in 1974," pp. 77–90, Math. Centrum, Amsterdam, 1974.
10. N. METROPOLIS AND G.-C. ROTA, Combinatorial structure of the faces of the n-cube, *SIAM J. Appl. Math. Ser. B*, in press.
11. S. MILNE, Restricted growth functions and incidence relations of the lattice of partitions of an n-set, *Advances in Math.*, in press.
12. S. MILNE, Rank row matchings of partition lattices, the maximum Stirling number, and Rota's conjecture, to appear.
13. R. MULLIN, On Rota's problem concerning partitions, *Aequationes Math.* 2 (1968), 88–104.
14. G.-C. ROTA, Research problem 2.1, *J. Combinatorial Theory* 2 (1967), 104.
15. E. SPERNER, Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* 27 (1928), 544–548.
16. D. WEST AND D. J. KLEITMAN, Skew chain orders and sets of rectangles, *Discrete Math.* 27 (1979), 99–102.