Interval numbers of powers of block graphs

Mingjang Chen*, Gerard J. Chang†, Douglas B. West‡

August 5, 2002

Abstract

The *interval number* of a graph G is the minimum t such that each vertex of G can be assigned a set that is the union of at most t intervals on the real line so that distinct vertices are adjacent if and only if their corresponding sets intersect. A graph with interval number one is an *interval graph*. We prove that the interval number of the kth power of a block graph is at most $k+1$. We also characterize block graphs whose kth powers are interval graphs. Since trees are block graphs and are their own first powers, these results generalize those of Trotter and Harary that the interval number of a tree is at most two, and a tree is an interval graph if and only if it is a caterpillar.

Keywords. Intersection graph, interval graph, interval number, power, tree, clique, path, block graph.

1 Introduction

The *intersection graph* of a family $\mathcal{F}$ of sets is the graph with a vertex for each set in $\mathcal{F}$ such that distinct vertices are adjacent if and only if their corresponding sets intersect. *Interval graphs* are the intersection graphs of families of intervals on a line. They have applications in many fields, such as biology and computer science.

To generalize the concept of interval graphs, Trotter and Harary [28] introduced interval numbers of graphs. The *interval number* $i(G)$ of a graph G is the least integer t such that G is the intersection graph of a family $\mathcal{F}$ in which each set is the union of

*Center of General Education, National Chiao Tung University, Hsinchu 300, Taiwan, mjchen@cc.nctu.edu.tw.

†Department of Mathematics, National Taiwan University, Taipei 106, Taiwan, gjchang@math.ntu.edu.tw. Research partially supported by the National Science Council under grant NSC85-2121-M-009-024.

‡Department of Mathematics, University of Illinois, Urbana, IL 61801, west@math.uiuc.edu.

at most t intervals. Such a family $\mathcal{F}$ is a *multi-interval representation* or a *t-interval representation* of G. The interval graphs are the graphs with interval number 1. Previous results on interval numbers include upper bounds [1–3, 5, 6, 8–11, 13, 20, 24–29], lower bounds [8, 9, 17, 18], exact values [5, 8, 14, 28], and NP-completeness [30].

Trotter and Harary [28] proved that the interval number of a tree is at most 2 and that a tree is an interval graph if and only if it is a caterpillar (a tree whose edges are incident to a single path). Our paper generalizes these results to powers of block graphs. A *block* of a graph is a maximal subgraph that has no cut-vertices. A graph is a *block graph* if it is the intersection graph of the family of blocks of some graph. The *kth power* of a graph G is the graph G^k whose vertex set is $V(G)$ and whose edge set is $\{xy\colon 1 \le d_G(x,y) \le k\}$, where $d_G(x,y)$ denotes the *distance* between x and y, meaning the least length (number of edges) of an x,y-path.

Harary [12] proved that a graph is a block graph if and only if all its blocks are complete graphs; thus trees are block graphs. We prove that the interval number of the kth power of a block graph is at most $k+1$. For the sharpness of this upper bound, we present a tree T with $i(T^2) = 3$. We also characterize block graphs whose kth powers are interval graphs. Our results reduce to those of [28] when $k = 1$ and the given block graph is a tree.

An *interval ordering* of a graph G is an ordering $v_1, \ldots, v_n$ of its vertices such that

$$i < \ell < j \text{ and } v_i v_j \in E(G) \text{ imply } v_\ell v_j \in E(G).$$

It is easy to show that a graph is an interval graph if and only if it has an interval ordering. This was observed by Ramalingam and Pandu Rangan [21] (studying domination), by Jacobson, McMorris, and Mulder [15] (studying tolerance intersection graphs), and by Olariu [19]. Many papers study algorithms or theorems for interval graphs by using interval representations directly, often working with the endpoints of the intervals. For us, it is more convenient to use interval orderings.

We sometimes use a more difficult characterization due to Lekkerkerker and Boland [16]:

a graph is an interval graph if and only it is a chordal graph with no asteroidal triple. A *chordal graph* is a graph with no "chordless cycle", meaning a cycle of length at least 4 as an induced subgraph. An *asteroidal triple* is a set of three distinct vertices such that each pair lies on some path containing no neighbor of the third.

2 Upper bounds for powers of block graphs

This section proves that the interval number of the kth power of a block graph G is at most $k+1$. We may assume without loss of generality that G is connected. The proof then consists of two main steps.

(1) Append a path of length $\lceil k/2 \rceil$ to get a supergraph B that is also a block graph. Prove that $i(G^k) \le i(B^k)$ so that proving $i(B^k) \le k+1$ will suffice.

(2) Define $k+1$ edge sets $E(0), E(1), \ldots, E(k)$ whose union is $E(B^k)$, and prove that each $E(j)$ induces an interval graph.

We first discuss step (1). In general, the kth power of an induced subgraph H of G need not be an induced subgraph of G^k, and $i(H^k) \le i(G^k)$ may not hold. Consider the 6-cycle C_6 and the graph W formed by adding a new vertex adjacent to all of $V(C_6)$. The graph C_6^2 has a chordless 4-cycle and hence is not an interval graph, but W^2 is a complete graph and hence is an interval graph.

For some kind of graphs G, we do have $i(H^k) \le i(G^k)$ when H is an induced subgraph of G. A graph G is *distance-hereditary* if $d_C(x,y) = d_G(x,y)$ for every two vertices x and y in each connected induced subgraph C of G.

Proposition 1 *If H is an induced subgraph of a distance-hereditary graph G, then $i(H^k) \le i(G^k)$, where k is a positive integer. This holds when G is a block graph.*

Proof. If x and y are vertices in a component C of H, then $d_C(x,y) = d_G(x,y)$. Hence $xy \in E(C^k)$ if and only if $xy \in E(G^k)$. Thus C^k is an induced subgraph of G^k and $i(C^k) \le i(G^k)$. Consequently, $i(H^k) \le i(G^k)$.

Since every block of a block graph G is a complete subgraph, vertices x and y in the same component are joined by a unique shortest path whose internal vertices are all cut-vertices. Hence x and y lie in the same component of an induced subgraph H if and only if all those cut-vertices also lie in H, which yields $d_H(x,y) = d_G(x,y)$. ■

To discuss distances in block graphs, we introduce an auxiliary graph. The *block-vertex graph* of a graph G is the graph G^* whose vertex set is $V(G) \cup \{A\colon A$ is a block of $G\}$ and whose edge set is $\{xA\colon A$ is a block and x is a vertex in $A\}$. The vertices of degree 1 in the block-vertex graph correspond the vertices in G belonging to only one block, and deleting them yields the more common "block-cutpoint graph" of G.

The block-vertex graph is bipartite, since edges join blocks to vertices. It is also acyclic, since the union of the blocks on a cycle in G^* would form a larger subgraph of G with no cut-vertex. Hence the block-vertex graph of a connected graph is a tree.

We have observed that any two vertices u and v in a connected block graph B are joined by a unique shortest path P and that the internal vertices on P are cut-vertices. Furthermore, successive vertices on P lie in a common block of B, and these blocks are distinct. Including these blocks yields the unique u,v-path in B^*. Therefore, $d_{B^*}(u,v) = 2d_B(u,v)$.

Let T be a tree rooted at r. For any vertex v in T, the vertices on the unique r,v-path in T (including v) are the *ancestors* of v; and v is a *descendant* of its ancestors. The common ancestors of any two vertices u and v form a path $r,\ldots,w$, and w is their *least common ancestor*. The ancestor adjacent to a vertex is its *parent*, and the descendants adjacent to a vertex are its *children*.

We are now ready to consider the main result of this section.

Theorem 2 *If G is a block graph and k is a positive integer, then $i(G^k) \le k+1$.*

Proof. We may assume that G is connected. Let B be the block graph obtained from G by identifying a vertex of G with one end of a path P of length $\lceil k/2 \rceil$. Let r denote the other end of P. Since G is an induced subgraph of B, Proposition 1 yields

$i(G^k) \le i(B^k)$, and it suffices to prove that $i(B^k) \le k+1$. Our plan is to express B^k as the union of $k+1$ subgraphs that are interval graphs. We begin by associating a subgraph G_x of B^k with each vertex x of B and showing that these graphs are interval graphs.

View the block-vertex graph B^* of B as a tree T rooted at r. (See Figure 1 for an example of B and T and the analysis of such a graph G_x as presented next.)

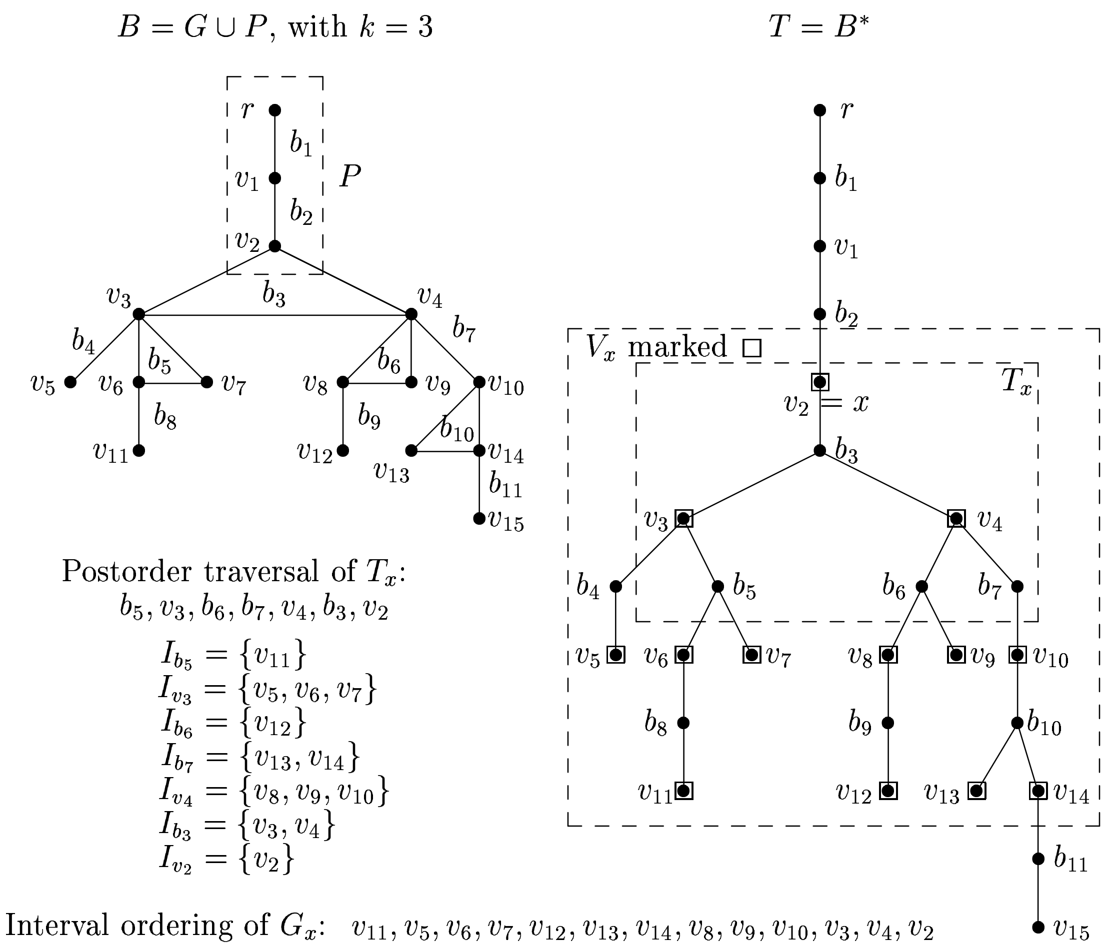


Figure 1: An example for Theorem 2, with $x = v_2$ and $k = 3$.

Recall that $d_T(u,v) = 2d_B(u,v)$ whenever $u, v \in V(B)$. Let $D(u)$ denote the set of descendants of u in T; this generally contains vertices of B and blocks of B. For $x \in V(B)$, let $V_x = D(x) \cap N_{B^k}(x)$, where $N_G(x)$ denotes the set of neighbors of x in G. Thus V_x consists of the descendants of x in T that are vertices of B within distance k of x in B; these are marked with squares in Figure 1.

For $v \in V_x$, let $h_x(v)$ be the vertex on the x,v-path in T that is halfway from x to v; this exists since $d_T(x,v)$ is even. Let G_x be the graph whose vertex set is V_x and whose edge set E_x is $\{uv\colon$ one of $\{h_x(u), h_x(v)\}$ is a descendant of the other in $T\}$. If $h_x(u)$ is a descendant of $h_x(v)$, for example, then $d_T(v, h_x(v)) = d_T(h_x(v), x)$ yields $d_T(u,v) \le d_T(u,x) \le 2k$. Hence $G_x \subseteq B^k$.

We claim that G_x is an interval graph. Let T_x be the subtree of T induced by $\{h_x(v)\colon v \in V_x\}$. Perform a *postorder traversal* of T_x; this visits the vertices in the subtree with a given root by recursively visiting the vertices in the subtrees rooted at its children from left to right and then visiting the root last. This yields an ordering σ of $V(T_x)$ such that (PT1) and (PT2) below hold.

(PT1) If $u \in D(v) - \{v\}$, then $\sigma(u) < \sigma(v)$.

(PT2) If $\sigma(u) < \sigma(v) < \sigma(w)$ and $u \in D(w)$, then $v \in D(w)$.

For $y \in V(T_x)$, let $I_y = \{v \in V_x\colon h_x(v) = y\}$; these sets partition V_x. Let σ' be the ordering of V_x obtained from σ by replacing each vertex y in σ with the vertices of I_y in any order. Show that σ' is an interval ordering of G_x will prove that G_x is an interval graph.

Consider $v_1, v_2, v_3 \in V_x$ such that $\sigma'(v_1) < \sigma'(v_2) < \sigma'(v_3)$ and $v_1v_3 \in E_x$. The construction of σ' yields $\sigma(h_x(v_1)) \le \sigma(h_x(v_2)) \le \sigma(h_x(v_3))$. Since $v_1v_3 \in E_x$, the definition of E_x and (PT1) imply that $h_x(v_1) \in D(h_x(v_3))$. Now (PT2) yields $h_x(v_2) \in D(h_x(v_3))$, and hence $v_2v_3 \in E_x$. Thus σ' is an interval ordering of G_x, and G_x is an interval graph.

Let $\mathcal{G} = \{G_x\colon x \in V(B)\}$. We use these graphs to express B^k as the union of $k+1$ interval graphs. If $d_T(r,x) \equiv d_T(r,y) \bmod (2k+2)$, then $V_x \cap V_y = \varnothing$, since V_x contains only descendants of x in T within distance $2k$ of x in T. Therefore, with $H_j = \bigcup\{G_x\colon d_T(r,x) \equiv 2j \bmod (2k+2)\}$, we have that each H_j is an interval graph. Since we have observed that $G_x \subseteq B^k$ for all $x \in V(B)$, it remains only to show that

$B^k \subseteq \bigcup_{j=0}^{k-1} H_j$.

Given an edge $uv \in E(B^k)$, let y be the least common ancestor of u and v in T. Without loss of generality, we may assume that $d_T(y,v) \le d_T(y,u)$. We claim that $d_T(y,v) \le d_T(y,r)$. If $v \in V(P)$, then $y = v$ and the claim holds. Otherwise, y is a descendant of all the vertices in P, and $d_T(y,r) \ge k$. This suffices, since the computation $2d_T(y,v) \le d_T(y,v) + d_T(y,u) = d_T(u,v) = 2d_B(u,v) \le 2k$ yields $d_T(y,v) \le k$.

With $d_T(y,r) \ge d_T(y,v)$, we may let x be the ancestor of y in T such that $d_T(x,y) = d_T(y,v)$. Since $d_T(x,v)$ is even and $v \in V(B)$, also $x \in V(B)$. Since $d_T(x,v) \le d_T(x,u) = d_T(u,v) = 2d_B(u,v) \le 2k$, we have $u,v \in V_x$. Also, $h_x(u) \in D(y) = D(h_x(v))$. Thus $uv \in E_x$. This yields $B^k \subseteq \bigcup H_j$, completing the proof. ■

The following theorem shows that Theorem 2 is sharp for $k = 2$. We have not been able to show that the theorem is sharp for larger k. Indeed, it remains unknown whether there exists any power of any block graph whose interval number exceeds 3. Some of the arguments in the proof hold for more general k.

Theorem 3 $i(T^2) = 3$, *where T is the rooted tree in which the root r has distance 4 from the leaves and every non-leaf vertex has nine children.*

Proof Let L_t be the set of 9^t vertices at distance t from r in T. Suppose that $i(T^2) \le 2$, and let f be a 2-interval representation of T^2, with $f(x)$ being the set of points in the intervals assigned to x. For $S \subseteq V(T^2)$, let $f(S) = \bigcup_{v \in S} f(v)$. Let $D(x)$ denote the set of descendants of x, as before.

We observe first that

$$\text{If } x \in L_1 \cup L_2, \text{ then } f(x) \not\subseteq \bigcup_{y \in V(T)-D(x)} f(y). \qquad (*)$$

If the containment holds, then choose $z \in D(x) \cap L_{d(x,r)+2}$. Now $d_T(z,x) = 2$ requires $f(z) \cap f(x) \neq \varnothing$, but $d(z,y) > 2$ requires $f(z) \cap f(y) = \varnothing$ for all $y \in V(T) - D(x)$.

Next we define a special set $S \subseteq L_2$. Note that $f(v) \cap f(r) \neq \varnothing$ when $v \in L_2$. For each maximal interval I in $f(r)$, choose a vertex $v \in L_2$ such that $f(v)$ contains the leftmost point of $I \cap f(L_2)$ and a vertex $v' \in L_2$ such that $f(v')$ contains the rightmost point of $I \cap f(L_2)$ (if $I \cap f(L_2) \neq \varnothing$). Let S be the set of all vertices so chosen; since $f(r)$ is the union at most two intervals, we have $|S| \leq 4$.

Let L_t^* be the set of vertices in L_t whose least common ancestor with all vertices of S is r. For $u \in L_t^*$ with $1 \leq t \leq 2$, we have $d_T(u, S) > 2$, and hence every maximal interval in $f(u)$ that intersects $f(r)$ is entirely contained in $f(r)$.

We next prove that for $u \in L_1^*$, some portion of $f(u)$ contained in $f(r)$ intersects $f(D(u) \cap L_2)$. Otherwise, the children of u all have intervals outside $f(r)$ that intersect $f(u)$ in addition to their intervals contained entirely in $f(r)$ that do not intersect $f(u)$. Since $k = 2$, there is only one interval I for u outside $f(r)$. Let w and z be the children of u whose intervals intersecting I have leftmost and rightmost endpoints, respectively. Now u has a third child x, and we have shown that $f(x) \subseteq f(r) \cup f(u) \cup f(w) \cup f(z)$. This contradicts $(*)$ for $x \in L_2$.

For each $u \in L_1^*$, we now have one child $u' \in L_2^*$ for which there is an interval intersecting $f(u)$ inside $f(r)$. These intervals generated by distinct $u \in L_1^*$ are pairwise disjoint, since children of distinct elements of L_1^* have distance 4 in T. Since $|S| \leq 4$, we have $|L_1^*| \geq 5$. We index these vertices as $u_1, \ldots, u_5$ in the left-to-right order of the intervals for $u'_1, \ldots, u'_5$ in $f(r)$. Since u_t and u'_s are nonadjacent in T^2 when $t \neq s$, within $f(r)$ we must end $f(u_1)$ before $f(u'_2)$ starts, end $f(u'_2)$ before $f(u_3)$ starts, etc. Hence u_1, u_3, u_5 are assigned pairwise disjoint intervals within $f(r)$.

On the other hand, u_1, u_3, u_5 are pairwise adjacent in T^2. Therefore, their assigned intervals outside $f(r)$ are pairwise intersecting. Let w and z be the vertices among $\{u_1, u_3, u_5\}$ whose intervals among these three are assigned the leftmost and rightmost points (possibly $w = z$), and choose $x \in \{u_1, u_3, u_5\} - \{w, z\}$. Now $f(x) \subseteq f(w) \cup f(z) \cup f(r)$, which contradicts $(*)$ for $x \in L_1$. ■

Note that we did not need the full tree in Theorem 3. It suffices for the root to have nine children, each vertex of L_1 to have three children, and each vertex of $L_2 \cup L_3$ to have one child.

3 Powers of block graphs that are interval graphs

This section characterizes block graphs whose kth powers are interval graphs.

Given a graph G, let G' denote the graph obtained from G by deleting all *simplicial vertices*, which are the vertices whose neighbors form a clique. If G is a block graph, then G' is the graph obtained by deleting all the non-cut vertices, and G' is a block graph. Define $G^{(n)}$ recursively by letting $G^{(0)}$ be G and letting $G^{(n)}$ be $(G^{(n-1)})'$ for $n \geq 1$. Note that G' is analogous to Harary's notion of "derived graph", which is the graph obtained by deleting all leaves. Our notation thus evokes the notation for iterative derivations of functions.

A *clique-path* is a connected block graph in which every cut-vertex is in exactly two blocks and every block contains at most two cut-vertices. A path is simply a clique-path in which every block has two vertices.

Theorem 4 *If B is a connected block graph and k is a positive odd [even] integer, then B^k is an interval graph if and only if $B^{(\lceil k/2 \rceil)}$ is a path [clique-path].*

Proof. Let $m = \lceil k/2 \rceil$.

Necessity. We prove the contrapositive. Let k be odd [even], and suppose that $B^{(m)}$ is not a path [clique-path]. We prove that B^k is not an interval graph. For odd k, since $B^{(m)}$ is not a path, it has an induced subgraph isomorphic to G_1 or G_2 in Figure 2. For even k, since $B^{(m)}$ is not a clique-path, it has an induced subgraph isomorphic to G_2 or G_3 in Figure 2.

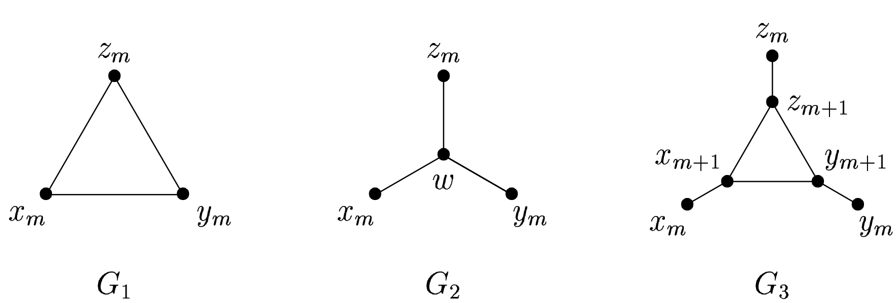


Figure 2. G_1, G_2, and G_3.

Every vertex of $B^{(m)}$ is non-simplicial in $B^{(m-1)}$, since it appears in $B^{(m)}$. Hence x_m is a cut-vertex in $B^{(m-1)}$, and there exists $x_{m-1} \in V(B^{(m-1)})$ whose only neighbor in $V(B^{(m)})$ is x_m. Similarly, for y_m and z_m we obtain y_{m-1} and z_{m-1}. Furthermore, $\{x_{m-1}, y_{m-1}, z_{m-1}\}$ is independent. Repeating the argument $m-1$ more times yields in B an induced subgraph isomorphic to H_1 or H_2 [H_2 or H_3] in Figure 3 when k is odd [even]. The resulting set $\{x_0, y_0, z_0\}$ is an asteroidal triple in B, and hence B is not an interval graph, by the Lekkerkerker–Boland characterization [16].

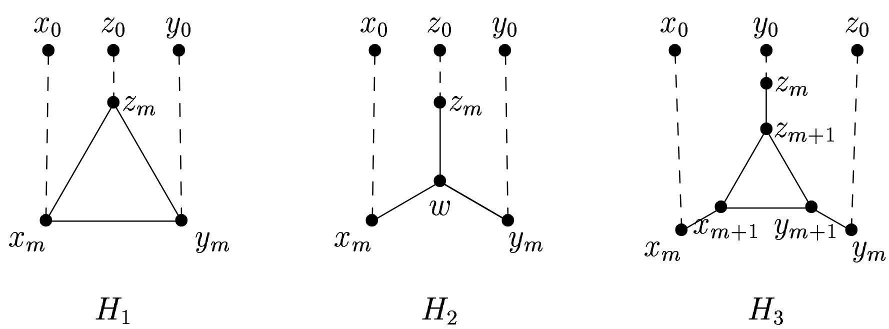


Figure 3. H_1, H_2, and H_3.

Sufficiency. Given that $B^{(m)}$ is a path [clique-path] and k is odd [even], we produce an interval ordering for B^k. Let T be the block-vertex graph B^* of B; recall that $2d_B(u,v) = d_T(u,v)$ for $u, v \in V(B)$. Hence B^k is an induced subgraph of T^{2k}.

Deleting the simplicial (non-cut) vertices from a block graph eliminates peripheral blocks (those with at most one cut-vertex). Thus $(B')^* = (B^*)^{(2)}$. Induction now yields $\left(B^{(m)}\right)^* = T^{(2m)}$.

The block-vertex graph of a clique-path is a caterpillar, and the derived graph of a caterpillar is a path. Hence whether k is even or odd the tree $T^{(k+1)}$ is a path. Let $X = V(T^{(k+1)})$, consisting of vertices $x_0, \dots, x_t$ in order along the path.

View T as a tree rooted at x_0. We construct an ordering σ of $V(T)$ that contains an interval ordering of B^k. Perform a BFS (Breadth-First Search) of T, starting at x_0, extending the ordering at each iteration by adding the children of the oldest unexplored reached node, and listing each $x_i \in V(P)$ as the last vertex reached in level i. Let σ denote the reverse of this ordering, listing $V(T)$ as $v_1, \ldots, v_n$.

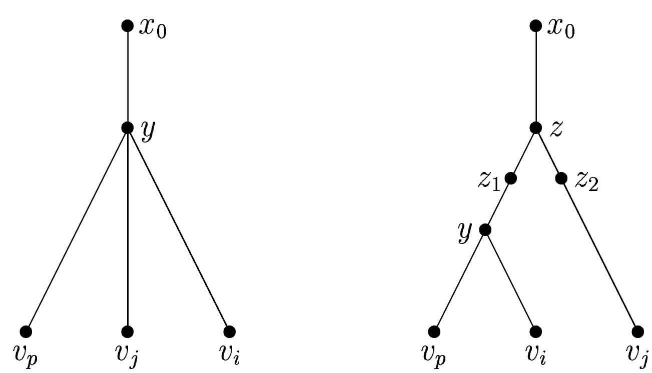
For each $v \in V(T)$, let $s(v)$ be the index j such that the path from x_0 to v in T leaves X at x_j. By its construction, the ordering σ satisfies the following two conditions, where RBFS stands for "Reverse BFS".

(RBFS1) If $d_T(x_0, v_i) > d_T(x_0, v_j)$ or $d_T(x_0, v_i) = d_T(x_0, v_j)$ with $s(v_i) > s(v_j)$, then $i < j$.

(RBFS2) If $i < j < p$ and $d_T(x_0, v_i) = d_T(x_0, v_j) = d_T(x_0, v_p)$ and $s(v_i) = s(v_j) = s(v_p)$, then v_j is a descendant of the least common ancestor of v_i and v_p.

We claim that the order in which vertices of B appear in σ is an interval ordering of B^k. Suppose that $i < j < p$ and $v_i v_p \in E(B^k)$. Let y be the least common ancestor of v_i and v_p. We prove in each case that $v_j v_p \in E(B^k)$.

Case 1. *y is an ancestor of v_j* (see Figure 4(a)). Since $i < j$, (RBFS1) implies that $d_T(x_0, v_j) \le d_T(x_0, v_i)$. Now $d_T(v_j, y) \le d_T(v_i, y)$, and hence $d_T(v_j, v_p) \le d_T(v_j, y) + d_T(y, v_p) \le d_T(v_i, y) + d_T(y, v_p) = d_T(v_i, v_p) = 2d_B(v_i, v_p) \le 2k$, as desired.



(a) (b)

Figure 4. Case 1 and Case 2 for Theorem 4

Case 2. *y is not an ancestor of v_j* (see Figure 4(b)). Let z be the least common ancestor of y and v_j; it is a proper ancestor of y. Let z_1 and z_2 be the children of z that are ancestors of y and v_j, respectively. Since $i < j < p$, we have

$d_T(x_0, v_p) \leq d_T(x_0, v_j) \leq d_T(x_0, v_i)$, and hence $d_T(v_p, z) \leq d_T(v_j, z) \leq d_T(v_i, z)$. To obtain $d_B(v_j, v_p) \leq k$ and $v_j v_p \in E(B^k)$, it suffices to show that $d_T(v_j, z) \leq k$ or $d_T(v_p, z) < d_T(v_j, z) \leq k+1$, since $d_T(v_j, v_p) = 2d_B(v_j, v_p)$. By the construction of X, for each $v \in V(T)$ we have $d_T(v, x_{s(v)}) \leq k+1$.

Subcase 2.1. *X does not contain z.* By the construction of X, $d_T(v_j, z) \leq k$.

Subcase 2.2. *X contains z and z_1 but not z_2.* Here $s(v_j) < s(v_p)$ and $d_T(v_j, z) \leq k+1$. Since $j < p$, (RBFS1) implies that $d_T(v_p, z) < d_T(v_j, z)$.

Subcase 2.3. *X contains z and z_2 but not z_1.* Here $s(v_i) < s(v_j)$ and $d_T(v_i, z) \leq k+1$. Since $i < j$, (RBFS1) implies that $d_T(v_j, z) < d_T(v_i, z)$.

Subcase 2.4. *X contains z but neither z_1 nor z_2.* Here $s(v_i) = s(v_j) = s(v_p)$ and $d_T(v_j, z) \leq d_T(v_i, z) \leq k+1$. Since v_j is not an ancestor of the least common ancestor of v_i and v_p, (RBFS2) implies that $d_T(v_p, z) < d_T(v_j, z)$ or $d_T(v_j, z) < d_T(v_i, z)$. ■

Acknowledgments. We thank the referees for many constructive suggestions.

References

[1] T. Andreae, On an extremal problem concerning the interval number of a graph, *Discrete Applied Math.* **14** (1986) 1–9.

[2] T. Andreae, On the interval number of a triangulated graph, *J. Graph Theory* **14** (1987) 273–280.

[3] J. Balogh and A. Pluhar, A sharp edge bound on the interval number of a graph, *J. Graph Theory* **32** (1999) 153–159.

[4] J. A. Bondy and U. S. R. Murty, *Graph Theory With Applications*, North-Holland, New York (1976).

[5] Y. W. Chang, *Graph Representations Using Stars, Trees, Intervals and Boxes*, Ph.D. Thesis, Dept. of Math., Univ. of Illinois at Urbana-Champaign (1995).

[6] Y. W. Chang, M. S. Jacobson, C. L. Monma, and D. B. West, Subtree and substar intersection numbers, *Discrete Applied Math.* **44** (1993) 205–220.

[7] G. Chartrand and L. Lesniak, *Graph and Digraphs* Wadsworth, Monterey CA, (1981).

[8] M. J. Chen, *Interval Numbers and Related Topics*, Ph.D. Thesis, Dept. of Applied Math., National Chiao Tung Univ., Hsinch, Taiwan (1999).

[9] P. Erdős and D. B. West, A note on interval number of a graph, *Discrete Math.* **55** (1985) 129–133.

[10] J. R. Griggs, Extremal values of the interval number of a graph, II, *Discrete Math.* **28** (1979) 37–47.

[11] J. R. Griggs and D. B. West, Extremal values of the interval of a graph, *SIAM J. Algebraic Discrete Methods* **1** (1980) 1–7.

[12] F. Harary, A characterization of block-graphs, *Canadian Math. Bulletin* **1** (1963) 113–121.

[13] L. B. Hopkins and W. T. Trotter, Jr., A bound on the interval number of a complete multipartite graph, *The Theory and Applications of Graphs*, G. Chartrand et al. (ed.), John Wiley & Sons (1981) 391–407.

[14] L. B. Hopkins, W. T. Trotter, Jr., and D. B. West, The interval number of a complete multipartite graph, *Discrete Applied Math.* **8** (1984) 163–187.

[15] M. S. Jacobson, F. R. McMorris, and H. M. Mulder, An introduction to tolerance intersection graphs, *Graph Theory, Combinatorics, and Applications, Volume 2, Proceedings of the Sixth Quadrennial International Conference on the Theory and Applications of Graphs at Western Michigan University, 1988*, Y. Alavi et al. (ed.) John Wiley & Sons (1991) 705–723.

[16] C. G. Lekkerkerker and J. Ch. Boland, Representation of a finite graph by a set of intervals on the real line, *Fund. Math.* **51** (1962) 45–64.

[17] C. Maas, Some results about the interval number of a graph, *Discrete Applied Math.* **6** (1983) 99–102.

[18] C. Maas, A lower bound for the interval number of a graph, *J. Comput. Applied Math.* **10** (1984) 65–69.

[19] S. Olariu, An optimal greedy heuristic to color interval graphs, *Inform. Proc. Letters* **37** (1991) 65–80.

[20] A. Pluhár, Remarks on the interval number of graphs, *Acta Cybernet.* **12** (1995) 125–129.

[21] G. Ramalingam and C. Pandu Rangan, A unified approach to domination problems on interval graphs, *Inform. Proc. Letters* **27** (1988) 271–274.

[22] E. R. Scheinerman, Irrepresentability by multiple intersection, why the interval number is unbounded, *Discrete Math.* **55** (1985) 195–211.

[23] E. R. Scheinerman, Irredundancy in multiple interval representations, *Discrete Math.* **63** (1987) 101–108.

[24] E. R. Scheinerman, The maximum interval number of graphs with given genus, *J. Graph Theory* **11** (1987) 441–446.

[25] E. R. Scheinerman, On the interval number of a chordal graph, *J. Graph Theory* **12** (1988) 311–316.

[26] E. R. Scheinerman and D. B. West, The interval number of a planar graph: three intervals suffice, *J. Comb. Theory, Series B* **35** (1983) 224–239.

[27] J. Spinrad, G. Vijayan, and D. B. West, An improved edge bound on the interval number of a graph, *J. Graph Theory* **11** (1987) 447–449.

[28] W. T. Trotter, Jr. and F. Harary, On double and multiple interval graphs, *J. Graph Theory* **3** (1979) 205–211.

[29] D. B. West, A short proof of the degree bound for interval number, *Discrete Math.* **73** (1989) 309–310.

[30] D. B. West and D. B. Shmoys, Recognizing graphs with fixed interval number is NP-complete, *Discrete Applied Math.* **8** (1984) 295–305.