The Game Saturation Number of a Graph

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Abstract

Given a family $\mathcal{F}$ and a host graph H, a graph $G \subseteq H$ is *$\mathcal{F}$-saturated relative to H* if no subgraph of G lies in $\mathcal{F}$ but adding any edge from $E(H) - E(G)$ to G creates such a subgraph. In the *$\mathcal{F}$-saturation game* on H, players *Max* and *Min* alternately add edges of H to G, avoiding subgraphs in $\mathcal{F}$, until G becomes $\mathcal{F}$-saturated relative to H. They aim to maximize or minimize the length of the game, respectively; $\mathrm{sat}_g(\mathcal{F}; H)$ denotes the length under optimal play (when Max starts).

Let $\mathcal{O}$ denote the family of odd cycles and $\mathcal{T}_n$ the family of n-vertex trees, and write F for $\mathcal{F}$ when $\mathcal{F} = \{F\}$. Our results include $\mathrm{sat}_g(\mathcal{O}; K_n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$, $\mathrm{sat}_g(\mathcal{T}_n; K_n) = \binom{n-2}{2} + 1$ for $n \geq 6$, $\mathrm{sat}_g(K_{1,3}; K_n) = 2 \left\lfloor \frac{n}{2} \right\rfloor$ for $n \geq 8$, and $\mathrm{sat}_g(P_4; K_n) \in \{\left\lfloor \frac{4n}{5} \right\rfloor, \left\lceil \frac{4n}{5} \right\rceil\}$ for $n \geq 5$. We also determine $\mathrm{sat}_g(P_4; K_{m,n})$; with $m \geq n$, it is n when n is even, m when n is odd and m is even, and $m + \lfloor n/2 \rfloor$ when mn is odd. Finally, we prove the lower bound $\mathrm{sat}_g(C_4; K_{n,n}) \geq \frac{1}{21} n^{13/12} - O(n^{35/36})$. The results are very similar when Min plays first, except for the P_4-saturation game on $K_{m,n}$.

1 Introduction

The archetypal question in extremal graph theory asks for the maximum number of edges in an n-vertex graph that does not contain a specified graph F as a subgraph. The answer is called the *extremal number* of F, denoted $\mathrm{ex}(F; n)$. The celebrated theorem of Turán [22] gives the answer when F is the complete graph K_r and determines the largest n-vertex

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graphs not containing K_r (the *size* of a graph is its number of edges, and by *largest* we mean "of maximum size").

We consider maximal graphs not containing F. The concept extends to a family $\mathcal{F}$ of graphs. A graph G is *$\mathcal{F}$-saturated* if no subgraph of G belongs to $\mathcal{F}$ but $G+e$ contains a member of $\mathcal{F}$ whenever $e \in E(\overline{G})$. The extremal number $\mathrm{ex}(\mathcal{F}; n)$ is the maximum size of an $\mathcal{F}$-saturated n-vertex graph. (In all notation involving families of graphs, we write $\mathcal{F}$ as F when $\mathcal{F}$ consists of a single graph F.)

One may also ask for the minimum size of an $\mathcal{F}$-saturated n-vertex graph; this is the *saturation number* of $\mathcal{F}$, denoted $\mathrm{sat}(\mathcal{F}; n)$. Erdős, Hajnal, and Moon [6] initiated the study of graph saturation by determining $\mathrm{sat}(K_r; n)$.

Generalizing further, a subgraph G of a host graph H is *$\mathcal{F}$-saturated relative to H* if no subgraph of G lies in $\mathcal{F}$ but adding any edge of $E(H) - E(G)$ to G completes a subgraph belonging to $\mathcal{F}$. Although $\mathrm{ex}(\mathcal{F}; n)$ and $\mathrm{sat}(\mathcal{F}; n)$ concern saturation relative to K_n, saturation has also been studied relative to other graphs. For example, Zarankiewicz's Problem, which asks for the largest subgraph of $K_{n,n}$ not containing $K_{t,t}$, involves saturation relative to $K_{n,n}$. When two agents have opposing interests in creating a large or a small $\mathcal{F}$-saturated graph, we obtain the "saturation game".

Definition 1.1. The *$\mathcal{F}$-saturation game* on a host graph H has players *Max* and *Min*. The players jointly construct a subgraph G of H by iteratively adding one edge of H, constrained by G having no subgraph that lies in $\mathcal{F}$. The game ends when G becomes $\mathcal{F}$-saturated relative to H. Max aims to maximize the length of the game; Min aims to minimize it. When both players play optimally, the length of the game is the *game $\mathcal{F}$-saturation number* of H, denoted $\mathrm{sat}_g(\mathcal{F}; H)$ when Max starts the game and by $\mathrm{sat}'_g(\mathcal{F}; H)$ when Min starts it. For clarity and for consistency with the extremal and saturation numbers, we write the values as $\mathrm{sat}_g(\mathcal{F}; n)$ and $\mathrm{sat}'_g(\mathcal{F}; n)$ when playing on K_n.

The saturation game generalizes to any hereditary family of sets. Let D be a family of subsets of a set X such that every subset of a member of D also belongs to D. The saturated subsets are the maximal elements of D. Max and Min alternately add elements of X to a set that always lies in D. The game ends when a saturated set is reached, with Max and Min having the same goals as before. We denote the length of the game under optimal play by $\mathrm{sat}_g(D; X)$ when Max starts and by $\mathrm{sat}'_g(D; X)$ when Min starts. In the $\mathcal{F}$-saturation game on H, we have $X = E(H)$, and avoiding subgraphs in $\mathcal{F}$ defines the hereditary family D.

Patkós and Vizer [21] introduced this general model and studied the case where X is the family of k-element subsets of $\{1, \ldots, n\}$ and D is the set of intersecting families of k-sets. View X as the n-vertex complete k-uniform hypergraph $K_n^{(k)}$. Let M denote the forbidden subgraph consisting of two disjoint edges. The Erdős–Ko–Rado Theorem [7] then states $\mathrm{ex}(M; K_n^{(k)}) = \binom{n-1}{k-1} \sim \frac{1}{(k-1)!} n^{k-1}$. Füredi [12] proved that $\mathrm{sat}(M; K_n^{(k)}) \le \frac{3}{4}k^2$ when a projective plane of order $k/2$ exists. For $k \geq 2$, Patkós and Vizer [21] proved $\Omega(n^{\lfloor k/3 \rfloor - 5}) \leq \mathrm{sat}_g(M; K_n^{(k)}) \leq O(n^{k-\sqrt{k}/2})$.

The saturation game is also related to other well-studied graph games. In a *Maker-Breaker* game, the players Maker and Breaker take turns choosing edges of a host graph H, typically K_n. Maker wins by claiming all of the edges in a subgraph of H having some specified property $\mathcal{P}$, and Breaker wins by preventing this. For example, Hefetz, Krivelevich, Stojaković, and Szabó [16] studied Maker-Breaker games played on K_n in which Maker seeks to build non-planar graphs, non-k-colorable graphs, or K_t-minors. Several papers have considered the minimum number of turns needed for Maker to win (see [10, 17]). In this context, Breaker behaves like Max in the saturation game, making the game last as long as possible. In the saturation game both players contribute edges, but here Maker cannot use the edges taken by Breaker.

The $\mathcal{F}$-saturation game on H is also related to the process in which edges are added at random, which in the literature has been called *the $\mathcal{F}$-free process*; the length of the process is the number of moves to reach a graph that is $\mathcal{F}$-saturated relative to H. Usually $H = K_n$ (see [3, 4, 8, 20]), but [1] is more general. When $\mathcal{F} = \{C_4\}$ and $H = K_{n,n}$, the lower bound of [1] specializes to $\Omega(n^{4/3} \log^{1/3} n)$.

The saturation game on graphs was introduced by Füredi, Reimer, and Seress [14]; they studied $\mathrm{sat}_g(K_3; n)$, calling it "a variant of Hajnal's triangle-free game". In Hajnal's original "triangle-free game", the players aim only to avoid creating triangles, and the loser is the player first forced to create one (Ferrara, Jacobson, and Harris [11] considered the generalization of Hajnal's loser criterion to arbitrary $\mathcal{F}$ and G). Since the F-saturation game always ends with an F-saturated graph, $n - 1 = \mathrm{sat}(K_3; n) \leq \mathrm{sat}_g(K_3; n) \leq \mathrm{ex}(K_3; n) = \lfloor n^2/4 \rfloor$; hence $\mathrm{sat}_g(K_3; n) \in \Omega(n) \cap O(n^2)$. Füredi et al. [14] proved $\mathrm{sat}_g(K_3; n) \in \Omega(n \log n)$. Erdős claimed $\mathrm{sat}_g(K_3; n) \leq n^2/5$, but his proof has been lost; the strongest bound appearing in print is the recent result of Biró, Horn, and Wildstrom [2], who proved $\mathrm{sat}_g(K_3; n) \leq \frac{26}{121} n^2 + o(n^2)$. The correct order of growth remains unknown.

The P_3-saturation game was studied by Cranston, Kinnersley, O, and West [5]; here P_k denotes the k-vertex path. The subgraphs of H that are P_3-saturated relative to H are precisely the maximal matchings in H. Thus the game P_3-saturation number is just the *game matching number*, with $\alpha'_g(G)$ and $\hat{\alpha}'_g(G)$ denoting the values of the Max-start and Min-start games since $\alpha'(G)$ denotes the maximum size of a matching in G. They proved $\alpha'_g(G) \geq \frac{2}{3}\alpha'(G)$ for every graph G (with equality for some split graphs) and $\alpha'_g(G) \geq \frac{3}{4}\alpha'(G)$ when G is a forest (with equality for some trees). They also showed that the minimum of $\alpha'_g(G)$ over n-vertex 3-regular graphs is between $n/3$ and $7n/18$.

We have mentioned bounds on α'_g but not $\hat{\alpha}'_g$ because the two parameters never differ by more than 1 (see [5]). This does not hold for $\mathcal{F}$-saturation in general. For example, when the host graph is obtained from a star with m edges by subdividing one edge, the

Max-start $2K_2$-saturation number is m, but the Min-start $2K_2$-saturation number is 2. As a less artificial example, we will show that $|\operatorname{sat}_g(P_4; K_{m,n}) - \operatorname{sat}'_g(P_4; K_{m,n})|$ can be large, where $K_{m,n}$ is the complete bipartite graph with part-sizes m and n. In most instances that we study, the choice of the starting player matters little.

In Section 2, we study the $\mathcal{F}$-saturation games on K_n for $\mathcal{F} \in \{\mathcal{O}, \mathcal{T}_n, \{K_{1,3}\}, \{P_4\}\}$, where $\mathcal{O}$ is the family of all odd cycles and $\mathcal{T}_n$ is the family of all n-vertex trees. We first prove $\operatorname{sat}_g(\mathcal{O}; n) = \operatorname{sat}'_g(\mathcal{O}; n) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, achieving the trivial upper bound $\operatorname{ex}(\mathcal{O}; n)$. For $n \geq 3$, we prove $\operatorname{sat}_g(\mathcal{T}_n; n) = \operatorname{sat}'_g(\mathcal{T}_n; n) = \binom{n-2}{2} + 1$, except $\operatorname{sat}_g(\mathcal{T}_5; 5) = 6$ and $\operatorname{sat}'_g(\mathcal{T}_4; 4) = 3$; note that $\operatorname{ex}(\mathcal{T}_n; n) = \binom{n-1}{2}$. Hefetz et al. [15] have since studied more general versions of both of these problems. They studied $\operatorname{sat}_g(\mathcal{C}_k; n)$ and $\operatorname{sat}_g(\mathcal{X}_k; n)$ where $\mathcal{C}_k$ is the family of k-connected graphs with n vertices and $\mathcal{X}_k$ is the family of non-k-colorable graphs. In both cases, the value is close to the extremal number. Lee and Riet [19] have generalized the tree problem in a different direction, studying $\operatorname{sat}_g(\mathcal{T}_k; n)$ for $k \leq n$.

Since $\operatorname{sat}(K_{1,3}; n) = n-1$ and $\operatorname{ex}(K_{1,3}; n) = n$, both $\operatorname{sat}_g(K_{1,3}; n)$ and $\operatorname{sat}'_g(K_{1,3}; n)$ always lie in $\{n, n-1\}$. The two values differ when $n \notin \{2, 3, 4, 7\}$, with $\operatorname{sat}_g(K_{1,3}; n)$ being even and $\operatorname{sat}'_g(K_{1,3}; n)$ being odd. That is, $\operatorname{sat}_g(K_{1,3}; n) = 2\lfloor n/2 \rfloor$ and $\operatorname{sat}'_g(K_{1,3}; n) = 2\lceil n/2 \rceil - 1$ when $n \geq 8$. For $n > r > 2$, we have checked by computer that $\operatorname{sat}_g(K_{1,r+1}; n) = \left\lfloor \frac{rn-1}{2} \right\rfloor$ when $n \leq 8$. We ask whether this holds for larger n; note that $\operatorname{ex}(K_{1,r+1}; n) = \left\lfloor \frac{rn}{2} \right\rfloor$. Kászonyi and Tuza [18] proved $\operatorname{sat}(K_{1,r+1}) = \left\lceil \frac{rn}{2} - \frac{(r+1)^2}{8} \right\rceil$ for $n \geq 3r/2$. Lee and Riet [19] proved $\operatorname{sat}_g(K_{1,r+1}; n) \geq \frac{rn}{2} - r + 1$.

For the P_4-saturation game on K_n, the value is not asymptotic to the extremal number. We prove that $\operatorname{sat}_g(P_4; n)$ and $\operatorname{sat}'_g(P_4; n)$ lie in $\{\left\lfloor \frac{4n}{5} \right\rfloor, \left\lceil \frac{4n}{5} \right\rceil\}$ when $n \geq 5$, while $\operatorname{ex}(P_4; n) \in \{n, n-1\}$. Lee and Riet [19] proved $n - 1 \leq \operatorname{sat}_g(P_5; n) \leq n + 2$.

In Section 3, we study the P_4-saturation game on $K_{m,n}$, for $m \geq n \geq 2$. The choice of who starts the game can matter a lot, as do the parities of m and n. The value of $\operatorname{sat}_g(P_4; K_{m,n})$ is n when n is even, m when m is even and n is odd, and $m + \lfloor n/2 \rfloor$ when mn is odd. The value of $\operatorname{sat}'_g(P_4; K_{m,n})$ is m when $n = 2$ and $m + \lfloor n/2 \rfloor - \epsilon$ when $n > 2$, where $\epsilon = 0$ when mn is even and $\epsilon = 1$ when mn is odd.

Note that the difference is $m-2$ when $n = 2$; for larger n the difference is $m - n/2$ when n is even. In fact, when $n = o(m)$, we have $\operatorname{sat}_g(P_4; K_{m,n}) = o(\operatorname{sat}'_g(P_4; K_{m,n}))$ when n is even, but $\operatorname{sat}_g(P_4; K_{m,n}) = (1 - o(1))(\operatorname{sat}'_g(P_4; K_{m,n}))$ when n is odd and m is even. Note also that $\operatorname{sat}(P_4; K_{m,n}) = n$, so when n is even we obtain an example where $\operatorname{sat}_g(\mathcal{F}; H) = \operatorname{sat}(\mathcal{F}; H)$. We ask whether there are other interesting examples where $\operatorname{sat}_g(\mathcal{F}; H)$ or $\operatorname{sat}'_g(\mathcal{F}; H)$ equals $\operatorname{sat}(\mathcal{F}; H)$; [9] provides a survey of saturation numbers as of 2011.

In Section 4, we study the C_4-saturation game on $K_{n,n}$. This game is the natural bipartite analogue of the triangle-saturation game on K_n studied by Füredi, Reimer, and Seress [14]. A subgraph that is C_4-saturated relative to $K_{n,n}$ must be connected, and the spanning tree having adjacent vertices of degree n is C_4-saturated, so $\operatorname{sat}(C_4; K_{n,n}) = 2n - 1$. On the other hand, Füredi [13] proved $\mathrm{ex}(C_4; K_{n,n}) = n^{3/2} + O(n^{4/3})$, so $2n-1 \le \mathrm{sat}_g(C_4; K_{n,n}) \le O(n^{3/2})$. Our main result increases the exponent from the trivial lower bound: $\mathrm{sat}_g(C_4; K_{n,n}) \ge \frac{1}{21}n^{13/12} - O(n^{35/36})$.

Our results suggest many further questions. The most interesting specific question is the order of growth of $\mathrm{sat}_g(C_4; K_{n,n})$. One would also like to understand the conditions under which $\mathrm{sat}_g(\mathcal{F}; n)$, $\mathrm{sat}_g(\mathcal{F}; K_{m,n})$, or $\mathrm{sat}_g(\mathcal{F}; H)$ does not differ much from the value of the corresponding Min-start game.

We mention some general terminology. A graph is *nontrivial* if it has at least one edge. The disjoint union of graphs G and H is denoted $G + H$, and the graph kG is the disjoint union of k copies of G. When we say that a player *creates* F, *makes* F, or *forms* F, we mean that the player selects an edge whose addition to the graph of selected edges produces a copy of F as a component.

2 Saturation games on complete graphs

We begin with saturation games on the complete graph K_n.

Theorem 2.1. $\mathrm{sat}_g(\mathcal{O}; n) = \mathrm{sat}'_g(\mathcal{O}; n) = \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil = \mathrm{ex}(\mathcal{O}; n)$.

Proof. A bipartite graph is *balanced* if the parts of the bipartition have equal size; *nearly balanced* if the sizes differ by 1. Every $\mathcal{O}$-saturated graph is a complete bipartite graph. The largest such n-vertex graph is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. It therefore suffices to give Max a strategy that ensures the following condition after each move by Max:

($*$) Each nontrivial component of the selected graph is balanced, except when n is odd and no isolated vertices remain, in which case all components are balanced except for one nearly balanced component.

Condition ($*$) ensures that when only one component remains, it is balanced or nearly balanced. When $n \ge 4$, Max can ensure ($*$) after his first move by choosing an isolated edge, no matter who moves first, and the forced copy of P_3 when $n = 3$ and Min moves first also satisfies ($*$).

If ($*$) holds after a move by Max, then Min cannot produce a component that fails to be balanced or nearly balanced, and Min can increase the number of nontrivial nearly balanced components only by absorbing an isolated vertex into a balanced component. If Min does that, then there were still isolated vertices in the graph, all components were balanced before Min's move, and Max can restore ($*$) by absorbing another isolated vertex into the nearly balanced component, unless n is odd and no isolated vertices remain.

In the remaining case, ($*$) holds after the move by Min. If Max plays within a component that is not complete bipartite, or connects two nontrivial components, or connects two isolated vertices, then ($*$) again holds. If no such move is available, then there is only one nontrivial component, it is complete bipartite, and there is at most one isolated vertex. If there is an isolated vertex, then the nontrivial component is balanced, and Max connects the isolated vertex to it. Otherwise, the game is over. $\square$

Theorem 2.2. *If $n \geq 3$, then $\mathrm{sat}_g(\mathcal{T}_n; n) = \mathrm{sat}_g'(\mathcal{T}_n; n) = \binom{n-2}{2} + 1$, except that $\mathrm{sat}_g(\mathcal{T}_5; 5) = 6 = \mathrm{ex}(\mathcal{T}_5; 5)$ and $\mathrm{sat}_g'(\mathcal{T}_4; 4) = 3 = \mathrm{ex}(\mathcal{T}_4; 4)$.*

Proof. The value is 1 when $n = 3$, so we may assume $n \geq 4$. Every $\mathcal{T}_n$-saturated subgraph of K_n is $K_r + K_{n-r}$ for some r. The largest is $K_{n-1} + K_1$, with $\binom{n-1}{2}$ edges, and next is $K_{n-2} + K_2$, with $\binom{n-2}{2} + 1$ edges. We show that the final subgraph under optimal play is $K_{n-2} + K_2$ except when $n = 5$ and Max starts or when $n = 4$ and Min starts. Let s and t denote the current numbers of nontrivial components and isolated vertices, respectively.

Max can force the smaller component to have at most two vertices by always leaving $s = 1$ after his move while $t \geq 1$. Min can create a second nontrivial component only by joining two isolated vertices, and Max can then absorb it into the first component. Only when Min connects the last two isolated vertices can a second nontrivial component survive.

Max does better in two cases. When $n = 4$ and Min starts, Max forms $P_3 + K_1$ on his first move. When $n = 5$ and Max starts, the second move by Max forms $P_4 + K_1$. In both cases, the final graph is $K_{n-1} + K_1$, and we may henceforth exclude these cases.

In the remaining cases, it suffices to show that Min can ensure $s = 2$ at the end. Suppose first that some move by Max leaves $s \geq 2$. If $t \geq 2$, then Min connects two isolated vertices to obtain $s \geq 3$, and the move by Max again leaves $s \geq 2$. If $t = 1$, then Min plays an edge incident to the isolated vertex, leaving $s \geq 2$, and guarantees $s = 2$ at the end of the game.

Therefore, to have a chance of reaching $K_{n-1} + K_1$, Max must always leave $s = 1$ after his move. If Min increases s, then Max must connect the new isolated edge to the existing nontrivial component. If at some point t is even and Min is to move, then Min can enforce $s = 2$ at the end by repeatedly making isolated edges, which Max must absorb, until Max cannot absorb the last edge. It thus suffices to show that Min can achieve even t before one of her moves when the first move by Max (with $s = 1$) leaves t odd.

If Max starts, then having t odd requires n odd. Min forms P_3. Max now must enlarge this component to P_4 or $K_{1,3}$ to keep t odd and $s = 1$. There remain three unplayed edges among these four vertices, and Min plays next. Eventually, Max will be forced to absorb an isolated vertex or make an isolated edge, reaching t odd or $s = 2$, respectively.

If Min starts, then Max must form P_3 to keep $s = 1$, so having t odd requires n even. Min next forms K_3, and again Max must leave t odd or $s = 2$ after his next move. $\square$

Theorem 2.3. *For $n \in \mathbb{N}$,*

$$\operatorname{sat}_g(K_{1,3}; n) = \begin{cases} n & \text{when } n \in \{3,7\} \cup 2\mathbb{N} - \{2\} \\ n-1 & \text{otherwise} \end{cases}$$

$$\operatorname{sat}_g'(K_{1,3}; n) = \begin{cases} n-1 & \text{when } n \in \{1\} \cup 2\mathbb{N} - \{4\} \\ n & \text{otherwise} \end{cases}$$

Proof. All $K_{1,3}$-saturated graphs are disjoint unions of cycles plus possibly one isolated vertex or one isolated edge (not both). Hence the only possible outcomes are n (call this *Max wins*) or $n-1$ (call this *Min wins*). Let $X(n)$ and $Y(n)$ denote the Max-start and Min-start $K_{1,3}$-saturation games on K_n, respectively.

For $n \geq 5$, our claim is that the first player wins when n is even and the second player wins when n is odd, except that Max wins $X(7)$. After giving specific strategies for $n \leq 8$, we provide general strategies for $n \geq 9$ that reduce the problem to the case $n \in \{5, 6, 8\}$.

When $n \leq 3$, no copy of $K_{1,3}$ is possible; Min wins when $n \leq 2$ and Max wins when $n = 3$, no matter who starts.

Max wins $X(4)$ by forming P_4 (second round) and $Y(4)$ by forming $2K_2$ (first round).

Min wins $X(5)$ by forming $2K_2$ first and then C_3 or C_4.

Max wins $Y(5)$ by forming $2K_2$ first and then P_5, forcing C_5 as the final graph.

Max wins $X(6)$. If Min first makes P_3, then Max completes the triangle, reducing to $Y(3)$, which Max wins. If Min first makes $2K_2$, then Max forms $3K_2$ and next P_6 to win.

Min wins $Y(6)$ by making P_4 on the second move and then C_4 or C_5.

Max wins $X(7)$. If Min first makes P_3, then Max completes the triangle and wins $Y(4)$. If Min first makes $2K_2$, then Max makes $3K_2$. After Min's move, Max next completes C_3 or C_4, and the remainder of the game completes a cycle on the other vertices.

Max wins $Y(7)$. Max first makes P_3. If Min completes the cycle, then Max wins the remaining $X(4)$. If Min forms P_4, then Max completes the cycle and wins $Y(3)$. If Min forms $P_3 + P_2$, then Max can form $P_3 + P_3$ and force the game to end with $C_3 + C_4$ or C_7.

Max wins $X(8)$. Max makes P_4 on his second move. If Min closes the cycle or plays an isolated edge, then Max forms $C_4 + P_2$, and the remaining game is equivalent to $X(4)$, which Max wins. If Min extends the path to P_5, then Max closes the cycle and wins $Y(3)$.

Min wins $Y(8)$. If after four edges the selected graph contains a triangle, then the remaining game is $X(5)$ in which the first move has been played, which is won by Min. To avoid this, Max must make $2K_2$, and after Min makes $P_3 + P_2$ Max must make $P_3 + 2P_2$, extend one of the paths, or join the paths to form P_5. If $P_3 + 2P_2$, then again Min closes the triangle, and the remainder is a copy of $X(5)$ in which Min has played a winning $2K_2$. In the other cases, Min forms P_6; the final graph is $C_6 + P_2$ or $C_7 + P_1$, with Min winning.

Now assume $n \geq 9$. We denote by Player 1 and Player 2 the player who moves first and the player who moves second, respectively. We begin by showing that Player 1 wins when n

is even. Player 1 always leaves an even number of isolated vertices, some number of cycles, and one nontrivial path, until the number of isolated vertices is 6. This is true after the initial move. If Player 2 extends the path or makes an isolated edge, then Player 1 further extends the path or combines it with the new isolated edge to restore the condition. If Player 2 completes the path to a cycle, then Player 1 starts a new path. In each case, there is still one nontrivial path component, and the number of isolated vertices has decreased by 2.

When six isolated vertices remain, Player 2 is ready to move. If the nontrivial path is a single edge, then the remaining game is a copy of the game on eight vertices in which Player 1 has made the arbitrary first move, and we have proved that Player 1 wins this game. Hence we may assume that the nontrivial path is longer. If Player 2 completes it to a cycle (and Player 1 chooses an isolated edge) or Player 2 chooses an isolated edge and lets Player 1 close the cycle, then the remaining game is equivalent to the game with $n = 6$ in which Player 1 has made the first move; we have shown that Player 1 wins this game. If Player 2 extends the path, then Player 1 closes the cycle to leave a game with $n = 5$ started by Player 2. Since 5-vertex games are won by the player who makes the second move, again Player 1 wins.

Suppose now that n is odd; we claim that Player 2 wins. In the first round, Player 2 makes P_3. The graph now consists of one nontrivial path and some isolated vertices. Moreover, the number of isolated vertices is even and at least 6. Player 2 now wins by following the strategy outlined above for Player 1. $\square$

In the $K_{1,3}$-saturation game on K_n, each player prefers to move first when n is even and second when n is odd. Hefetz, Krivelevich, Naor, and Stojaković [15] showed that the kP_2-saturation game on K_n behaves similarly (where $k \le n/2$). They showed that when k is even, $\mathrm{sat}_g(kP_2; n) \ge n - 1$ and $\mathrm{sat}'_g(kP_2; n) \le \binom{2k-1}{2}$. In contrast, when k is odd, $\mathrm{sat}_g(kP_2; n) \le \binom{2k-1}{2}$ and $\mathrm{sat}'_g(kP_2; n) \ge n - 1$. Thus when n is much bigger than k^2, the identity of the first player makes a dramatic difference in the outcome. Furthermore, whether a player does better by moving first depends on the parity of k; each player prefers to move first if and only if k is even.

Because there are only two possible (consecutive) lengths of the $K_{1,3}$-saturation game on K_n, given the identity of the first player, the outcome is determined by who plays last. Ferrara, Jacobson, and Harris [11] studied that question explicitly; in their game, the player who moves last wins. Although their analysis is similar to ours due to the structure of $K_{1,3}$-saturated graphs, their result is different: in their game, for $n \ge 5$, the first player wins if and only if n is even, except $n = 7$. In particular, under their criterion for winning, the number of moves played will always be $n - 1$ (except $n = 7$).

Our final game on K_n is the P_4-saturation game. In a graph not containing P_4, all components must be stars or triangles. Since Max seeks a large ratio of number of edges to number of vertices, triangles and large stars are beneficial to Max, while small stars are beneficial to Min. However, stars with two edges are dangerous for Min, since they may become triangles. This intuition motivates the strategies for the players. It turns out that optimal play by both players produces components isomorphic to $K_{1,4}$, which explains the form of the answer.

Theorem 2.4. *For $n \geq 5$, both $\mathrm{sat}_g(P_4; n)$ and $\mathrm{sat}'_g(P_4; n)$ lie in $\{\lfloor 4n/5 \rfloor, \lceil 4n/5 \rceil\}$.*

Proof. Every P_4-saturated subgraph of K_5 contains exactly four edges, so we may suppose $n \geq 6$. As noted above, all components of the graph being constructed must be stars or triangles. Let the *value* ρ of the current position count a contribution for each component: 0 for an isolated vertex or triangle, $\frac{1}{2}$ for P_2 or P_3, and 1 for a larger star. The value can decrease only by a move that turns a copy of P_3 into a triangle. In this proof every mention of a subgraph refers to a component of the current selected graph. As in the proof of Theorem 2.2, let t denote the current number of isolated vertices during the game. Let m be the final number of edges.

Upper bound: Min strategy. While $t \geq 1$, if a copy of P_3 exists, then Min converts it to $K_{1,3}$. If there is no P_3 and $t \geq 2$, then Min makes P_2. Otherwise, $t = 1$ and there is no copy of P_3, in which case Min adds the isolated vertex to a largest star, or $t = 0$, in which case the only possible move completes a copy of P_3 to a triangle.

With this strategy, each move by Min increases ρ by $\frac{1}{2}$, except when $t \leq 1$. Min never forms a 3-vertex component, except when $t = 1$ and all other components are isolated edges, yielding $K_3 + \frac{n-3}{2}P_2$ with $\frac{n+3}{2}$ edges. Any earlier copy of P_3 formed by Max is converted immediately to $K_{1,3}$. If Max makes P_3 with the last isolated vertex, then Min must complete the triangle. Hence except for one possible triangle, all components are stars. Regardless of whether a triangle appears at the end, the number of stars is $n - m$.

Since the strategy also prevents Max from decreasing ρ (with the exception of the final triangle when Max stupidly makes isolated edges as mentioned above), the final value of ρ is at least $\frac{1}{2}\frac{m-2}{2} - \frac{1}{2}$, which equals $\frac{m-4}{4}$. (When Min absorbs the last isolated vertex, ρ reaches at least $\frac{m}{4}$.) Also, at the end ρ is at most the number of components that are stars. We obtain $\frac{m-4}{4} \leq n - m$, which simplifies to $m \leq \frac{4n+4}{5}$. The same computation yields $m \leq \frac{4n+3}{5}$ in the Min-start game, so in both cases $m \leq \lceil 4n/5 \rceil$.

Lower bound: Max strategy. Max never forms an isolated edge, except on the first turn of the Max-start game or when all nontrivial components are triangles. If there is an isolated edge and $t \geq 1$, then Max turns it into P_3. Otherwise, if some component is P_3, then Max completes the triangle. If none of these moves is available (and the game is not over), then Max enlarges a star with at least three edges.

Note that Max increases ρ only when all nontrivial components are triangles. Except for the first move in the Max-start game, each move by Max that increases ρ is thus preceded by a move by Min that decreases ρ by completing a triangle. With each Min move increasing ρ by at most $\frac{1}{2}$, the upper bound on ρ is $\frac{m+2}{4}$ (or $\frac{m+1}{4}$ in the Min-start game). Moreover, if

Min ever completes a triangle, then ρ is at most $\frac{m-2}{4}$ in the Max-start game or $\frac{m-3}{4}$ in the Max-start game.

Any isolated edge formed by Min is immediately converted to P_3 by Max (provided $t \geq 1$). Max himself forms an isolated edge only when none are already present and all nontrivial components are triangles (including on the first turn of the game). When $t = 2$, Min may create a second copy of P_2 that Max cannot convert to P_3. Thus, the final graph contains at most two copies of P_2. If Min always makes P_2, then Max turns these components into P_3 and they become triangles at the end, yielding at least $n - 2$ edges, which is good for Max since $n - 2 \geq \left\lfloor \frac{4n}{5} \right\rfloor$ when $n \geq 6$. If at some point Min does not make P_2, then Max turns any remaining isolated edge into P_3. Max later creates another isolated edge only if every nontrivial component of the graph is a triangle; in this case, Min's preceding move must have completed a triangle. Hence we may assume that either the final graph has at most one isolated edge, or it has two but at some point Min completed a triangle.

Components that are triangles have no net effect on ρ or on $n - m$. Also, $n - m$ is the number of stars. When the game ends with at most one isolated edge, we thus have $\rho \geq n - m - \frac{1}{2}$ at the end, no matter who starts. Using the weaker upper bound on ρ from the Max-start game, we obtain $\frac{m+2}{4} \geq n - m - \frac{1}{2}$, which simplifies to $m \geq \frac{4n-4}{5}$. When the game ends with two isolated edges and Min having completed a triangle, we have the weaker lower bound $\rho \geq n - m - 1$ but a stronger upper bound on ρ. This leads to $\frac{m-2}{4} \geq n - m - 1$, hence $m \geq \frac{4n-2}{5}$. In either case, $m \geq \lfloor 4n/5 \rfloor$. $\square$

A referee pointed out that a more delicate analysis shows that for $n \geq 8$ the answer is always $\lfloor 4n/5 \rfloor$, except for the Min-start game when n is congruent to 1 or 3 modulo 5, where the game lasts one additional move.

3 The P_4-saturation game on $K_{m,n}$

Now we study the P_4-saturation game on the complete bipartite graph $K_{m,n}$. Since $K_{m,n}$ contains no triangles, during the game all components are stars. Throughout this section, X and Y are the partite sets of $K_{m,n}$, with $|X| = m \geq n = |Y|$. Let an X-*star* or a Y-*star* be a star having at least two leaves in X or in Y, respectively. We use $\alpha'(G)$ for the maximum size of a matching in G.

Lemma 3.1. *A graph G that is P_4-saturated relative to $K_{m,n}$ has at most $m + n - \alpha'(G)$ edges. If it contains both an X-star and a Y-star (or an isolated edge), then equality holds.*

Proof. Any even cycle contains P_4, so G is a forest. To avoid P_4, edges of a matching must lie in distinct components, so $\alpha'(G)$ is the number of nontrivial components. Since G is a forest, $|E(G)|$ is the number of vertices minus the number of components, so $|E(G)| \leq m+n-\alpha'(G)$.

A saturated subgraph containing both an X-star and a Y-star (or an isolated edge) cannot have isolated vertices. All components are then nontrivial stars, so equality holds in the computation above. $\square$

Call a P_4-saturated subgraph that contains both an X-star and a Y-star a *full subgraph.* A P_4-saturated subgraph that is not full has stars of only one of these types (plus isolated edges, possibly) and thus has only m or n edges. Hence Max wants to make a full subgraph. When m or n is even, Min can prevent this in the Max-start game; in particular, when n is even, we obtain $\mathrm{sat}_g(P_4; K_{m,n}) = n = \mathrm{sat}(P_4; K_{m,n})$. When Max can make a full subgraph, Lemma 3.1 encourages Min to create a large matching.

Theorem 3.2. *For $m \geq n \geq 2$, the game P_4-saturation numbers of $K_{m,n}$ are given by*

$$\mathrm{sat}_g(P_4; K_{m,n}) = \begin{cases} n & \text{when } n \text{ is even,} \\ m & \text{when } n \text{ is odd and } m \text{ is even,} \\ m + \left\lfloor \frac{n}{2} \right\rfloor & \text{when } mn \text{ is odd.} \end{cases}$$

and

$$\mathrm{sat}'_g(P_4; K_{m,n}) = \begin{cases} m & \text{when } n \leq 2\ , \\ m + \left\lfloor \frac{n}{2} \right\rfloor & \text{when } n > 2 \text{ and } mn \text{ is even,} \\ m + \left\lfloor \frac{n}{2} \right\rfloor - 1 & \text{when } n > 2 \text{ and } mn \text{ is odd.} \end{cases}$$

Proof. We will consider cases based on who moves first and the parity of m and n. Let G denote the P_4-saturated subgraph built during the game. Again "making" a subgraph means producing it as a component of the current graph.

Upper bounds. We give strategies for Min. If Max moves first and n is even, then Min ensures that only Y-stars are created, by immediately extending isolated edges made by Max to such stars and otherwise enlarging such stars. If m is even and n is odd, then Min similarly creates X-stars. The final number of edges is then $|Y|$ or $|X|$ in these two cases.

In the other cases, Min just ensures a large matching. If Max moves first and mn is odd, or Min moves first and mn is even, then Min makes isolated edges until a matching of size $\left\lceil \frac{n}{2} \right\rceil$ is built, later playing any legal move. By Lemma 3.1, at most $m + \left\lfloor \frac{n}{2} \right\rfloor$ moves are played.

If Min moves first and mn is odd, then Min can do slightly better. If Max responds to the first move by making an X-star or a Y-star, then the parity allows Min to ensure that only X-stars or Y-stars, respectively, will be played, yielding an outcome of $|X|$ or $|Y|$. Hence Max must immediately make another isolated edge. The moves by Min still yield a matching of size $\left\lceil \frac{n}{2} \right\rceil$, and with the extra edge made by Max the bound improves by 1.

Lower bounds. We give strategies for Max. Since the game cannot end with an isolated vertex in each part, at least n moves are played, which completes the proof for the Max-start game with n even.

For the remaining arguments, we distinguish cases. Except in Case 1, Max will want to force a full subgraph and keep $\alpha'(G)$ small. Since $\alpha'(G)$ equals the number of components in G, in those cases Max will avoid making isolated edges.

Case 1: *Max-start game with n odd and m even, or Min-start with $n \le 2$.* If an X-star is made, then no isolated vertex can be left in X, and at least m moves are made. In the Min-start game with $n \le 2$, Max makes an X-star immediately, completing the proof.

Min can prevent an X-star only by ensuring that, after each of her moves, every nontrivial component is a Y-star. In the Max-start game with n odd, Max can present isolated edges that Min converts to Y-stars. After $n-1$ moves, Max makes an isolated edge using the last isolated vertex of Y, and then Min must make an X-star. Hence Max can force an X-star, which establishes the desired lower bound.

Case 2: *Min-start game with n even.* Max responds to the first move by making a Y-star. While at least two isolated vertices remain in Y, Max enlarges the Y-star. If no isolated vertices remain in Y, then Max makes any legal move.

If exactly one isolated vertex remains in Y when Max is to move, then Max again adds to the Y-star if the graph now contains an X-star or an isolated edge; otherwise, Max forms an isolated edge. After this move, either the graph is already full or it contains an isolated edge. If it contains an isolated edge but no X-start, then note that every move in the game has reduced the number of isolated vertices in Y, but not all moves have reduced the number of isolated vertices in X. Consequently, although Y no longer contains isolated vertices, X does. Min now must transform an isolated edge into an X-star, making the graph full.

On the other hand, if it is Min's turn when exactly one isolated vertex remains in Y, then since n is even at least one of Min's moves did not reduce the number of isolated vertices in Y. This means that Min created an X-star. Hence in all cases the graph becomes full.

Throughout the game, Max makes at most one isolated edge. If in fact Max makes no isolated edges, then Min can make at most $n/2$, since on each turn the number of isolated vertices in Y decreases. Moreover, if Min makes an isolated edge on each of her turns, then Max is not forced to make one himself. Consequently, if Max does make an isolated edge, then Min makes fewer than $n/2$. In either case, the graph G at the end has at most $n/2$ components. Thus $\alpha'(G) \le n/2$ so, by Lemma 3.1, the number of edges in G is at least $m+n-n/2$, which equals $m+\lfloor n/2 \rfloor$.

Case 3: *Max-start game with mn odd.* A similar strategy suffices. Max plays any edge on his first turn. If Min responds by making a second isolated edge, then Max forms an X-star. Otherwise, Min creates a star. In either case, Max repeatedly enlarges the star that has been created. If Min ever makes an isolated edge, then Max makes a star of the opposite type; if not, then Max makes an isolated edge at the time when exactly one isolated vertex remains in the partite set containing the leaves of the star, and then Min must create a star of the opposite type. Thus the graph becomes full, and arguments like those in the preceding paragraph establish $|E(G)| \geq m + \lfloor n/2 \rfloor$.

Case 4: *Min-start game with n odd and m even.* Max responds to Min's initial move by creating an X-star. After this initial move, Min may either enlarge the X-star or play an isolated edge. If Min always enlarges the X-star, then Max does the same, until exactly one isolated vertex remains in X. Since m is even, it is now Max's turn, and he plays an isolated edge. Since $n \geq 3$, at least one isolated vertex remains in Y, while no isolated vertices remain in X; consequently, Min must transform the isolated edge into a Y-star. The graph is full, there are only two components, and no new components can be added. Hence $\alpha'(G) = 2$, so the number of edges in G is $m + n - 2$, which is at least $m + \lfloor n/2 \rfloor$.

Suppose instead that Min does eventually play an isolated edge. Since $n \geq 3$, at least one isolated vertex remains in Y. Max transforms the isolated edge into a Y-star, making the graph full. Thereafter, as long as it remains possible, Max always enlarges this Y-star. When the Y-star is first created, the graph has two components, $n - 3$ isolated vertices remain in Y, and it is Min's turn. While Y still has isolated vertices, each of Max's moves decreases the number by one, as does each move by Min that creates a new component. Thus at most $(n-3)/2$ more components are created throughout the game, so $\alpha'(G) \leq 2 + (n-3)/2 = (n+1)/2$. Now by Lemma 3.1, the number of edges in G is at least $m + n - (n+1)/2$, which equals $m + \lfloor n/2 \rfloor$.

Case 5: *Min-start game with mn odd.* Here Max cannot do quite as well. As noted when discussing upper bounds, if Max makes an X-star or Y-star on move 2, then Min can limit the final number of edges to m or n, respectively. If $n = 3$, then the claimed upper bound reduces to m, so Max makes an X-star. Otherwise, Max makes an isolated edge on move 2. If Min makes $K_{1,2}$, then Max makes the other type of star. If Min makes an isolated edge, then Max makes an X-star and can make a Y-star on the next round.

Hence the graph becomes full. Max subsequently enlarges Y-stars until Y has no more isolated vertices. After Max's second turn, the graph has at most three components and Y has exactly $n-3$ isolated vertices. As before, at most $(n-3)/2$ more components are created throughout the game. Now $\alpha'(G) \leq 3 + (n-3)/2 = (n+3)/2$, so $|E(G)| \geq m + n - (n+3)/2 = m + \lfloor n/2 \rfloor - 1$. $\square$

4 The C_4-saturation game on $K_{n,n}$

In this section, we study the C_4-saturation game on $K_{n,n}$, the natural bipartite analogue of the Füredi-Reimer-Seress problem. As we have noted, the trivial lower bound and the result of [13] yield $2n - 1 \leq \mathrm{sat}_g(C_4, K_{n,n}) \leq O(n^{3/2})$.

Our main result is a polynomial improvement of the lower bound: $\mathrm{sat}_g(C_4, K_{n,n}) = \Omega(n^{13/12})$. We first prove a technical lemma giving a lower bound on the size of a restricted type of graph that is also C_4-saturated graph relative to $K_{n,n}$. Here our interest is the exponent on n; we make no attempt to optimize lower-order terms or the leading coefficient.

Lemma 4.1. *Let G be a graph that is C_4-saturated relative to $K_{n,n}$, and let c and d be positive constants. If there exists $S \subseteq V(G)$ with at least cn vertices in each partite set such that $|N(v) \cap S| \le d\sqrt{n}$ for all $v \in V(G)$, then $|E(G)| \ge an^{13/12} - O(n^{35/36})$, where $a = \min\{\frac{1}{2}(\frac{c^2}{2d^2})^{2/3}, \frac{c^2}{2d}\}$.*

Proof. Let S_X and S_Y be the subsets of S in the two partite sets. Consider $x \in S_X$ and $y \in S_Y$ such that $xy \notin E(G)$. Since G is C_4-saturated relative to $K_{n,n}$, it contains a copy of P_4 with endpoints x and y. Each vertex in S_X has at most $d\sqrt{n}$ neighbors in S_Y and hence at least $cn - d\sqrt{n}$ nonneighbors in S_Y. Thus G contains at least $c^2n^2 - cdn^{3/2}$ copies of P_4 with endpoints in S_X and S_Y; call such paths *essential paths*. Since each essential path has endpoints in S_X and S_Y, and since no vertex has more than $d\sqrt{n}$ neighbors in S, no edge is the central edge of more than d^2n essential paths.

Let T be the set of vertices of G with degree at least $n^{5/12}$, and let $b = (\frac{c^2}{2d^2})^{2/3}$. If $|T| \ge bn^{2/3}$, then $\sum_{v\in T} d(v) \ge bn^{13/12}$, which yields $|E(G)| \ge \frac{b}{2}n^{13/12}$. Otherwise, let H be the subgraph of G induced by T. Since $C_4 \not\subseteq H$, the result of Füredi [13] yields $|E(H)| \le (bn^{2/3})^{3/2} + O((bn^{2/3})^{4/3})$, which simplifies to $|E(H)| \le \frac{c^2}{2d^2}n + O(n^{8/9})$. Multiplying by d^2n, we conclude that at most $\frac{c^2}{2}n^2 + O(n^{17/9})$ essential paths have central edges in H.

Thus at least $\frac{c^2}{2}n^2 - O(n^{17/9})$ essential paths have central edges incident to a vertex with degree less than $n^{5/12}$. Each such edge is the central edge of at most $dn^{11/12}$ essential paths; hence G has at least $\frac{c^2}{2d}n^{13/12} - O(n^{35/36})$ such edges. $\square$

Though the hypotheses of Lemma 4.1 seem technical, they apply whenever $\Delta(G) \le d\sqrt{n}$. Hence we obtain a corollary for ordinary saturation (using $c = 1$).

Corollary 4.2. *If G is C_4-saturated relative to $K_{n,n}$ and $\Delta(G) \le d\sqrt{n}$, then $|E(G)| \ge an^{13/12} - O(n^{35/36})$, where $a = \min\{\frac{1}{2}(\frac{1}{2d^2})^{2/3}, \frac{1}{2d}\}$. (If $d \le \frac{1}{4}$, then $a = \frac{1}{2d}$).*

Our main result for the C_4-saturation game on $K_{n,n}$ follows easily from Lemma 4.1.

Theorem 4.3. $\mathrm{sat}_g(C_4, K_{n,n}) \ge \frac{1}{21}n^{13/12} - O(n^{35/36})$*, and similarly for* $\mathrm{sat}'_g(C_4; K_{n,n})$.

Proof. We provide a strategy for Max that forces the final subgraph of $K_{n,n}$ to satisfy the hypotheses of Lemma 4.1. This strategy governs almost the first $2n/3$ moves for Max, after which Max plays arbitrarily.

Let $k = \left\lfloor \sqrt{n/3} \right\rfloor - 1$. Max arranges to give degree k to k specified vertices in each partite set. Each move by Max makes an isolated vertex adjacent to one of the specified vertices; hence it cannot complete a 4-cycle. Fewer than $n/3$ vertices are needed by Max in each part, so Min cannot exhaust the isolated vertices in either part with fewer than $2n/3$ moves. After this phase, Max may play any legal move.

At the end of the first phase, let S be the set of leaves attached to the $2k$ specified stars. By construction, these stars are disjoint, so S has about $n/3-2\sqrt{n/3}$ vertices in each part.

In the final subgraph G, no vertex has more than $2\sqrt{n/3}$ neighbors in S, since each vertex other than the center of a star is adjacent to at most one leaf of the star. (However, in addition to the leaves of its own star, the center of a star may be adjacent to one leaf in each of the other stars whose center is in the same partite set.) Thus G satisfies the hypotheses of Lemma 4.1 with c being any constant less than $1/3$ and $d=2\sqrt{1/3}$, from which the claim follows. $\square$

While Theorem 4.3 does establish a nontrivial asymptotic lower bound for $\mathrm{sat}_g(C_4; K_{n,n})$, the correct order of growth remains undetermined. Theorem 4.3 can perhaps be strengthened by improving the bounds in Lemma 4.1. This suggests the following question: What is the minimum number of edges in a graph with maximum degree D that is C_4-saturated relative to $K_{n,n}$?

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