Game matching number of graphs

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slides available on DBW preprint page

Joint work with
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Compute $\alpha'_g(G)$ on special classes.

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Thm. For 3-regular connected G, $\frac{n}{3} \le \min \alpha'_g(G) \le \frac{7n}{18}$.

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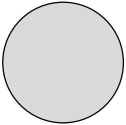
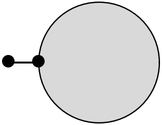
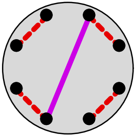




For $(2, 3)$, use 6-cycle.

For $(2k, 2k + 1)$, use K_{4k+2} minus perfect matching.



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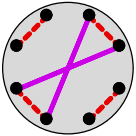

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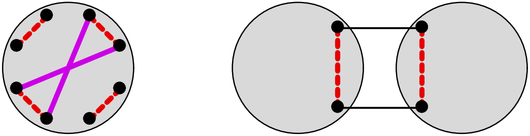

For $(2k-1, 2k)$, use $2K_{2k}$ modified by 2-switch. ■

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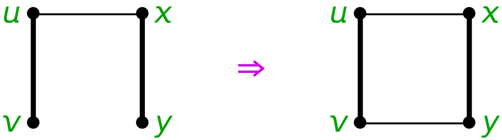
Sharpness: $rK_2 + C_6$, with v in isolated edge. No one wants to start on C_6. (Odd r for α'_g, even r for $\hat{\alpha}'_g$.)

When Max Can't be Beaten

Thm. If $ux \in E(G) \Rightarrow vy \in E(G)$ for $uv, xy \in M$, where M is a perfect matching in G, then $\alpha'_g(G) = \hat{\alpha}'_g(G) = n/2$.



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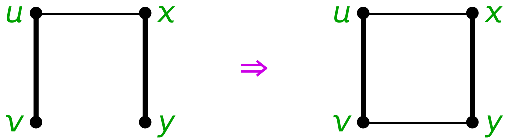

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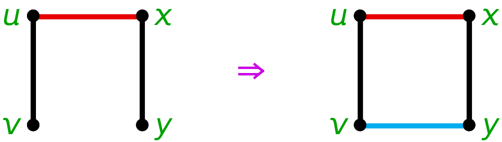

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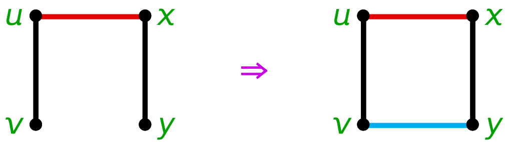

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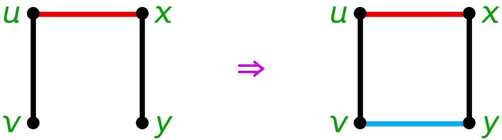

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- The condition is preserved under cartesian product with any graph. Hence Max can force a perfect matching in any $K_{r,r} \square H$, such as hypercubes.

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Ex. $G = K_{3k} \oplus \overline{K}_{n-3k}$: $\alpha'(G) = 3k$ and $\alpha'_g(G) = 2k$.

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Hence $\hat{\alpha}'_g(G) \le 2\mu(G) - k \le (2 - \frac{1}{2})\mu(G)$. ∎

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Pf. Let T be the set of $2\mu(G)$ vertices covered by a smallest maximal matching M.

Min plays in M when possible, using two vertices of T.

Max plays some edge, using at least one vertex of T.

This continues for k rounds, where $k \ge \mu(G)/2$, using at least $3k$ vertices of T and making $2k$ moves.

Subsequently, at most $2\mu(G) - 3k$ moves remain.

Hence $\hat{\alpha}'_g(G) \le 2\mu(G) - k \le (2 - \frac{1}{2})\mu(G)$. ■

Ex. Equality holds when $G = rP_4$ with r even.
Here $\mu(G) = r$ and $\hat{\alpha}'_g(G) = \frac{3}{2}r$.

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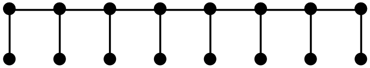

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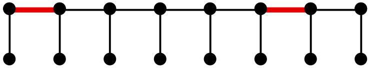

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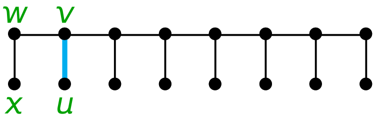

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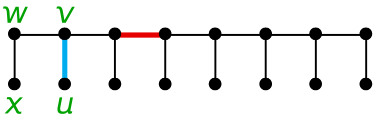

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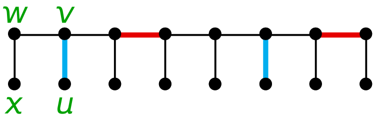

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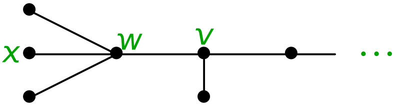

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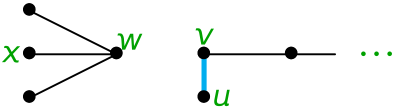

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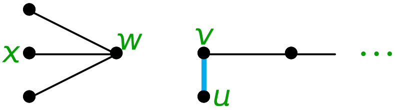

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The response by Min reduces α' by at most 2. Such moves by Min can occur only after moves by Max that guarantee two good moves. Hence $m \ge 4k$. ∎

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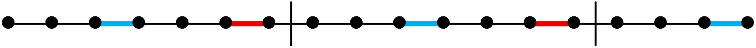

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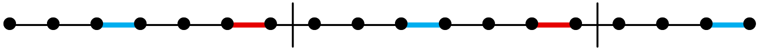

These strategies ensure these bounds against any strategy by the opponent. ■

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What construction provides an upper bound?

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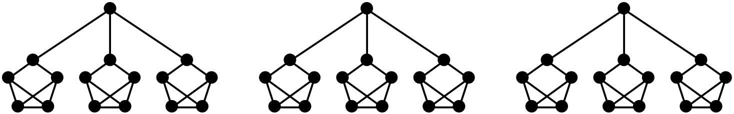

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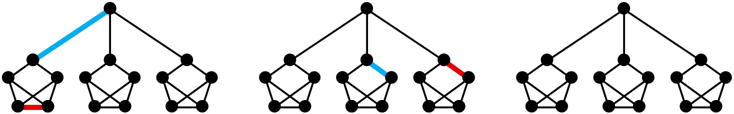

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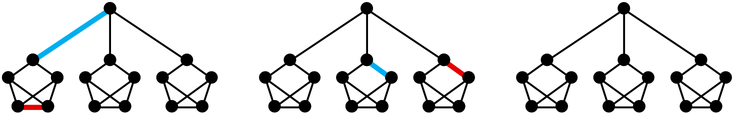

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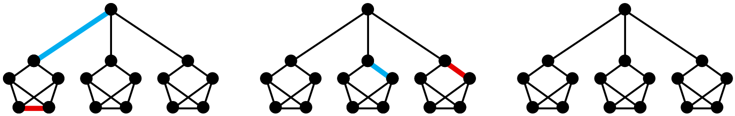

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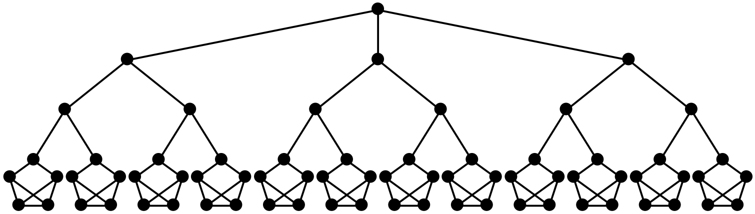

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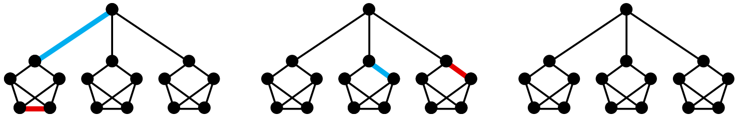

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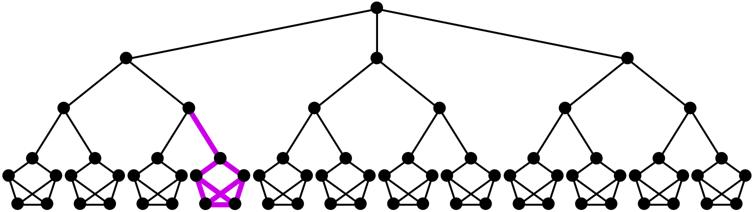

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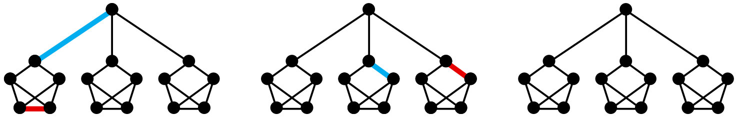

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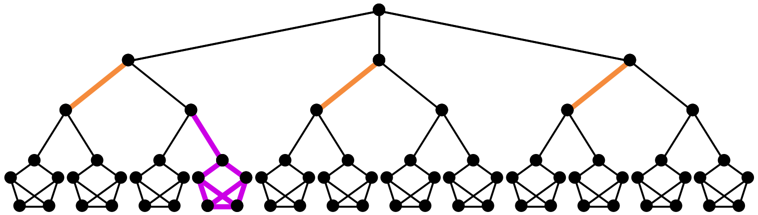

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Together with a maximum matching in the tree above copies of B, fewer than $\frac{7}{18}n$ edges are played.

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Ques. The matching game is the "F-saturation game" with $F = P_3$. What can be proved for other F?