A short proof of the Berge–Tutte Formula and the Gallai–Edmonds Structure Theorem

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Abstract

We present a short proof of the Berge–Tutte Formula and the Gallai–Edmonds Structure Theorem from Hall's Theorem.

The fundamental theorems on matchings in graphs have been proved in many ways. The most famous of these results is Hall's Theorem [6], characterizing when a bipartite graph has a matching that covers one partite set. Anderson [1] used Hall's Theorem to prove Tutte's 1-Factor Theorem [9], characterizing when a graph has a perfect matching. Berge [2] extended Tutte's 1-Factor Theorem to a min-max formula (known as the Berge–Tutte Formula) for the maximum size of a matching in a general graph.

In fact, Anderson's approach proves the Berge–Tutte Formula as easily as it proves Tutte's 1-Factor Theorem. Using the Berge–Tutte Formula, it then also yields the Gallai–Edmonds Structure Theorem [3, 4, 5], which describes all the maximum matchings in a given graph. Our proof by this method is shorter than earlier inductive proofs (see Theorem 3.2.1 of [8], for example) by not needing a characterization of factor-critical graphs or a "Stability Lemma" (Lemma 3.2.2 in [8]). We are indebted to the referee for pointing out the paper by Kotlov [7], which gives another short proof along similar lines to that given here.

For a set S of vertices in a graph G, let $N_G(S)$ or $N(S)$ denote the set of vertices having at least one neighbor in S. An X,Y*-bigraph* is a bipartite graph with partite sets X and Y. A *matching* is a set of pairwise non-incident edges. In an X,Y-bigraph, an obvious necessary condition for a matching that covers X is that $|N(S)| \geq |S|$ for all $S \subseteq X$. This is *Hall's Condition*, and Hall's Theorem [6] states that it is also sufficient.

A 1-*factor* is a spanning 1-regular subgraph; its edge set is a *perfect matching*. In a graph H, let $o(H)$ be the number of odd components (those having an odd number of vertices). In a graph G, an obvious necessary condition for a 1-factor is that $o(G-S) \leq |S|$ whenever $S \subseteq V(G)$. This is *Tutte's Condition*; Tutte proved that it is also sufficient.

In a graph G, the *deficiency* $\mathrm{def}_G(S)$ or $\mathrm{def}(S)$ is $o(G-S)-|S|$. Covering all vertices in an odd component of $G-S$ by a matching in G requires matching one of its vertices with a vertex

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of S. A *Tutte set* is a vertex subset with positive deficiency. Let $\mathrm{def}(G) = \max_{S\subseteq V(G)} \mathrm{def}(S)$. In an n-vertex graph G, every matching leaves at least $n - \mathrm{def}(G)$ vertices uncovered. By applying Tutte's Theorem to the graph obtained from G by adding $\mathrm{def}(G)$ vertices with no non-neighbors, Berge observed that the maximum size of a matching is $\frac{1}{2}(n - \mathrm{def}(G))$.

Anderson [1] proved Tutte's Theorem by applying Hall's Theorem to a bipartite graph derived from a maximal set of maximum deficiency. We show that the same approach directly yields the Berge–Tutte Formula. It also yields a short proof of the Gallai–Edmonds Structure Theorem, which describes all the maximum-sized matchings in a graph G.

The first two lemmas are well known; we include them for completeness.

Lemma 1 *(Parity Lemma) If G is an n-vertex graph and $S \subseteq V(G)$, then $o(G-S)-|S| \equiv n \mod 2$. In particular, if S is a Tutte set and n is even, then $o(G-S) \geq |S| + 2$.*

Proof. Counting vertices shows that $o(G-S) + |S| \equiv n \mod 2$. □

Lemma 2 *Let T be a maximal set among the vertex sets of maximum deficiency in a graph G. If u is a vertex of an odd component C of $G-T$, then the graph $C-u$ satisfies Tutte's Condition. Also, all components of $G-T$ are odd.*

Proof. For $S \subseteq V(C-u)$, we have

$$\begin{aligned}
\mathrm{def}_G(T \cup u \cup S) &= o(G - T - u - S) - (|T| + 1 + |S|) \\
&= o(G - T) - 1 + o(C - u - S) - |T| - 1 - |S| \\
&= \mathrm{def}_G(T) - 2 + \mathrm{def}_{C-u}(S)
\end{aligned}$$

The choice of T yields $\mathrm{def}_G(T \cup u \cup S) < \mathrm{def}_G(T)$. By the Parity Lemma, they have the same parity. Hence $\mathrm{def}_{C-u}(S) \leq 0$. Since S is arbitrary, $C-u$ satisfies Tutte's Condition.

If $G-T$ has a component with even order, then adding to T any leaf of a spanning tree of that component creates a larger set with the same deficiency as T. □

For $T \subseteq V(G)$, define an auxiliary bipartite graph $H(T)$ by contracting each component of $G-T$ to a single vertex and deleting edges within T. With Y denoting the set of components of $G-T$, the graph $H(T)$ is a T,Y-bigraph having an edge ty for $t \in T$ and $y \in Y$ if and only if t has a neighbor in G in the component of $G-T$ corresponding to y.

Lemma 3 *If T is a maximal set of maximum deficiency in a graph G, then $H(T)$ contains a matching that covers T.*

Proof. For $S \subseteq T$, all vertices of $Y - N_{H(T)}(S)$ are odd components of $G - (T - S)$. By the choice of T, we have $(|Y| - |N_H(S)|) - |T - S| \le \text{def}(T)$. Since $\text{def}(T) = |Y| - |T|$, the inequality simplifies to $|S| \le |N_H(S)|$. Thus Hall's Condition holds, and $H(T)$ has a matching that covers T. □

Edges of G corresponding to the matching obtained in Lemma 3 match T into vertices of distinct odd components of $G-T$. This enables us to build a matching of size $\frac{1}{2}(n-\text{def}(G))$. As noted, Tutte's 1-Factor Theorem is the special case $d = 0$.

Theorem 4 *(Berge–Tutte Formula; Berge [2]). If G is an n-vertex graph, then the maximum size of a matching in G is $\frac{1}{2}(n - \text{def}(G))$.*

Proof. We have noted that $\frac{1}{2}(n - \text{def}(G))$ is an upper bound. To build a matching of this size, we use induction on n. The claim is trivial for $n = 0$; consider $n > 0$.

Let T be a maximal set with deficiency $\text{def}(G)$. By Lemma 2, all components of $G - T$ are odd, and $C - u$ satisfies Tutte's Condition whenever u is a vertex in an odd component C of $G - T$. By the induction hypothesis, $C - u$ has a perfect matching.

Since $G - T$ has $|T| + \text{def}(G)$ odd components, it thus suffices to cover T using edges to distinct components of $G - T$. Lemma 3 guarantees this. □

In a graph G, let B be the set of vertices covered by every maximum matching in G, and let $D = V(G) - B$. Further partition B by letting A be the subset consisting of vertices with at least one neighbor outside B, and let $C = B - A$. The *Gallai–Edmonds Decomposition* of G is the partition of $V(G)$ into the three sets A, C, D.

A graph G is *factor-critical* if every subgraph obtained by deleting one vertex has a 1-factor. A matching in G is *near-perfect* if it covers all but one vertex of G. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S.

Theorem 5 *(Gallai–Edmonds Structure Theorem) Let A, C, D be the sets in the Gallai–Edmonds Decomposition of a graph G. Let $G_1, \ldots, G_k$ be the components of $G[D]$. If M is a maximum matching in G, then the following properties hold.*
a) M covers C and matches A into distinct components of $G[D]$.
b) Each G_i is factor-critical, and M restricts to a near-perfect matching on G_i.
c) If $\varnothing \ne S \subseteq A$, then $N_G(S)$ has a vertex in least $|S| + 1$ of $G_1, \ldots, G_k$.
d) $\text{def}(A) = \text{def}(G) = k - |A|$.

Proof. Let T be a maximal set of deficiency $\text{def}(G)$. By the Berge–Tutte Formula, M leaves $\text{def}(G)$ vertices uncovered and matches T into vertices of distinct components of $G-T$ (all are odd), and the rest of M forms a near-perfect matching in each component of $G-T$.

We use T to find the sets A, C, D of the Gallai–Edmonds Decomposition. Since $H(T)$ has a matching covering T (Lemma 3), Hall's Condition holds: $\left|N_{H(T)}(S)\right| \geq |S|$ for $S \subseteq T$. Since $\left|N_{H(T)}(\varnothing)\right| = 0$, we may let R be a maximal subset of T for which equality holds.

The crucial point is that $C = R \cup R'$, where R' consists of all vertices of all components of $G - T$ in $N_{H(T)}(R)$. Since $\left|N_{H(T)}(R)\right| = |R|$, the edges of M match R into vertices of distinct components of $G[R']$. We have observed that M covers the rest of R'. Since M covers T and no vertex of R or R' has a neighbor in the other odd components of $G-T$, we conclude that $R \cup R' \subseteq C$.

Let $D' = V(G) - T - R'$. It suffices to show that $D = D'$ and $A = T - R$. That is, we show that every vertex in D' is omitted by some maximum matching and that every vertex of $T - R$ has a neighbor in D'.

Let $H' = H(T) - (R \cup N_{H(T)}(R))$. For $S \subseteq T - R$ with S nonempty, we have $|N_{H'}(S)| > |S|$, since otherwise R could be enlarged to include S. Therefore, deleting any vertex of $N_{H'}(T-R)$ from H' leaves a subgraph of H' satisfying Hall's Condition, so H' has a maximum matching omitting any such vertex. By Lemma 2 and Theorem 4, each component of $G-T$ is factor-critical, so each vertex in D' is avoided by some maximum matching. This completes (a), (b), and (c).

For (d), since $o(G[D]) = k$, we have $\operatorname{def}(T) = o(G-T) - |T| = k + |R| - |A \cup R| = k - |A|$. Since $G[C]$ has a perfect matching, its components have even order, so $o(G - A) = k$. Hence A is another set with maximum deficiency. □

References

[1] I. Anderson, Perfect matchings of a graph. *J. Combinatorial Theory Ser. B* 10 (1971), 183–186.

[2] C. Berge, Sur le couplage maximum d'un graphe. *C. R. Acad. Sci. Paris* 247 (1958), 258–259.

[3] J. Edmonds, Paths, trees, and flowers. *Canad. J. Math.* 17 (1965), 449–467.

[4] T. Gallai, Kritische Graphen II. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 8 (1963), 373–395.

[5] T. Gallai, Maximale Systeme unabhängiger Kanten. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 9 (1964), 401–413.

[6] P. Hall, On representatives of subsets. *J. London Math. Soc.* 10 (1935), 26–30.

[7] A. Kotlov, Short proof of the Gallai–Edmonds Structure Theorem, http://arxiv.org/PS_cache/math/pdf/0011/0011204v1.pdf (2000), 3 pages.

[8] L. Lovász and M. D. Plummer, *Matching Theory. Ann. Discrete Math. Vol. 29*, (North-Holland, Amsterdam), 1986.

[9] W. T. Tutte, The factorization of linear graphs. *J. London Math. Soc.* 22 (1947), 107–111.