Forbidden Subposets for Fractional Weak Discrepancy at Most k

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Abstract

The *fractional weak discrepancy* of a poset P, written $\mathrm{wd}_F(P)$, is the least k such that some $f\colon P \to \mathbb{R}$ satisfies $f(y) - f(x) \geq 1$ for $x \prec y$ and $|f(y) - f(x)| \leq k$ for $x \| y$. We determine the minimal forbidden subposets for the property $\mathrm{wd}_F(P) \leq k$ when k is an integer.

1 Introduction

Various applications require a linear ordering of the elements of a partially ordered set (poset) P, with x before y if x is less than y in P. Which such orderings are "good"?

One may want to keep incomparable elements of P close together. Tanenbaum, Trenk, and Fishburn [7] list several such scenarios. For example, patients in an emergency room must be treated in some linear order, but the relation of "more urgent" is just a partial order. (We use $\preceq$ for the order relation in a poset P and $\|$ for incomparability, with $\prec$, $\succ$, $\succeq$ having the natural meanings). Certainly x should be treated before y if $x \prec y$ in the poset. For fairness, patients with incomparable urgencies should be treated not long apart.

In other contexts, the elements must receive values on a linear scale, but the values need not be equally spaced and may be repeated. As an example, [7] considers salary assignments. The employees form a poset P, with $x \prec y$ if y is more valuable than x. Employees are incomparable when one cannot tell which is more valuable. Clearly y should

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be paid more than x if $x \prec y$, perhaps at least \$100 more. Also, to avoid discontent, salaries of incomparable employees should not differ by much.

The problem of minimizing the maximum difference between values assigned to incomparable elements leads to natural poset parameters. A *consistent labeling* of a poset P is a real-valued function f on P such that $f(y) \geq f(x) + 1$ when $x \prec y$ in P. A *k-weak labeling* of P is a consistent labeling f such that $|f(x) - f(y)| \leq k$ whenever $x \parallel y$. The *weakness* of a consistent labeling f is $\max\{|f(x) - f(y)| \colon x \parallel y\}$. In a 0-weak labeling, elements have the same label if and only if they are incomparable. The distinct values of f define levels, with elements on the same level if and only if they are incomparable. Such posets are called *weak orders*, which motivates the terminology used here.

The *weak discrepancy* of P, denoted $\mathrm{wd}(P)$, is the least integer k such that P has an integer-valued k-weak labeling. The *fractional weak discrepancy*, denoted $\mathrm{wd}_F(P)$, is the least weakness of any consistent labeling of P. Weak discrepancy was introduced by Trenk in 1998 [8]; fractional weak discrepancy later by Shuchat, Shull, and Trenk [6]. A consistent labeling with weakness $\mathrm{wd}_F(P)$ or a consistent integer-valued labeling with weakness $\mathrm{wd}(P)$ is an *optimal* labeling for the corresponding parameter. (When the values in a labeling f are required to be distinct integers, $\min_f \max\{|f(x) - f(y)| \colon x \parallel y\}$ is the *linear discrepancy*; this parameter models our first example when patients are treated one-by-one.)

Shuchat et al. [6] noted that wd_F can be formulated as a linear program. The dual maximization problem bounds $\mathrm{wd}_F(P)$ from below using special subposets of P. A *forcing cycle* C in a poset P is a list $x_1, \ldots, x_m$ of elements of P, such that for each i (with indices modulo m) either $x_i \prec x_{i+1}$ or $x_i \parallel x_{i+1}$. When C is a forcing cycle, let $u(C)$ be the number of indices i such that $x_i \prec x_{i+1}$, and let $s(C)$ count those such that $x_i \parallel x_{i+1}$. That is, $u(C)$ counts the "up" steps and $s(C)$ counts the "side" steps. Since the up steps force increases in f totalling at least $u(C)$ along the cycle, the pigeonhole principle yields $\mathrm{wd}_F(P) \geq u(C)/s(C)$. Call $u(C)/s(C)$ the *ratio* of the cycle C. Forcing cycles with maximum ratio are *optimal* cycles.

These definitions come from [8] and [6] except for our use of "weakness" and "ratio". The term "weakness" originally meant the weak discrepancy of a poset (see [1]); now that fractional weak discrepancy is also studied, "weakness" is no longer used that way.

If P has two incomparable elements, then a forcing cycle exists and provides a lower bound on $\mathrm{wd}_F(P)$. Strong duality is expressed by the following results.

Theorem 1.1 (Shuchat, Shull, and Trenk [6]). *For a poset P that is not a chain,* $\mathrm{wd}_F(P) = \max_C \frac{u(C)}{s(C)}$, *where the ratio is maximized over all forcing cycles C in P.*

Theorem 1.2 (Gimbel and Trenk [1]). *For a poset P that is not a chain,* $\mathrm{wd}(P) = \max_C \lceil \frac{u(C)}{s(C)} \rceil$, *where the maximum is taken over all forcing cycles C in P.*

These results imply for all P that $\mathrm{wd}(P) = \lceil \mathrm{wd}_F(P) \rceil$ and that $\mathrm{wd}_F(P)$ is rational. In [3], the possible values of $\mathrm{wd}_F(P)$ are explored (it can be any rational value at least 1, or $r/(r+1)$ for integer r). Shuchat et al. ([4, 5]) studied the values of $\mathrm{wd}_F(P)$ within various families described by forbidden subposets.

In particular, let $\mathbf{r}_1 + \cdots + \mathbf{r}_t$ denote the poset consisting of chains of sizes $r_1, \ldots, r_t$ and no additional comparabilities. Weak orders (defined earlier) are characterized by forbidding $\mathbf{2}+\mathbf{1}$ as a subposet. Similarly characterized families include *interval orders* (no $\mathbf{2}+\mathbf{2}$) and *semiorders* (no $\mathbf{2}+\mathbf{2}$ or $\mathbf{3}+\mathbf{1}$). Semiorders are the posets for which wd_F can be less than 1. Shuchat et al. [5] studied the possible values of wd_F for interval orders, non-interval orders, posets containing $\mathbf{2}+\mathbf{2}$ but not $\mathbf{3}+\mathbf{1}$, and posets with no $\mathbf{t}+\mathbf{1}$ for each $t \geq 3$.

Forbidden subposet characterizations of hereditary families of posets have both structural and algorithmic value. Minimal forbidden subposets are often called *obstructions*. For linear discrepancy at most 1, Tanenbaum, Trenk, and Fishburn [7] determined the list of obstructions to be $\{\mathbf{2}+\mathbf{2}, \mathbf{3}+\mathbf{1}, \mathbf{1}+\mathbf{1}+\mathbf{1}\}$ (also they note that $\mathbf{1}+\mathbf{1}$ is the only obstruction for linear discrepancy 0, trivially). Howard, Keller, and Young [2] determined the obstructions for linear discrepancy at most 2.

In this paper, we study the obstructions for the poset families defined by $\mathrm{wd}_F(P) \leq k$ or $\mathrm{wd}_F(P) < k$ for a given natural number k. Analogous results for (integer) weak discrepancy follow as corollaries. In this language, our remark about weak orders was that $\mathbf{2}+\mathbf{1}$ is the unique obstruction to $\mathrm{wd}_F(P) \leq 0$. Shuchat et al. [3] solved one of the problems for $k = 1$.

Theorem 1.3 (Shuchat, Shull, and Trenk [3]). *A poset P satisfies $\mathrm{wd}_F(P) < 1$ if and only if P is a semiorder; that is, the obstructions to $\mathrm{wd}_F(P) < 1$ are $\mathbf{2}+\mathbf{2}$ and $\mathbf{3}+\mathbf{1}$.*

We characterize the obstructions completely for $\{P\colon\ \mathrm{wd}_F(P) \leq k\}$, where k is an integer. In Section 2, we construct the forbidden subposets, show that their fractional weak discrepancy exceeds k, and show that none contains another. Unfortunately, the list is infinite. In Section 3, we prove that $\mathrm{wd}_F(P) \leq k$ when P contains none of these posets. In Section 4, we consider the analogous problem for the family where $\mathrm{wd}_F(P) < k$. Although the list is finite when $k = 1$, we provide infinitely many examples for larger k, but not a full description.

The following observation is a tool for our characterizations. It holds because Theorem 1.1 implies that an optimal consistent labeling never decreases along an optimal cycle by more than the difference imposed by the pigeonhole principle, and hence the difference must equal that value everywhere.

Proposition 1.4 (Shuchat, Shull, and Trenk [3]). *Let P be a poset with $\mathrm{wd}_F(P) = r$. If C is a forcing cycle with elements $x_1, \ldots, x_m$ and $\frac{u(C)}{s(C)} = r$, and f is an r-weak labeling of P, then $f(x_{i+1}) = f(x_i) + 1$ whenever $x_i \prec x_{i+1}$ and $f(x_{i+1}) = f(x_i) - r$ whenever $x_i \parallel x_{i+1}$.*

2 Obstructions to $\mathrm{wd}_F(P) \le k$

For positive integers k and q, let $(b_1, \cdots, b_q)$ be a q-tuple of positive integers with sum $(k+1)q+1$. We view two q-tuples as equivalent if one is a cyclic shift of the other; this defines an equivalence relation on the q-tuples. We abuse notation by writing $(b_1, \ldots, b_q)$ to refer to the class containing $(b_1, \ldots, b_q)$. From each class, we will construct a family $\mathcal{F}(b_1, \ldots, b_q)$ of posets with fractional weak discrepancy $k + 1/q$, and we will show that these are the obstructions to $\mathrm{wd}_F(P) \le k$.

From $(b_1, \ldots, b_q)$, first form a disjoint union of chains with sizes $b_1, \ldots, b_q$. Let x_i^j denote the jth element on the ith chain, for $1 \le j \le b_i$, and let B_i denote the chain with elements $x_i^1, \ldots, x_i^{b_i}$ in order. To $B_1 + \cdots + B_q$, we will add additional comparabilities to form P. Order the elements by the indices (i, j) in lexicographic order, running through all $B_1, \ldots, B_q$ successively. The added comparabilities will make this list an optimal forcing cycle C in the resulting poset. The top element of each chain B_i will remain incomparable to the bottom element of the next chain (cyclically), and hence we will have $u(C) = kq + 1$ and $s(C) = q$.

In order to show that C is optimal, we first produce a labeling f that satisfies the properties listed in the conclusion of Proposition 1.4 for the elements of C. Let

$$f(x_i^j) = (j-1) - (i-1) \cdot \frac{kq+1}{q} + \sum_{r=1}^{i-1} (b_r - 1). \qquad (*)$$

Note that $f(x_i^{j+1}) = f(x_i^j) + 1$ for $1 \le j \le b_i - 1$. Also $f(x_i^{b_i}) - f(x_{i+1}^1) = \frac{kq+1}{q}$ for all i, with x_{q+1}^1 taken to be x_1^1. Hence f behaves as in Proposition 1.4 for C, but to guarantee that C is optimal we still must add comparabilities to prohibit forcing cycles with larger ratio.

In terms of f, we define a family of posets on the elements of C by requiring the properties in Definition 2.1 below; we will then prove that f is an optimal labeling for each poset in the family. Properties (1) and (2) in the definition guarantee that C remains a forcing cycle in P with ratio $\frac{kq+1}{q}$. Property (3) guarantees that f is order-preserving on P. Given (3), the contrapositives of properties (4) and (5) guarantee that f is a $\frac{kq+1}{q}$-weak labeling of P.

Definition 2.1. Fix $k, q \in \mathbb{N}$, and let $(b_1, \ldots, b_q)$ be a q-tuple with sum $(k+1)q+1$. Let $\mathcal{F}(b_1, \ldots, b_q)$ denote the family of all posets P with elements $\{x_i^j \colon 1 \le i \le q, 1 \le j \le b_i\}$ whose order relation $\prec$ satisfies the following properties, where f is as defined in $(*)$ or is shifted from $(*)$ by a uniform constant.

(1) $x_i^j \prec x_i^{j+1}$ for $1 \le j \le b_i - 1$.
(2) $x_i^{b_i} \parallel x_{i+1}^1$ for all i (with $x_{q+1}^1 = x_1^1$).
(3) $f(x_i^j) < f(x_{i'}^{j'})$ if $x_i^j \prec x_{i'}^{j'}$.
(4) $x_i^j \prec x_{i'}^{j'}$ whenever $f(x_{i'}^{j'}) - f(x_i^j) > \frac{kq+1}{q}$.
(5) $x_i^j \parallel x_{i'}^{j'}$ whenever $|f(x_{i'}^{j'}) - f(x_i^j)| < 1$.

The family $\mathcal{F}(b_1, \dots, b_q)$ may be large, since we have not specified whether $x_i^j \prec x_{i'}^{j'}$ when $1 \le f(x_{i'}^{j'}) - f(x_i^j) \le k + 1/q$. We may add any set of comparabilities of that form (not putting any x_i^j above $x_{i+1}^{j'}$) and then impose transitivity. We may add none, so $\mathcal{F}(b_1, \dots, b_q)$ is nonempty. Note that k is specified by the q-tuple $(b_1, \dots, b_q)$, since $\sum_{i=1}^q b_i = (k+1)q+1$.

Lemma 2.2. *Each poset in $\mathcal{F}(b_1, \dots, b_q)$ has fractional weak discrepancy $k + 1/q$, and the function f defined in $(*)$ is an optimal labeling for it.*

Proof. In constructing such a poset P, we start with the comparabilities in the forcing cycle C. Property (4) may add comparabilities of the form $x_i^j \prec x_{i'}^{j'}$. Additional comparabilities not forbidden by the combination of (5) and (3) may also be added. All added comparabilities of the form $x_i^j \prec x_{i'}^{j'}$ satisfy $f(x_{i'}^{j'}) - f(x_i^j) \ge 1$. Hence they create no inconsistency, and taking the transitive closure to complete P yields a poset on which f is a consistent labeling.

The requirement $x_i^{b_i} \parallel x_{i+1}^1$ enforces that no element of B_i is put above any element of B_{i+1}. Together, (4) and (5) imply that f is $(k + 1/q)$-weak. By (1) and (2), C is a forcing cycle in P, with $u(C) = kq+1$ and $s(C) = q$. Hence $\mathrm{wd}_F(P) = k+1/q$, and f is optimal. □

In the definition, each poset P in $\mathcal{F}(b_1, \dots, b_q)$ is built from a particular forcing cycle C and optimal labeling f. We call these the *fundamental cycle* and *fundamental labeling* of P.

Lemma 2.3. *If C and f are a fundamental cycle and a fundamental labeling for a poset $P \in \mathcal{F}(b_1, \dots, b_q)$ such that $\mathrm{wd}_F(P) = r$, then $|f(x) - f(y)| \in \{1, r\}$ if and only if x and y are consecutive on C.*

Proof. Let $k = (-1 + \sum b_i)/q - 1$. For P, we have $r = (kq+1)/q$. By Proposition 1.4, consecutivity on C implies $|f(x) - f(y)| \in \{1, r\}$. For the converse, choose $x, y \in P$ arbitrarily. Following C from x to y traverses some number a of up steps and some number b of side steps. By Proposition 1.4, $f(y) - f(x) = a - br$. Multiplying by q yields $|aq - b(kq+1)| \in \{q, kq+1\}$.

When r and s are relatively prime, integer solutions to the equation $ar - bs = p$ for a fixed integer p have the form $(a_0 + js, b_0 + jr)$ for integer j. In our case, $r = q$ and $s = kq+1$, and we seek solutions where $a \ge 0$, $b \ge 0$, and $a + b \le (k+1)q$. Since the sums $a + b$ for solution pairs (a, b) are equally spaced by $(k+1)q+1$, there is at most one solution in range. For $p = \pm q$, the unique solution is $(a, b) = (\pm 1, 0)$. For $p = \pm(kq+1)$, it is $(0, \mp 1)$. Hence the specified differences occur only when x and y are consecutive along C. □

Let $\mathcal{F}_k$ be the union of all families $\mathcal{F}(b_1, \dots, b_q)$ such that $\sum_{i=1}^q b_i = (k+1)q + 1$.

Lemma 2.4. *The fundamental cycle and labeling for a poset $P \in \mathcal{F}_k$ are well-defined. That is, each such P arises from exactly one equivalence class of integer lists defining a fundamental cycle, and in exactly one way from the lexicographically least element of the equivalence class.*

Proof. Since the number of elements depends on q, we need only consider posets generated by fundamental cycles with the same number of chains. Suppose that $P' \cong P$, where P arises from cycle C and labeling f, and P' arises from cycle C' and labeling f'. If P' arises from a different class of q-tuples, then under the isomorphism $\phi\colon P \to P'$, some $x, y \in P$ that are consecutive along C are mapped by ϕ to elements $\phi(x)$ and $\phi(y)$ that are not consecutive along C'. Since $\mathrm{wd}_F(P) = \mathrm{wd}_F(P') = r$, Lemma 2.3 implies that $|f(x) - f(y)| \in \{1, r\}$, but $|f'(\phi(x)) - f'(\phi(y))| \notin \{1, r\}$.

This contradicts Proposition 1.4, because elements consecutive along C must be mapped by any automorphism to elements that are consecutive along some optimal cycle. Hence their values under any optimal labeling must differ by 1 or r.

Hence ϕ can do nothing other than shift along C. After this shift, we have chosen a canonical element of the equivalence class of q-tuples, the fundamental cycles are the same, and the comparabilities added to form P and P' are the same. □

Lemma 2.5. *No poset in $\mathcal{F}_k$ is contained in another poset in $\mathcal{F}_k$.*

Proof. By Lemma 2.4, $P \subseteq P'$ only if P is smaller than P'. This requires $q < q'$, where q and q' are the numbers of chains in the fundamental cycles of P and P'. Now $\mathrm{wd}_F(P) = k + 1/q > k + 1/q' = \mathrm{wd}_F(P')$. However, $\mathrm{wd}_F(P) \le \mathrm{wd}_F(P')$ when $P \subseteq P'$, since every forcing cycle in P is also a forcing cycle in P'. □

Note that $\mathcal{F}_k$ is an infinite family; we construct distinct members of $\mathcal{F}_k$ for each value of q. The fact that the posets in $\mathcal{F}_k$ are minimal posets with fractional weak discrepancy greater than k will follow from the proof in the next section that every poset containing no poset in $\mathcal{F}_k$ has fractional weak discrepancy at most k.

3 Characterization of posets with $\mathrm{wd}_F(P) \le k$

Consider $P \in \mathcal{F}(b_1, \ldots, b_q)$. When we treat the fundamental cycle C of P as a forcing cycle, we consider only the comparabilities along the q special chains and the incomparabilities involving the top of one chain and the bottom of the next. In this section, we will view a forcing cycle C' in P also as a subposet of P whose elements are the elements along C'; thus we can sensibly write $\mathrm{wd}_F(C') \le \mathrm{wd}_F(P)$. Note that a *subposet* is a subset with the inherited partial order (some authors use the term "induced subposet" instead of "subposet").

The next lemma is the crux of the sufficiency argument for the characterization. We consider an arbitrary optimal cycle in an arbitrary poset P.

Lemma 3.1. *Let C be an optimal cycle in a poset P, with $s(C) = q$ and $u(C) = qk + t$ for positive integers k and t. If $t \ge 2$, then P contains a forcing cycle C' such that $\mathrm{wd}_F(C') = k + a/b$ for some pair (a, b) with $1 \le a < t$ and $1 \le b \le q$.*

Proof. We may assume that C is a shortest optimal cycle, since a shorter optimal cycle in P has the same ratio and therefore serves as the desired C'. Since C is a shortest optimal cycle, every subposet of C has smaller fractional weak discrepancy than C.

Since $\mathrm{wd}_F(C) > 1$, some chain in C (when C is viewed as a forcing cycle) has at least three elements. Let x be the middle element in a chain of three consecutive elements in C. Let Q be the subposet obtained by deleting x from C. If the elements of Q are viewed in the same order as in C, then Q is a forcing cycle with $u(Q) = qk + r - 1$ and $s(Q) = q$, since the consecutive incomparabilities in C were preserved by deleting x. Since Q is a subposet of C, we conclude that $k + (t-1)/q \le \mathrm{wd}_F(Q) < \mathrm{wd}_F(C) = k + t/q$.

Let C' be an optimal cycle in Q (C' need not be all of Q). Let $b = s(C')$, and define a by $\mathrm{wd}_F(Q) = k + a/b$. Now C' has $(k+1)b + a$ elements, and $\mathrm{wd}_F(C') = k + a/b$ with $(t-1)/q \le a/b < t/q$. It remains to show that $a < t$ and $b \le q$. In fact, it suffices to show that $b \le q$, since then $a/b < t/q$ implies $a < t$.

Note that $(k+1)b + a = |C'| \le |Q| = (k+1)q + t - 1$. If $b > q$, then $(k+1)b + a \le (k+1)q + t - 1$ yields $a < t - 1$, which yields $a/b < (t-1)/q$, a contradiction. □

In Lemma 3.1, there is no upper bound on t in the hypothesis. Hence we can apply the lemma for all P such that $\mathrm{wd}_F(P) > k$, not just those with $k < \mathrm{wd}_F(P) \le k + 1$.

Theorem 3.2. *For $k \in \mathbb{N}$, every poset with fractional weak discrepancy greater than k contains a poset in $\mathcal{F}_k$, and hence $\mathcal{F}_k$ is the complete list of obstructions for $\{P\colon \mathrm{wd}_F(P) \le k\}$.*

Proof. Let P be a poset with $\mathrm{wd}_F(P) > k$. Choose an optimal cycle C_0 in P. Let $t = u(C_0) - ks(C_0)$. By repeated application of Lemma 3.1, we reduce t to 1, producing a cycle C in P such that $u(C) = qk+1$ and $s(C) = q$, for some q, and furthermore $\mathrm{wd}_F(C) = k+1/q$, where C is viewed as a subposet. Note that C need not be an optimal cycle in P.

Let $(b_1, \ldots, b_q)$ be the sizes of the successive chains in C (between side-steps). Viewing C as a poset, we can apply Proposition 1.4. Since $\mathrm{wd}_F(C) = k + 1/q$, the cycle C itself is an optimal cycle. Therefore, every optimal numbering f of C has differences along C as specified by Proposition 1.4. Shifting the values of f by a constant, if necessary, yields f satisfying $(*)$. Now, since both C and f have been chosen to be optimal, $f(x) - f(y) > k + 1/q$ requires $x \prec y$. Incorporating these comparabilities forces the subposet C to lie in $\mathcal{F}_k$. □

For weak discrepancy, the characterization is the same.

Corollary 3.3. *For each integer k, the family of obstructions to $\{P\colon \mathrm{wd}(P) \le k\}$ is $\mathcal{F}_k$.*

Proof. If k is an integer and $\mathrm{wd}(P) > k$, then $\mathrm{wd}_F(P) > k$, since $\mathrm{wd}(P) = \lceil \mathrm{wd}_F(P) \rceil$. By Theorem 3.2, P contains a poset in $\mathcal{F}_k$. Conversely, if P contains such a poset P', then $\mathrm{wd}(P) \ge \mathrm{wd}_F(P) \ge \mathrm{wd}_F(P') > k$. □

Thus the posets with weak discrepancy k are those that contain a poset in $\mathcal{F}_{k-1}$ but no poset in $\mathcal{F}_k$.

4 Obstructions to $\mathrm{wd}_F(P) < k$

The set $\{P\colon\ \mathrm{wd}_F(P) \le k\}$ has infinitely many obstructions because its obstructions may have fractional weak discrepancy equal to any number that exceeds k by the reciprocal of an integer. One may hope that every obstruction to $\mathrm{wd}_F(P) < k$ has fractional weak discrepancy equal to k and that the number of obstructions is finite. We have remarked that Shuchat et al. [3] proved this for $k = 1$, with $\mathbf{2}+\mathbf{2}$ and $\mathbf{3}+\mathbf{1}$ being the obstructions.

Unfortunately, for $k > 1$ the list of obstructions is infinite. We do not provide a complete characterization, but we construct infinitely many obstructions. These examples do not include all obstructions.

If $k \ge 2$, then the fundamental cycle for a poset in $\mathcal{F}_k$ contains a chain of size at least 3. Skipping an interior element of that chain leaves a forcing cycle with ratio k. Hence no poset in $\mathcal{F}_k$ is an obstruction to $\mathrm{wd}_F(P) < k$, and all obstructions have fractional weak discrepancy exactly k. Such posets do not lie in $\mathcal{F}_{k-1}$, since those posets have fractional weak discrepancy strictly less than k. We seek minimal posets Q such that $\mathrm{wd}_F(Q) = k$.

Let $M_k = \{\mathbf{r}+\mathbf{s}\colon\ r+s = 2k+2 \text{ and } r,s \ge 1\}$. For each $P \in M_k$, we have $\mathrm{wd}_F(P) = k$ and $\mathrm{wd}_F(P-x) = k - 1/2$ for each $x \in P$. Hence M_k is a family of obstructions. In fact, there are only $\lceil (k+1)/2 \rceil$ posets in M_k, so we have not yet disproved finiteness. Note also that M_1 is the complete list of obstructions for $k = 1$.

To construct obstructions, again we start by building a forcing cycle C, but this time the chains in C all have size $k+1$. That is, we start with the q-tuple $(b_1, \dots, b_q)$ with each $b_i = k+1$. Again let x_i^j denote the jth element on chain i. Minimality requires that the poset have no other elements. We require $x_i^{k+1} \parallel x_{i+1}^1$ (with $x_{q+1}^1 = x_1^1$) and have an optimal labeling f with $f(x_i^j) = j$ for all (i,j). This time we add more comparabilities to ensure that none of our examples contains another. We obtain a 3-parameter family, but there are many more examples where the chains in the original forcing cycle do not all have the same length and where the "added comparabilities" are defined in different ways.

Definition 4.1. *Fix integers k, q, t, all at least 2, with $t \le k$. Let $P_{k,q,t}$ be the poset with elements $\{x_i^j\colon\ 1 \le i \le q, 1 \le j \le k+1\}$ defined by putting $x_i^j \prec x_i^{j+1}$ for all (i,j) such that $1 \le j \le k$ and putting $x_i^1 \prec x_{i'}^j$ for all (i,j) such that $t < j \le k+1$ and $i' \ne i-1 \pmod q$.*

Note that every such relation $\prec$ is transitive. Furthermore, no two of these posets are isomorphic, except for degeneracy when $q = 2$, since already $\prec$ as specified is transitive. For

$q = 2$, the construction reduces to the obstruction $\mathbf{k} + \mathbf{k}$. For $k = 1$, the constructions would generate families of disjoint 2-chains, which all contain $\mathbf{2} + \mathbf{2}$.

Proposition 4.2. *Each poset $P_{k,q,t}$ constructed in Definition 4.1 is a minimal poset with fractional weak discrepancy k.*

Proof. Since $x_i^{k+1} \parallel x_{i+1}^1$ in $P_{k,q,t}$, the original cycle in order is a forcing cycle with ratio k. Letting $f(x_i^j) = j$ defines a consistent labeling with weakness k. Hence $\mathrm{wd}_F(P_{k,q,t}) = k$.

To prove minimality, consider deleting x_i^j from $P_{k,q,t}$. By symmetry, we may assume that $i = q$. The remaining elements, in the same order as before, form a forcing cycle with ratio $(qk - 1)/q$. We show that this is the fractional weak discrepancy by providing a consistent labeling with weakness $(qk - 1)/q$. Throughout the cycle, augment f by 1 with each step up a chain, but decrease f by $(qk - 1)/q$ when moving from the top of one chain to the bottom of the next. This is well-defined, since the net change while traversing the cycle is 0. Furthermore, the values of f at incomparable elements differ by no more than $(qk - 1)/q$. □

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