Fractional Separation Dimension

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slides available on DBW preprint page

Joint work with Sarah Loeb

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This is the least d such that the embedding exists.

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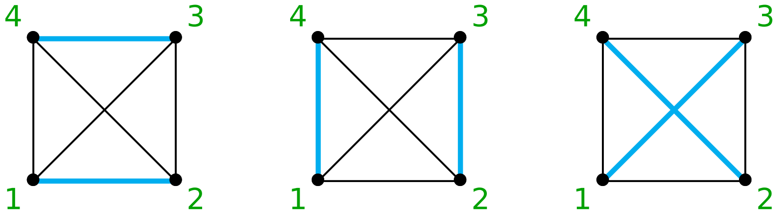
Ex. $\pi(K_4) = 3$.



$\sigma_1 = (1, 2, 3, 4)$ $\sigma_2 = (1, 4, 2, 3)$ $\sigma_3 = (1, 3, 2, 4)$

Other Notions of Separation

Ex. Barycentric representation - Assign each vertex v a nonnegative integer triple (v_1, v_2, v_3) with sum n so that for each edge xy and $z \notin \{x, y\}$ there is a coordinate k such that $z_k \geq \max\{x_k, y_k\}$.

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Application - $\min |\mathcal{F}|$ = dimension of the inclusion poset on 1-sets and k-sets. (Dushnik [1950])

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∴ separation dimension is a special case of boxicity.

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- (A-B-C-M-R) $\pi(G) \geq d/2$ for almost all d-regular graphs.

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Thm. (Loeb–West [2016+]) $\pi_f(G) \le 3$ for every G.

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In solving a linear program, the values are rational. When the gcd is b and the optimum is a/b, scaling by b yields an $(a:b)$-covering of H; the infimum is a min.

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Prop. The value of the matrix game on $M(G)$ is $\frac{1}{\pi_f(G)}$.

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This ordering strategy separates each nonincident pair with probability at least p, so $\pi_f(G) \le 1/p$. ■

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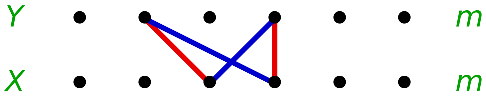

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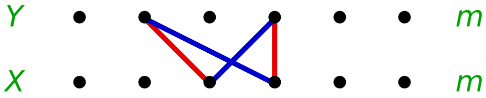

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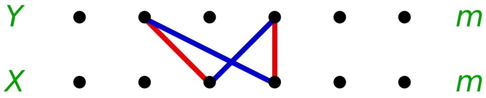

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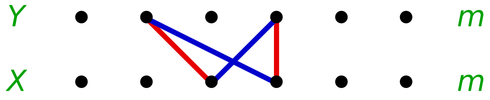

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Using three indices, $i < j < k$, one index contributes two vertices, completed two ways if it is i or k, only one way if it is j. Hence $\exists\, 5\binom{m}{3}$ pairs are separated.

Counting Separated Pairs

Order $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ separately, then order each $\{x_i, y_i\}$. How many edge pairs separated?

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Miraculously, $4\binom{m}{4} + 5\binom{m}{3} + \binom{m}{2} = \frac{m+1}{3m} 2\binom{m}{2}^2$.

No Ordering Separates More Pairs

For an ordering σ not of that form: by symmetry it orders X and Y as $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ but puts y_j immediately before x_i for some i and j with $j < i$.

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Hence the game has value $\frac{m+1}{3m}$ and $\pi_f(K_{m,m}) = 3\frac{m}{m+1}$. ■

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Thm. $\pi_f(K_{m+1,qm}) = 3\left(1 - \frac{(q+1)m-2}{(2m+1)mq-m-2}\right)$.

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Conj. For $n = 3m$, the n-vertex K_4-free graph maximizing π_f is $K_{m,m,m}$.

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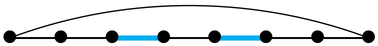

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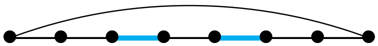

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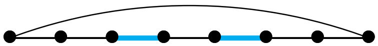

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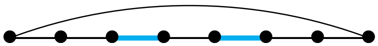

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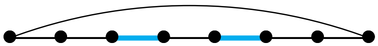

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Iterating, $v_{n-2} <_\sigma v_n$ and $v_{n-1} <_\sigma v_1$.
Now $v_1 <_\sigma \{v_3 \text{ or } v_4\} <_\sigma \cdots <_\sigma v_{n-1} <_\sigma v_1$. ∎

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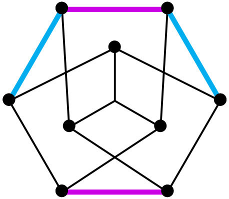

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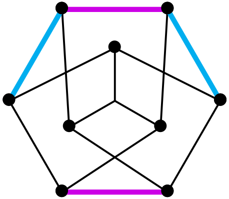

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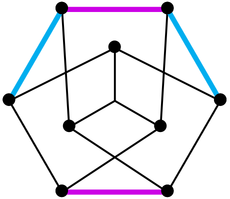

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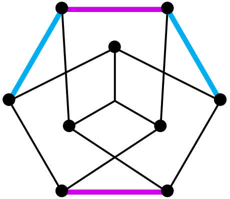

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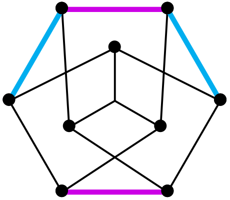

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Conj. $\pi_f(G) < 2$ if G has no cycle of length at most 4.

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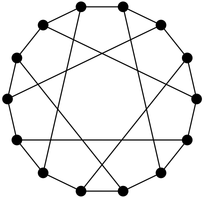

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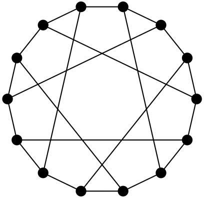

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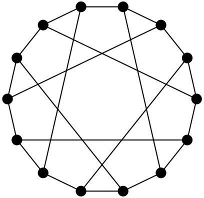

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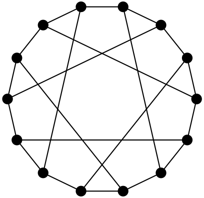

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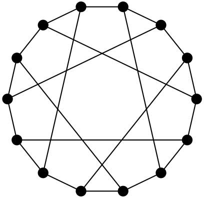

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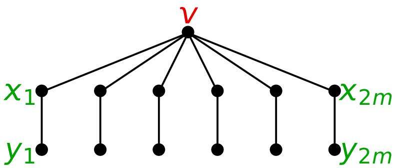
Increasing girth suggests better upper bounds on $\pi_f(G)$ (recall $\pi_f(C_n) = \frac{n}{n-2}$), but trees don't have $\pi_f(G) = 1$.

Subdivided Stars

Thm. $\pi_f(G) = \frac{4m-2}{3m-1}$, where G is obtained from the star with $2m$ edges by subdividing each edge.



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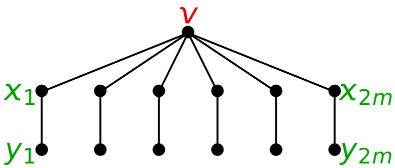

Pf. Ordering Player: Play all orderings having v in the middle and consecutive pairs x_i, y_i in any order, equally.

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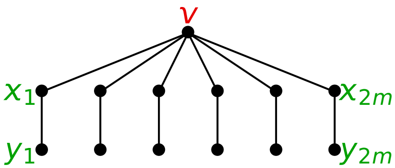

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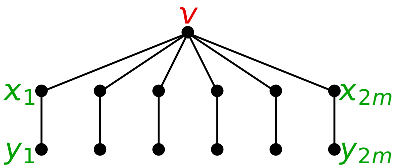

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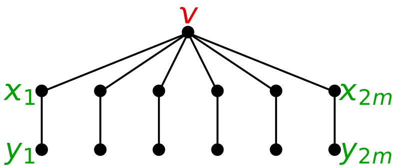

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Argue that the orderings above are the best. ■

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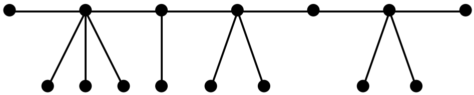

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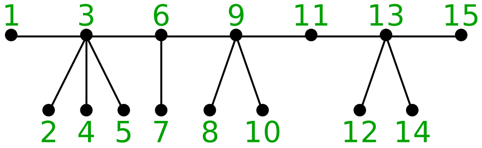

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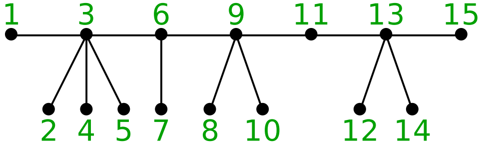

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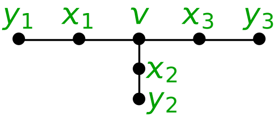
A non-caterpillar tree contains Y. Separating its pendant edges prevents inserting v. ∎



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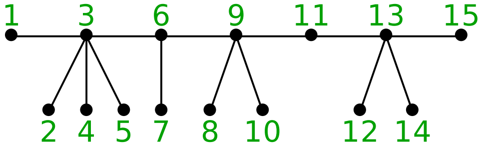

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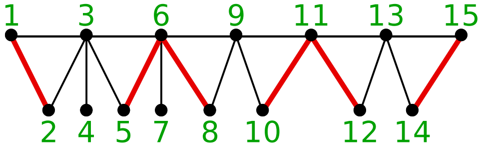

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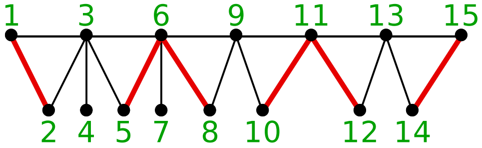

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This is the same as the characterization of $\pi(G) = 1$ in B–C–G–M–R [2014], since $\pi_f(G) = 1 \iff \pi(G) = 1$.

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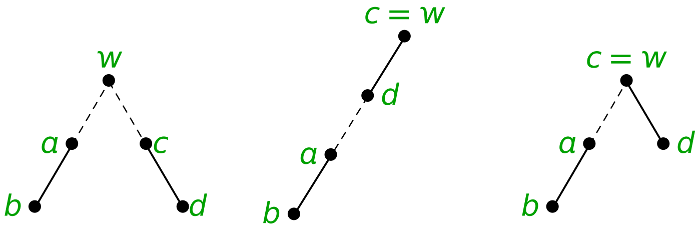

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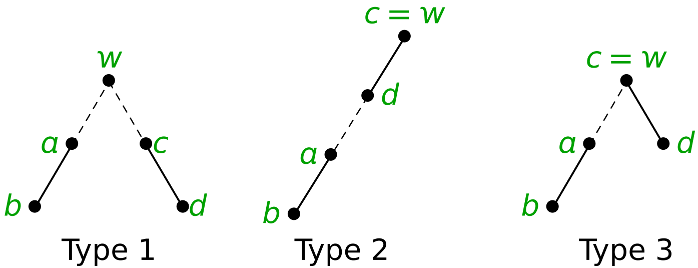

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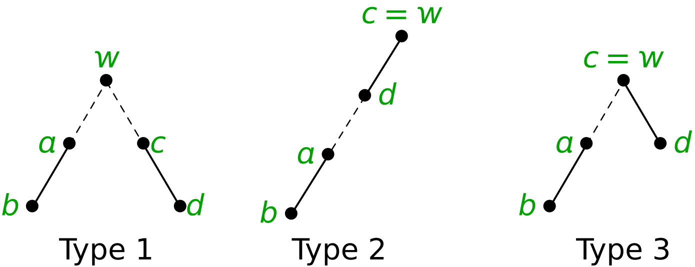

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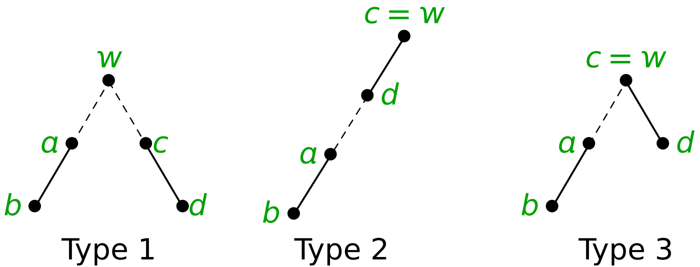

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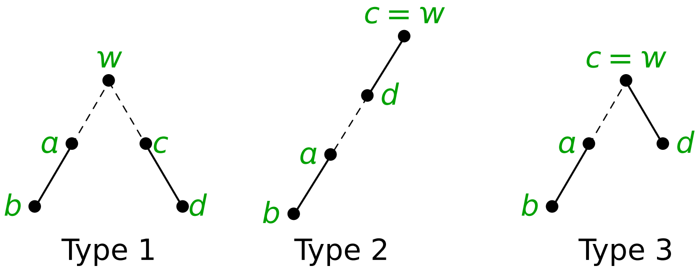

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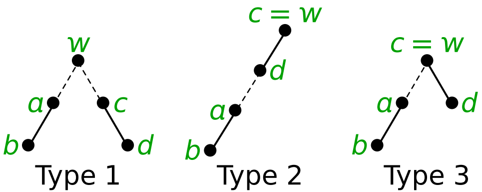
- Type 3 pairs separated with probability $\geq \frac{1}{\sqrt{2}}$

Algorithm for Upper Bound

Ordering Player generates an ordering with these properties at random, from the root v on down.



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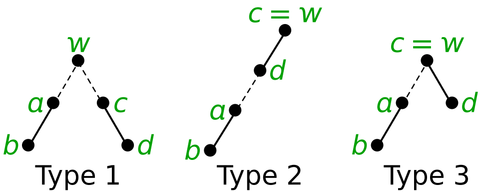

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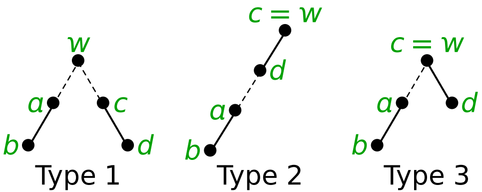

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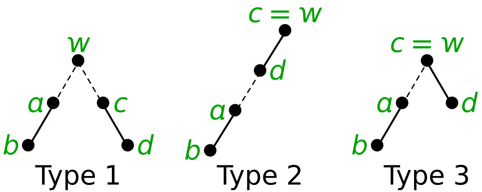

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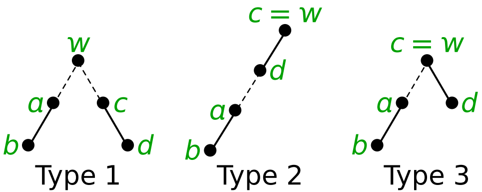

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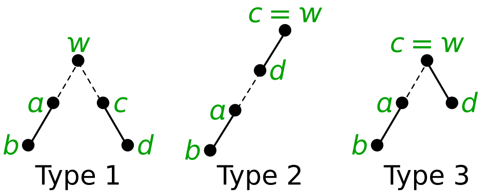

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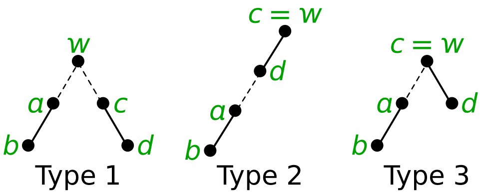
Thus Type 1 never fails to be separated.

Probability for Types 2 and 3



Let $\beta = \frac{1}{\sqrt{2}}$

Type 2 fails separation only if a is between c and d, meaning the child of d above a is placed between d and its parent: prob $1 - \beta$. Hence $\mathbb{P}(ab : cd) = \beta = \frac{1}{\sqrt{2}}$.

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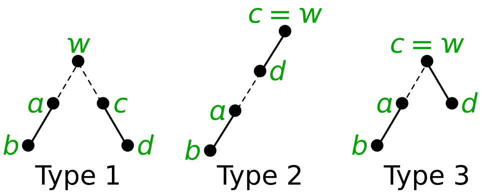

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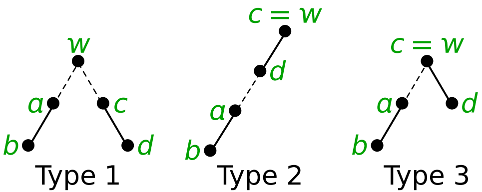

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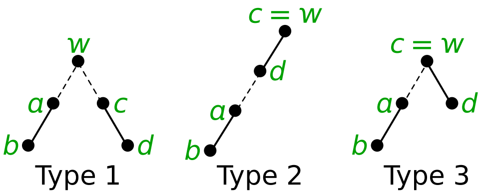

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If $c = w = v$, then a and d are on the same side with prob $\frac{1}{2}$, and $\mathbb{P}(ab:cd) = \frac{3}{4}$.

Open Questions

Ques. On n-vertex trees, what is $\max \pi_f$?

To exceed $\frac{4}{3}$, pair player must play some pairs that are Type 2 and some that are Type 3 without $w =$ root.

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Ques. Which rational numbers x satisfy $\pi_f(G) = x$ for some graph G?