Fractional Separation Dimension

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slides available on DBW preprint page

Joint work with Sarah Loeb
The Original Problem

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This is the least $d$ such that the embedding exists.
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Ex. \( \pi(K_4) = 3 \).

\[ \sigma_1 = (1, 2, 3, 4) \]
\[ \sigma_2 = (1, 4, 2, 3) \]
\[ \sigma_3 = (1, 3, 2, 4) \]
Other Notions of Separation

**Ex.** Barycentric representation - Assign each vertex \( v \) a nonnegative integer triple \((v_1, v_2, v_3)\) with sum \( n \) so that for each edge \( xy \) and \( z \notin \{x, y\} \) there is a coordinate \( k \) such that \( z_k \geq \max\{x_k, y_k\} \).
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Ex. $k$-suitable family for $[n]$ - A set $F$ of orderings of $[n]$ such that for each $k$-set $S$ and $y \not\in S$, there exists $\sigma \in F$ in which $y$ appears after all of $S$. 
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Ex. **\( k \)-suitable family** for \([n]\) - A set \( \mathcal{F} \) of orderings of \([n]\) such that for each \( k \)-set \( S \) and \( y \notin S \), there exists \( \sigma \in \mathcal{F} \) in which \( y \) appears after all of \( S \).

**Application** - \( \min |\mathcal{F}| = \) dimension of the inclusion poset on 1-sets and \( k \)-sets. (Dushnik [1950])
One More Separation Problem

**Ex.** A $k$-box representation of $H$ assigns each vertex an axis-parallel box in $\mathbb{R}^k$ so vertices are adjacent iff their boxes intersect. **Boxicity** $\text{box}(H)$ is the least such $k$. 
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\( \therefore \) separation dimension is a special case of boxicity.
Bounds on Separation Dimension

Basavaraju, Chandran, Golumbic, Mathew, Rajendraprasad [2014] (for hypergraphs [2016], incl. $\pi(H) = \text{box}(L(H))$).

- $\pi(G) \leq 6.84 \log n$ when $G$ has $n$ vertices.
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- (A-B-C-M-R) $\pi(G) \geq d/2$ for almost all $d$-regular graphs.
Covering Problems

Given a hypergraph $H$ with vertex set $U$, let
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Always $\tau_f(H) \leq \tau(H)$. 
Fractional vs. Ordinary

$\tau_f(H)$ may be much smaller than $\tau(H)$. 
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For a vertex-transitive graph, \( \chi_f(G) = |V(G)|/\alpha(G) \). Also, \( \alpha(K(n,k)) = \binom{n-1}{k-1} \), by Erdös–Ko–Rado, so \( \chi_f(K(n,k)) = \frac{n}{k} \).
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**Thm.** (B–C–G–M–R [2014]) \( \pi(G) \geq \log \left[ \frac{\omega(G)}{2} \right] \).
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**Thm.** (Loeb–West [2016+]) \( \pi_f(G) \leq 3 \) for every \( G \).
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Linear Programs

Covering problems are integer linear programs. $\tau(H)$ is the solution to:

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\begin{align*}
\text{minimize} & \quad \sum_{I \in E(H)} x_I \\
\text{subject to} & \quad x_I \in \{0, 1\} \quad \forall I \in E(H) \\
& \quad \sum_{I \ni v} x_I \geq 1 \quad \forall v \in V(H)
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In solving a linear program, the values are rational. When the gcd is \( b \) and the optimum is \( a/b \), scaling by \( b \) yields an \((a: b)\)-covering of \( H \); the infimum is a \( \text{min} \).
Converting to a Game

Given a list $\mathcal{F}$ of orderings of $V(G)$, let $a = |\mathcal{F}|$, and let all pairs of nonincident edges be separated $b$ times.
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Let $M(G)$ be the 0, 1-matrix with rows = orderings and columns = pairs of non-incident edges such that $M_{i,j} = 1 \Leftrightarrow$ ordering $i$ separates pair $j$. 
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Prop. The value of the matrix game on $M(G)$ is $\frac{1}{\pi_f(G)}$. 
The Game as a Linear Program

Let $S = \text{rows (orderings)}$, $P = \text{cols (edge pairs)}$.

$S_p = \text{set of orderings separating a pair } p$.

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Row player wants to choose weights $\chi_\sigma$ for $\sigma \in S$ to

maximize $t$

subject to $\chi_\sigma \geq 0 \quad \forall \sigma \in S$

$\sum_{\sigma \in S} \chi_\sigma = 1$ and $\sum_{\sigma \in S_p} \chi_\sigma \geq t \quad \forall p \in P$
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Prop. $\pi_f(G) = 1/t^*$, where $t^*$ is the value of the game.
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Cor. Always $\pi_f(G)$ is rational.
Strategy

Idea: To prove $\pi_f(G) = 1/t$, find distributions:
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**Idea:** To prove $\pi_f(G) = 1/t$, find distributions: $\chi$ over $S$ such that $\sum_{\sigma \in S_p} x_\sigma \geq t$ for all $p \in P$ (upper bd.).
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$\forall p \in P$, we have $\Pr(p \text{ is separated}) \geq t$. 
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$y$ over $P$ such that $\sum_{p \in P_\sigma} y_p \leq t$ for all $\sigma \in S$ (lower bd.).
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- $\gamma$ over $P$ such that $\sum_{p \in P_\sigma} \gamma_p \leq t$ for all $\sigma \in S$ (lower bd.).
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**Prop.** $\pi_f(G) \leq 3$ for every graph $G$. 
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Any four vertices appear in each order equally often.
For any $uv$ and $wz$, we have $\mathbb{P}(uv : wz) = 1/3$.  

\[\blacksquare\]
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Characterization of Extreme Graphs

**Thm.** If $K_4 \not\subseteq G$, then $\pi_f(G) \leq 3 \left(1 - \frac{12}{n^4} + O\left(\frac{1}{n^5}\right)\right)$. 

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The pairs $ab : cd$ and $ad : bc$ are separated 8 times in the usual 24 but 12 times in the new list. The increase is $\frac{4(n-4)!}{n!}$; separation probability now $\geq \frac{1}{3} + \frac{4(n-4)!}{n!} = p$. 
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This ordering strategy separates each nonincident pair with probability at least $p$, so $\pi_f(G) \leq 1/p$. □
Complete Bipartite Graphs

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- **Pair player:** Play all $2\binom{m}{2}\binom{m}{2}$ edge pairs equally.
- Show every $\sigma$ separates at most $\frac{m+1}{3m}2\binom{m}{2}^2$ pairs.
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- **Pair player:** Play all $2 \binom{m}{2} \binom{m}{2}$ edge pairs equally.
- **Ordering player:** Play all orderings $\nu_1, \ldots, \nu_{2m}$ such that $\nu_{2i-1} \nu_{2i} \in E(K_{m,m})$ for $1 \leq i \leq m$, equally likely.
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**Thm.** $\pi_f(K_{m,m}) = 3 \left(1 - \frac{1}{m+1}\right) = \frac{3m}{m+1}$.

**Pf.** Give game strategies for $t = \frac{m+1}{3m}$.

- **Pair player:** Play all $2\binom{m}{2}\binom{m}{2}$ edge pairs equally. Show every $\sigma$ separates at most $\frac{m+1}{3m}2\binom{m}{2}^2$ pairs.

- **Ordering player:** Play all orderings $\nu_1, \ldots, \nu_{2m}$ such that $\nu_{2i-1}\nu_{2i} \in E(K_{m,m})$ for $1 \leq i \leq m$, equally likely. If such an ordering separates $\frac{m+1}{3m}2\binom{m}{2}^2$ pairs, then by symmetry each pair is separated with probability $\frac{m+1}{3m}$. 
Counting Separated Pairs

Order $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ separately, then order each $\{x_i, y_i\}$. How many edge pairs separated?

$$\ldots, x_i, y_i, \ldots, y_j, x_j, \ldots, x_k, y_k, \ldots, x_l y_l, \ldots$$
Counting Separated Pairs

Order $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ separately, then order each $\{x_i, y_i\}$. How many edge pairs separated?

$$\ldots, x_i, y_i, \ldots, y_j, x_j, \ldots, x_k, y_k, \ldots, x_l y_l, \ldots$$

Pairs hitting four indices, $i < j < k < l$, must be $x_i y_j$ or $y_i x_j$ and $x_k y_l$ or $y_k x_l$. Hence $\exists 4\binom{m}{4}$ such pairs.
Counting Separated Pairs

Order $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ separately, then order each $\{x_i, y_i\}$. How many edge pairs separated?

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Using three indices, $i < j < k$, one index contributes two vertices, completed two ways if it is $i$ or $k$, only one way if it is $j$. Hence $\exists 5 \binom{m}{3}$ pairs are separated.
Counting Separated Pairs

Order \(x_1, \ldots, x_m\) and \(y_1, \ldots, y_m\) separately, then order each \(\{x_i, y_i\}\). How many edge pairs separated?

\[\ldots, x_i, y_i, \ldots, y_j, x_j, \ldots, x_k, y_k, \ldots, x_l y_l, \ldots\]

Pairs hitting four indices, \(i < j < k < l\), must be \(x_i y_j\) or \(y_i x_j\) and \(x_k y_l\) or \(y_k x_l\). Hence \(\exists 4 \binom{m}{4}\) such pairs.

Using three indices, \(i < j < k\), one index contributes two vertices, completed two ways if it is \(i\) or \(k\), only one way if it is \(j\). Hence \(\exists 5 \binom{m}{3}\) pairs are separated.

Using two indices \(i < j\), one pair \(x_i y_i : x_j y_j\) is separated.
Counting Separated Pairs

Order \( x_1, \ldots, x_m \) and \( y_1, \ldots, y_m \) separately, then order each \( \{x_i, y_i\} \). How many edge pairs separated?

\[
\ldots, x_i, y_i, \ldots, y_j, x_j, \ldots, x_k, y_k, \ldots, x_l y_l, \ldots
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Pairs hitting four indices, \( i < j < k < l \), must be \( x_i y_j \) or \( y_i x_j \) and \( x_k y_l \) or \( y_k x_l \). Hence \( \exists 4 \binom{m}{4} \) such pairs.

Using three indices, \( i < j < k \), one index contributes two vertices, completed two ways if it is \( i \) or \( k \), only one way if it is \( j \). Hence \( \exists 5 \binom{m}{3} \) pairs are separated.

Using two indices \( i < j \), one pair \( x_i y_i : x_j y_j \) is separated.

Hence \( 4 \binom{m}{4} + 5 \binom{m}{3} + \binom{m}{2} \) pairs are separated.
Counting Separated Pairs

Order $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ separately, then order each $\{x_i, y_i\}$. How many edge pairs separated?

$$\ldots, x_i, y_i, \ldots, y_j, x_j, \ldots, x_k, y_k, \ldots, x_l y_l, \ldots$$

Pairs hitting four indices, $i < j < k < l$, must be $x_i y_j$ or $y_i x_j$ and $x_k y_l$ or $y_k x_l$. Hence $\exists 4\binom{m}{4}$ such pairs.

Using three indices, $i < j < k$, one index contributes two vertices, completed two ways if it is $i$ or $k$, only one way if it is $j$. Hence $\exists 5\binom{m}{3}$ pairs are separated.

Using two indices $i < j$, one pair $x_i y_i : x_j y_j$ is separated.

Hence $4\binom{m}{4} + 5\binom{m}{3} + \binom{m}{2}$ pairs are separated.

Miraculously, $4\binom{m}{4} + 5\binom{m}{3} + \binom{m}{2} = \frac{m+1}{3m} 2\binom{m}{2}^2$. 
No Ordering Separates More Pairs

For an ordering $\sigma$ not of that form: by symmetry it orders $X$ and $Y$ as $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ but puts $y_j$ immediately before $x_i$ for some $i$ and $j$ with $j < i$.

$\sigma: \ldots, y_j, x_i, \ldots$
For an ordering \( \sigma \) not of that form: by symmetry it orders \( X \) and \( Y \) as \( x_1, \ldots, x_m \) and \( y_1, \ldots, y_m \) but puts \( y_j \) immediately before \( x_i \) for some \( i \) and \( j \) with \( j < i \).

\[
\sigma : \ldots, y_j, x_i, \ldots \quad \sigma' : \ldots, x_i, y_j, \ldots
\]

Form \( \sigma' \) from \( \sigma \) by interchanging \( y_j \) and \( x_i \).
No Ordering Separates More Pairs

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$$\sigma: \ldots, y_j, x_i, \ldots \quad \sigma': \ldots, x_i, y_j, \ldots$$

Form $\sigma'$ from $\sigma$ by interchanging $y_j$ and $x_i$.

Any pair separated by exactly one of $\sigma$ and $\sigma'$ has $x_i$ and $y_j$ as endpoints of distinct edges.
No Ordering Separates More Pairs

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Form $\sigma'$ from $\sigma$ by interchanging $y_j$ and $x_i$.

Any pair separated by exactly one of $\sigma$ and $\sigma'$ has $x_i$ and $y_j$ as endpoints of distinct edges.

There are $(i - 1)(m - j)$ such pairs in $\sigma$ and $(j - 1)(m - i)$ such pairs in $\sigma'$. 
No Ordering Separates More Pairs

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Since $m \geq 2$ and $j > i$, comparing $mi + j$ and $mj + i$ shows that $\sigma$ separates fewer pairs than $\sigma'$. 
No Ordering Separates More Pairs

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$$\sigma: \ldots, y_j, x_i, \ldots \quad \sigma': \ldots, x_i, y_j, \ldots$$

Form $\sigma'$ from $\sigma$ by interchanging $y_j$ and $x_i$.

Any pair separated by exactly one of $\sigma$ and $\sigma'$ has $x_i$ and $y_j$ as endpoints of distinct edges.

There are $(i - 1)(m - j)$ such pairs in $\sigma$ and $(j - 1)(m - i)$ such pairs in $\sigma'$.

Since $m \geq 2$ and $j > i$, comparing $mi + j$ and $mj + i$ shows that $\sigma$ separates fewer pairs than $\sigma'$.

Hence the game has value $\frac{m+1}{3m}$ and $\pi_f(K_{m,m}) = 3\frac{m}{m+1}$. □
Unbalanced Complete Bipartite Graphs

**Thm.** \( \pi_f(K_{m+1,q_m}) = 3 \left( 1 - \frac{(q+1)m-2}{(2m+1)mq-m-2} \right) \).
Unbalanced Complete Bipartite Graphs

**Thm.** \( \pi_f(K_{m+1,qm}) = 3 \left( 1 - \frac{(q+1)m-2}{(2m+1)mq-m-2} \right) \).

**Pf.** Pair player: nonincident pairs equally likely.
Unbalanced Complete Bipartite Graphs

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Pf. **Pair player**: nonincident pairs equally likely.

**Ordering player**: play all orderings that alternate one vertex of \( X \) with \( q \) vertices of \( Y \), equally likely.

\[ x, y, \ldots, y, x, y, \ldots, y, x \]
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\[ x, y, \ldots, y, x, y, \ldots, y, x \]

These orderings separate the most pairs, enough so that each nonincident pair is separated with probability

\[ \frac{1}{3} \left(1 - \frac{(q+1)m-2}{(2m+1)mq-m-2}\right)^{-1}. \]
\textbf{Thm.} \quad \pi_f(K_{m+1,qm}) = 3 \left( 1 - \frac{(q+1)m-2}{(2m+1)mq-m-2} \right).

\textbf{Pf.} \textbf{Pair player:} nonincident pairs equally likely.

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\[ x, y, \ldots, y, x, y, \ldots, y, x \]

These orderings separate the most pairs, enough so that each nonincident pair is separated with probability

\[ \frac{1}{3} \left( 1 - \frac{(q+1)m-2}{(2m+1)mq-m-2} \right)^{-1} . \]

\[ \lim_{q \to \infty} \pi_f(K_{m+1,qm}) = 3 \left( 1 - \frac{1}{2m+1} \right). \]
Complete Tripartite Graphs

Thm. \( \pi_f(K_{m,m,m}) = 3 \left( 1 - \frac{1}{2m+1} \right) \)
Complete Tripartite Graphs

**Thm.** $\pi_f(K_{m,m,m}) = 3 \left(1 - \frac{1}{2m+1}\right) = \frac{6m}{2m+1}$. 
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**Pf.** **Pair player:** play nonincident pairs that hit all three parts (such as \( x_i y_j \) with \( y_k z_l \)), equally likely.
Complete Tripartite Graphs

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Against this distribution, the expected number of pairs separated by any ordering is at most \( \frac{2m+1}{6m} \).
Complete Tripartite Graphs

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Against this distribution, the expected number of pairs separated by any ordering is at most $\frac{2m+1}{6m}$.

**Ordering player**: Play all orderings $\nu_1, \ldots, \nu_{3m}$ with $\nu_{3i-2}, \nu_{3i-1}, \nu_{3i}$ in distinct parts, equally likely.
Thm. \( \pi_f(K_{m,m,m}) = 3 \left(1 - \frac{1}{2m+1}\right) = \frac{6m}{2m+1}. \)

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**Ordering player:** Play all orderings \( v_1, \ldots, v_{3m} \) with \( v_{3i-2}, v_{3i-1}, v_{3i} \) in distinct parts, equally likely.

Each pair hitting three parts separated w. prob. \( \frac{2m+1}{6m} \).

Each pair hitting two parts separated w. prob. \( \frac{m+1}{3m} \).

** Conj. ** For \( n = 3m \), the \( n \)-vertex \( K_4 \)-free graph maximizing \( \pi_f \) is \( K_{m,m,m} \).
Cycles

**Thm.** $\pi_f(C_n) = \frac{n}{n-2}$ for $n \geq 4$. 
Cycles

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**Pf. Upper:** For $x$, use the $n$ cyclic orders equally.
Cycles

**Thm.** \( \pi_f(C_n) = \frac{n}{n-2} \) for \( n \geq 4 \).

**Pf. Upper:** For \( x \), use the \( n \) cyclic orders equally. Nonincident \( e \) and \( e' \) are separated unless \( e \) or \( e' \) consists of the first and last vertex: \( \mathbb{P}(e : e') = \frac{n-2}{n} \).
Thm.  \( \pi_f(C_n) = \frac{n}{n-2} \) for \( n \geq 4 \).

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Lower: For \( y \), play the \( n \) pairs \( v_{i-1}v_i, v_{i+1}v_{i+2} \) equally.
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**Thm.**  $\pi_f(C_n) = \frac{n}{n-2}$ for $n \geq 4$.

**Pf. Upper:** For $x$, use the $n$ cyclic orders equally.
Nonincident $e$ and $e'$ are separated unless $e$ or $e'$
consists of the first and last vertex: $\mathbb{P}(e : e') = \frac{n-2}{n}$.

![Diagram](image)

**Lower:** For $y$, play the $n$ pairs $v_{i-1}v_i, v_{i+1}v_{i+2}$
equally. We show any ordering separates at most $n-2$ of these.
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![Diagram of cyclic orders](image)

**Lower:** For \( y \), play the \( n \) pairs \( v_{i-1}v_i, v_{i+1}v_{i+2} \) equally. We show any ordering separates at most \( n-2 \) of these. Otherwise, by symmetry \( \sigma \) separates them for \( 2 \leq i \leq n \), with \( v_1v_2 \) before \( v_3v_4 \).
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Lower: For \( y \), play the \( n \) pairs \( \nu_{i-1} \nu_i, \nu_{i+1} \nu_{i+2} \) equally. We show any ordering separates at most \( n-2 \) of these. Otherwise, by symmetry \( \sigma \) separates them for \( 2 \leq i \leq n \), with \( \nu_1 \nu_2 \) before \( \nu_3 \nu_4 \).

If \( \nu_i <_\sigma \nu_{i+2} \), then \( \nu_i \nu_{i+1} : \nu_{i+2} \nu_{i+3} \) requires \( \nu_{i+1} <_\sigma \nu_{i+3} \).
Cycles

**Thm.** \( \pi_f(C_n) = \frac{n}{n-2} \) for \( n \geq 4 \).

**Pf. Upper:** For \( \pi \), use the \( n \) cyclic orders equally. Nonincident \( e \) and \( e' \) are separated unless \( e \) or \( e' \) consists of the first and last vertex: \( P(e : e') = \frac{n-2}{n} \).

\[ \text{• • • • • • • •} \]

**Lower:** For \( y \), play the \( n \) pairs \( v_{i-1} v_i, v_{i+1} v_{i+2} \) equally. We show any ordering separates at most \( n-2 \) of these. Otherwise, by symmetry \( \sigma \) separates them for \( 2 \leq i \leq n \), with \( v_1 v_2 \) before \( v_3 v_4 \).

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Iterating, \( v_{n-2} <_\sigma v_n \) and \( v_{n-1} <_\sigma v_1 \).

Now \( v_1 <_\sigma \{ v_3 \text{ or } v_4 \} <_\sigma \cdots <_\sigma v_{n-1} <_\sigma v_1 \). ■
Girth 5

**Prop.** $\pi_f(Petersen) = \frac{30}{17}$. 
Girth 5

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Idea: Of the 75 nonincident pairs of edges, 15 are opposite on 6-cycles (Type 1) and 60 are not (Type 2).
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**Idea:** Of the 75 nonincident pairs of edges, 15 are opposite on 6-cycles (Type 1) and 60 are not (Type 2).

**Pair player:** Play the 60 Type 2 pairs, equally. Every ordering separates at most 34 Type 2 pairs: $\pi_f(G) \geq \frac{30}{17}$. 
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**Ordering player:** Use orderings that separate 34 Type 2 pairs and 9 Type 1. Note $\frac{9}{15} > \frac{17}{30}$, so $\pi_f(G) \leq \frac{30}{17}$. 
Prop. \( \pi_f(Petersen) = \frac{30}{17} \).

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- \( \pi_f(C_5) = \frac{5}{3} < \frac{30}{17} < 2 = \pi_f(C_4) \)
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**Conj.** \( \pi_f(G) < 2 \) if \( G \) has no cycle of length at most 4.
Girth 6

**Prop.** \( \pi_f(Heawood) = \frac{28}{17} \).
Girth 6

Prop. $\pi_f(Heawood) = \frac{28}{17}$.

Like Petersen graph, but searching is harder.
Girth 6

**Prop.** \( \pi_f(\text{Heawood}) = \frac{28}{17} \).

Like Petersen graph, but searching is harder.

- \( \pi_f(C_6) = \frac{6}{4} < \frac{28}{17} < \frac{5}{3} = \pi_f(C_5) \).
Girth 6

**Prop.** $\pi_f(\text{Heawood}) = \frac{28}{17}$.

Like Petersen graph, but searching is harder.

- $\pi_f(C_6) = \frac{6}{4} < \frac{28}{17} < \frac{5}{3} = \pi_f(C_5)$.

**Ques.** $\pi_f(G) < \frac{5}{3}$ if $G$ has no cycle of length at most 5?
Girth 6

**Prop.** \( \pi_f(\text{Heawood}) = \frac{28}{17} \).

Like Petersen graph, but searching is harder.

- \( \pi_f(C_6) = \frac{6}{4} < \frac{28}{17} < \frac{5}{3} = \pi_f(C_5) \).

**Ques.** \( \pi_f(G) < \frac{5}{3} \) if \( G \) has no cycle of length at most 5?

Increasing girth suggests better upper bounds on \( \pi_f(G) \) (recall \( \pi_f(C_n) = \frac{n}{n-2} \)), but trees don’t have \( \pi_f(G) = 1 \).
Subdivided Stars

**Thm.** \( \pi_f(G) = \frac{4m-2}{3m-1} \), where \( G \) is obtained from the star with \( 2m \) edges by subdividing each edge.
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**Thm.**  \( \pi_f(G) = \frac{4m-2}{3m-1} \), where \( G \) is obtained from the star with \( 2m \) edges by subdividing each edge.

\[
\begin{array}{c}
\text{Pf. Ordering Player: } \text{Play all orderings having } \nu \text{ in the middle and consecutive pairs } x_i, y_i \text{ in any order, equally.} \\
\end{array}
\]

\[
\begin{array}{c}
\nu \\
\begin{array}{c}
x_1 \\
y_1 \\
x_2m \\
y_2m
\end{array}
\end{array}
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\[ \begin{array}{c}
& \text{\( \nu \)} \\
\text{x}_1 & \text{\( y_1 \)} & \text{\( \text{x}_{2m} \)} & \text{\( \text{y}_{2m} \)} \\
\text{y}_1 & \vdots & \vdots & \vdots \\
\end{array} \]

**Pf. Ordering Player:** Play all orderings having \( \nu \) in the middle and consecutive pairs \( \text{x}_i, \text{y}_i \) in any order, equally.

\[ \begin{array}{c}
\text{x}_2, \text{y}_2, \text{x}_1, \text{y}_1, \text{x}_{2m} \text{y}_{2m}, \nu, \text{x}_4, \text{y}_4, \text{x}_5, \text{y}_5, \text{x}_3, \text{y}_3 \\
\end{array} \]

\[ \mathbb{P}(x_i y_i : x_j y_j) = 1. \]

\[ \mathbb{P}(x_i y_i : \nu x_j) = [2m^2 + m(m - 1)] \frac{(2m-2)!}{(2m)!} = \frac{3m-1}{4m-2}. \]
Subdivided Stars

**Thm.** \( \pi_f(G) = \frac{4m-2}{3m-1} \), where \( G \) is obtained from the star with \( 2m \) edges by subdividing each edge.

**Pf.** **Ordering Player:** Play all orderings having \( \nu \) in the middle and consecutive pairs \( x_i, y_i \) in any order, equally.

\[
\begin{align*}
\mathbb{P}(x_iy_i : x_jy_j) &= 1. \\
\mathbb{P}(x_iy_i : \nu x_j) &= \left[2m^2 + m(m - 1)\right] \frac{(2m-2)!}{(2m)!} = \frac{3m-1}{4m-2}.
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**Row Player:** Play the pairs \( x_iy_i, \nu x_j \) equally.
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x_2, y_2, x_1, y_1, x_{2m} y_{2m}, \nu, x_4, y_4, x_5, y_5, x_3, y_3
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**Row Player:** Play the pairs \( x_i y_i, \nu x_j \) equally. Argue that the orderings above are the best.
What Graphs Have $\pi_f(G) = 1$?
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**Thm.** If $G$ is a tree, then $\pi_f(G) = 1$ if and only if $G$ is a caterpillar (a path plus pendant edges).
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A natural ordering separates all nonincident pairs.

---

**Diagram:**

```
     1          3          6          9          11         13         15
      |          |          |          |          |          |          |
  2 |  4 |  5 |  7 |  8 | 10 | 12 | 14
```
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A non-caterpillar tree contains $Y$. Separating its pendant edges prevents inserting $\nu$. 

$x_1, y_1, \ldots, x_2, y_2, \ldots, x_3, y_3$
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If $G$ contains $C_n$ with $n \geq 4$, then $\pi_f(G) \geq \frac{n}{n-2}$.

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This is the same as the characterization of $\pi(G) = 1$ in B–C–G–M–R [2014], since $\pi_f(G) = 1 \iff \pi(G) = 1$. 
Thm. If $G$ is a tree, then $\pi_f(G) < \sqrt{2}$. 
Upper Bound for Trees

**Thm.** If \( G \) is a tree, then \( \pi_f(G) < \sqrt{2} \).

**Pf.** Root \( G \) at some vertex \( v \). Three types of pairs:

- **Type 1:** \( a \rightarrow c \rightarrow b \)
- **Type 2:** \( a \rightarrow d \rightarrow b \)
- **Type 3:** \( a \rightarrow \) (or any other combination where \( c = w \))
Upper Bound for Trees

**Thm.** If $G$ is a tree, then $\pi_f(G) < \sqrt{2}$.

**Pf.** Root $G$ at some vertex $v$. Three types of pairs:

- **Type 1**:
  - $w$ is the root.
  - $a < c < b < d$

- **Type 2**: $c = w$.
  - $a < d < b$

- **Type 3**: $c = w$.
  - $a < d < b$

Build a probability distribution on orderings so that:
Upper Bound for Trees

**Thm.** If $G$ is a tree, then $\pi_f(G) < \sqrt{2}$.

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- **Type 1** pairs: $b$ and $c$.
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Build a probability distribution on orderings so that:

- Type 1 pairs separated with probability 1.
Upper Bound for Trees

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**Thm.** If $G$ is a tree, then $\pi_f(G) < \sqrt{2}$.

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- **Type 2**

- **Type 3**

Build a probability distribution on orderings so that:

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- Type 3 pairs separated with probability $\geq \frac{1}{\sqrt{2}}$. 
Algorithm for Upper Bound

**Ordering Player** generates an ordering with these properties at random, from the root $v$ on down.

![Tree Diagrams]

Type 1

Type 2

Type 3

Let $\beta = \frac{1}{\sqrt{2}}$
**Algorithm for Upper Bound**

**Ordering Player** generates an ordering with these properties at random, from the root $v$ on down.

- Children of the root $v$ are assigned to the left or right of $v$ with probability $\frac{1}{2}$, independently.

Let $\beta = \frac{1}{\sqrt{2}}$
Algorithm for Upper Bound

**Ordering Player** generates an ordering with these properties at random, from the root $v$ on down.

- Children of the root $v$ are assigned to the left or right of $v$ with probability $\frac{1}{2}$, independently.
- Children of a non-root $u$ are put between $u$ and its parent $u'$ with prob $1 - \beta$; opposite from $u'$ with prob $\beta$.

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Algorithm for Upper Bound

**Ordering Player** generates an ordering with these properties at random, from the root $\nu$ on down.

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Only descendants of $u$ lie between $u$ and a child of $u$. Thus Type 1 never fails to be separated.
Probability for Types 2 and 3

Type 2 fails separation only if $a$ is between $c$ and $d$, meaning the child of $d$ above $a$ is placed between $d$ and its parent: prob $1 - \beta$. Hence $\mathbb{P}(ab : cd) = \beta = \frac{1}{\sqrt{2}}$. 

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Type 3 also fails only if $a$ is between $c$ and $d$, needing $a$ and $d$ on the same side of $c$. This has prob $(1 - \beta)^2 + \beta^2$, and then $a$ is between $c$ and $d$ with prob $\frac{1}{2}$.
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If $c = w = \nu$, then $a$ and $d$ are on the same side with prob $\frac{1}{2}$, and $\mathbb{P}(ab : cd) = \frac{3}{4}$. 
Open Questions

**Ques.** On $n$-vertex trees, what is $\max \pi_f$?

To exceed $\frac{4}{3}$, pair player must play some pairs that are Type 2 and some that are Type 3 without $w = \text{root}$. 
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**Ques.** For $n$-vertex $K_4$-free graphs, is $\pi_f(G)$ maximized for $G$ in $\{K_m,m,m, K_{m+1},m,m, K_{m+2},m,m\}$?
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Ques. Which rational numbers $x$ satisfy $\pi_f(G)=x$ for some graph $G$?