Fractional and Circular Separation Dimension of Graphs

Sarah J. Loeb* and Douglas B. West†

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Abstract

The *separation dimension* of a graph G, written $\pi(G)$, is the minimum number of linear orderings of $V(G)$ such that every two nonincident edges are "separated" in some ordering, meaning that both endpoints of one edge appear before both endpoints of the other. We introduce the *fractional separation dimension* $\pi_f(G)$, which is the minimum of a/b such that some a linear orderings (repetition allowed) separate every two nonincident edges at least b times.

In contrast to separation dimension, fractional separation dimension is bounded: always $\pi_f(G) \le 3$, with equality if and only if G contains K_4. There is no stronger bound even for bipartite graphs, since $\pi_f(K_{m,m}) = \pi_f(K_{m+1,m}) = \frac{3m}{m+1}$. We also compute $\pi_f(G)$ for cycles and some complete tripartite graphs. We show that $\pi_f(G) < \sqrt{2}$ when G is a tree and present a sequence of trees on which the value tends to 4/3.

Finally, we consider analogous problems for circular orderings, where pairs of nonincident edges are separated unless their endpoints alternate. Let $\pi^\circ(G)$ be the number of circular orderings needed to separate all pairs and $\pi^\circ_f(G)$ be the fractional version. Among our results: (1) $\pi^\circ(G) = 1$ if and only G is outerplanar. (2) $\pi^\circ(G) \le 2$ when G is bipartite. (3) $\pi^\circ(K_n) \ge \log_2 \log_3(n-1)$. (4) $\pi^\circ_f(G) \le \frac{3}{2}$ for every graph G, with equality if and only if $K_4 \subseteq G$. (5) $\pi^\circ_f(K_{m,m}) = \frac{3m-3}{2m-1}$.

1 Introduction

A pair of nonincident edges in a graph G is *separated* by a linear ordering of $V(G)$ if both vertices of one edge precede both vertices of the other. The *separation dimension* $\pi(G)$ of a graph G is the minimum number of vertex orderings that together separate every pair of nonincident edges of G. Graphs with at most three vertices have no such pairs, so their separation dimension is 0. We therefore consider only graphs with at least four vertices.

Introduced by Basavaraju (B), Chandran (C), Golumbic (G), Mathew (M), and Rajendraprasad (R) [4] (full version in [5]), separation dimension is motivated by a geometric

*College of William and Mary, Williamsburg, VA, sjloeb@wm.edu. Research supported in part by a gift from Gene H. Golub to the Mathematics Department of the University of Illinois.

†Zhejiang Normal University, Jinhua, China, and University of Illinois, Urbana, IL, west@math.uiuc.edu. Research supported by Recruitment Program of Foreign Experts, 1000 Talent Plan, State Administration of Foreign Experts Affairs, China.

interpretation. By viewing the orderings as giving coordinates for each vertex, the separation dimension is the least k such that the vertices of G can be embedded in $\mathbb{R}^k$ so that any two nonincident edges of G are separated by a hyperplane perpendicular to some coordinate axis (ties in a coordinate may be broken arbitrarily.)

The upper bounds on $\pi(G)$ proved by BCGMR [4, 5] include $\pi(G) \le 3$ when G is planar (sharp for K_4) and $\pi(G) \le 4\log_{3/2} n$ when G has n vertices. Since all pairs needing separation continue to need separation when other edges are added, $\pi(G) \le \pi(H)$ when $G \subseteq H$; call this fact *monotonicity*. By monotonicity, the complete graph K_n achieves the maximum among n-vertex graphs. In general, $\pi(G) \ge \log_2\lfloor \frac{1}{2}\omega(G)\rfloor$, where $\omega(G) = \max\{t\colon\ K_t \subseteq G\}$. This follows from the lower bound $\pi(K_{m,n}) \ge \log_2 \min\{m, n\}$ [4, 5] and monotonicity. Hence the growth rate of $\pi(K_n)$ is logarithmic.

BCMR [6] proved $\pi(G) \in O(k \log\log n)$ for the n-vertex graphs G in which every subgraph has a vertex of degree at most k. Letting K_n' denote the graph produced from K_n by subdividing every edge, they also showed $\pi(K_n') \in \Theta(\log\log n)$. Thus separation dimension is unbounded already on the family of graphs with average degree less than 4. In terms of the maximum vertex degree $\Delta(G)$, Alon and BCMR [2] proved $\pi(G) \le 2^{9\log_2^* \Delta(G)}\Delta(G)$. They also proved that almost all d-regular graphs G satisfy $\pi(G) \ge \lceil d/2\rceil$.

Separation dimension is equivalently the restriction of another parameter to the special case of line graphs. The *boxicity* of a graph G, written $\mathrm{box}(G)$, is the least k such that G can be represented by assigning each vertex an axis-parallel box in $\mathbb{R}^k$ (that is, a cartesian product of k intervals) so that vertices are adjacent in G if and only if their assigned boxes intersect. The initial paper [4] observed that $\pi(G) = \mathrm{box}(L(G))$, where $L(G)$ denotes the line graph of G (including when G is a hypergraph).

A variant of separation dimension was introduced by Dusart, Ziedan, and GMR [10]. *Induced separation dimension* requires separation only of pairs whose vertex sets induce exactly two edges. The parameter behaves quite differently from separation dimension. The authors note that monotonicity does not hold for this parameter and prove that it is NP-complete to determine which graphs have induced separation dimension 1.

We study a fractional version of separation dimension, using techniques that apply for hypergraph covering problems in general. Given a hypergraph H, the *covering number* $\tau(H)$ is the minimum number of edges in H whose union is the full vertex set. For separation dimension $\pi(G)$, the vertex set of H is the set of pairs of nonincident edges in G, and the edges of H are the sets of pairs separated by a single ordering of $V(G)$. Many minimization problems, including chromatic number, domination, poset dimension, and so on, can be expressed in this way.

Given a hypergraph covering problem, the corresponding fractional problem considers the difficulty of covering each vertex multiple times and measures the "long-term" average number of edges needed. More precisely, the *t-fold covering number* $\tau_t(H)$ is the least number of edges in a list of edges (repetition allowed) that covers each vertex at least t times, and the *fractional covering dimension* is $\liminf_t \tau_t(H)/t$. In the special case that H is the hypergraph associated with separation dimension, we obtain the *t-fold separation dimension* $\pi_t(G)$ and the *fractional separation dimension* $\pi_f(G)$.

Every list of s edges in a hypergraph H provides an upper bound on $\tau_f(H)$; if it covers each vertex at least t times, then it is called an *$(s\colon t)$-covering*, and $\tau_f(H) \le s/t$. This observation suffices to determine the maximum value of the fractional separation dimension. It is bounded, even though the separation dimension is not (recall $\pi(K_n) \ge \log\lfloor n/2 \rfloor$). Although fractional versions exist for all hypergraph covering parameters, this behavior of having the fractional version bounded while the integer version is unbounded seems unusual. In particular, although the ratio between the chromatic number and the fractional chromatic number is unbounded, the fractional chromatic number itself is also unbounded.

Theorem 1.1. *$\pi_f(G) \le 3$ for any graph G, with equality when $K_4 \subseteq G$.*

Proof. We may assume $|V(G)| \ge 4$, since otherwise there are no separations to be established and $\pi_f(G) \le \pi(G) = 0$. Now consider the set of all linear orderings of $V(G)$. For any two nonincident edges ab and cd, consider fixed positions of the other $n-4$ vertices in a linear ordering. There are 24 such orderings, and eight of them separate ab and cd. Grouping the orderings into such sets shows that ab and cd are separated $n!/3$ times. Hence $\pi_f(G) \le 3$.

Now suppose $K_4 \subseteq G$. In a copy of K_4 there are three pairs of nonincident edges, and every linear ordering separates exactly one of them. Hence to separate each at least t times, $3t$ orderings must be used. We obtain $\pi_t(G) \ge 3t$ for all t, so $\pi(G) \ge 3$. □

When G is disconnected, the value on G of π_t for any t (and hence also the value of π_f) is just its maximum over the components of G. We therefore focus on connected graphs. Also monotonicity holds for π_f just as for π.

Fractional versions of hypergraph covering problems are discussed in the book of Scheinerman and Ullman [9]. For every hypergraph covering problem, the fractional covering number is the solution to the linear programming relaxation of the integer linear program specifying $\tau(H)$. One can use this to express $\tau_f(G)$ in terms of a matrix game; we review this transformation in Section 2 to make our presentation self-contained. The resulting game yields a strategy for proving results about $\tau_f(H)$ and in particular about $\pi_f(G)$.

In Section 3, we characterize the extremal graphs for fractional separation dimension, proving that $\pi_f(G) = 3$ only when $K_4 \subseteq G$. No smaller bound can be given even for bipartite graphs; we prove $\pi_f(K_{m,m}) = \frac{3m}{m+1}$.

In Sections 4 and 5 we consider sparser graphs. The *girth* of a graph is the minimum length of its cycles (infinite if it has no cycles). In Section 4 we show $\pi_f(C_n) = \frac{n}{n-2}$. Also, the value is $\frac{30}{17}$ for the Petersen graph and $\frac{28}{17}$ for the Heawood graph. Although these results suggested asking whether graphs with fixed girth could admit better bounds on separation number, Alon [1] pointed out by using expander graphs that large girth does not permit bounding $\pi_f(G)$ by any constant less than 3 (see Section 4). Since expander graphs are nonplanar, the question remains open for planar graphs.

Question 1.2. *How large can $\pi_f(G)$ be when G is a planar graph with girth at least g?*

In Section 5, we consider trees. In contrast to Alon's result [1] permitting π_f to approach 3 for graphs with large girth, we prove that $\pi_f(G) < \sqrt{2}$ when G has no cycles. The bound improves to $\pi_f(T) \le \frac{4}{3}$ for trees obtained from a subdivision of a star by adding any number of pendant edges at each leaf. This is sharp; the tree with $4m + 1$ vertices obtained by subdividing every edge of $K_{1,2m}$ (replacing every edge of $K_{1,2m}$ with a path of length 2) has diameter 4 and fractional separation dimension $\frac{4m-2}{3m-1}$, which tends to $\frac{4}{3}$. We believe that the optimal bound for trees is strictly between $\frac{4}{3}$ and $\sqrt{2}$.

Question 1.3. *What is the supremum of $\pi_f(G)$ when G is a tree?*

In Section 6, we return to the realm of dense graphs with values of π_f near 3. We start with bipartite graphs, computing $\pi_f(K_{m+1,qm})$. The formula yields $\pi_f(K_{m,r}) < 3(1 - \frac{1}{2m-1})$ for all r, so both parts of the bipartite graph must grow to obtain a sequence of bipartite graphs on which π_f tends to 3. In the special case $q = 1$, we obtain $\pi_f(K_{m+1,m}) = \frac{3m}{m+1}$. Thus $\pi_f(K_{\lceil n/2\rceil,\lfloor n/2\rfloor}) = \frac{3m}{m+1}$, where $m = \lfloor n/2\rfloor$.

Conjecture 1.4. *Among bipartite n-vertex graphs, π_f is maximized by $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$.*

In seeking to maximize π_f over n-vertex graphs not containing K_4, we consider complete tripartite graphs. For the n-vertex K_4-free graph having the most edges, we prove $\pi_f(K_{m,m,m}) = \frac{6m+2}{2m+1}$. When $n = 6r$, we thus have $\pi_f(K_{2r,2r,2r}) > \pi_f(K_{3r,3r})$. Surprisingly, the value is larger for a quite different complete tripartite graph.

Conjecture 1.5. *For $n \ge 10$, the n-vertex graph not containing K_4 that maximizes π_f is $K_{1,\lfloor (n-1)/2\rfloor,\lceil (n-1)/2\rceil}$.*

Computer search verifies the conjecture for $n \ge 14$. For $n = 9$ there is an anomaly, with $\pi_f(K_{3,3,3}) > \pi_f(K_{1,4,4})$.

The sets of values achievable by fractional parameters have quite varied behavior. For example, all rational values at least 2 occur as the fractional chromatic number of some graph. On the other hand, the fractional matching number of any graph is half an integer. Motivated by the examples, we ask the following.

Question 1.6. *Which rational numbers (between 1 and 3) occur as the fractional separation dimension of some graph?*

Finally, in Section 7 we consider the analogues of π and π_f defined by using circular orderings of the vertices rather than linear orderings; we use the notation π° and π_f°. We show first that $\pi^\circ(G) = 1$ if and only if G is outerplanar. Surprisingly, $\pi^\circ(K_{m,n}) = 2$ when $m, n \geq 2$ and $mn > 4$, but π° is unbounded, with $\pi^\circ(K_n) > \log_2 \log_3(n-1)$. This contrasts with the linear version of separation dimension, which is unbounded on both $K_{n,n}$ and K_n. In a subsequent paper, Bharathi, De, and Lahiri [7] prove $\pi^\circ(G) = 2$ for 2-outerplanar graphs and $\pi^\circ(G) \leq 2$ for series-parallel graphs.

For the fractional context, we prove $\pi_f^\circ(G) \leq \frac{3}{2}$ for all G, with equality if and only if $K_4 \subseteq G$. Again no better bound holds for bipartite graphs; we prove $\pi_f^\circ(K_{m,qm}) = \frac{6(qm-1)}{4mq+q-3}$, which tends to $\frac{3}{2}$ as $m \to \infty$ when $q = 1$. It tends to $\frac{6m}{4m+1}$ when $q \to \infty$, so again both parts of the bipartite graph must grow to obtain a sequence on which π_f° tends to $\frac{3}{2}$. The proof is different from the linear case. The questions remaining are analogous to those for π_f.

Question 1.7. *How large can π_f° be when G is a planar graph with girth at least g? Which are the n-vertex graphs maximizing π_f° among bipartite graphs and among those not containing K_4? Which rational numbers between* 1 *and* $\frac{3}{2}$ *occur?*

2 Fractional Covering and Matrix Games

Given a hypergaph H with vertex set $V(H)$ and edge set $E(H)$, let $E_v = \{e \in E(H)\colon\ v \in e\}$ for $v \in V(H)$. The covering number $\tau(H)$ is the solution to the integer linear program "minimize $\sum_{e \in E(H)} x_e$ such that $x_e \in \{0,1\}$ for $e \in E(H)$ and $\sum_{e \in E_v} x_e \geq 1$ for $v \in V(H)$." The linear programming relaxation replaces the constraint $x_e \in \{0,1\}$ with $0 \leq x_e \leq 1$.

It is well known (see Theorem 1.2.1 of [9]) that the resulting solution τ^* equals $\tau_f(H)$. Multiplying the values in that solution by their least common multiple t yields a list of edges covering each vertex at least t times, and hence $\tau_f(H) \leq \tau^* t/t$. Similarly, normalizing an $(s:t)$-covering yields $\tau^* \leq s/t$. Note that since the solution to a linear program with integer constraints is always rational, always $\tau_f(H)$ is rational (when H is finite).

A subsequent transformation to a matrix game yields a technique for proving bounds on $\tau_f(H)$. The constraint matrix M for the linear program has rows indexed by $E(H)$ and columns indexed by $V(H)$, with $M_{e,v} = 1$ when $v \in e$ and otherwise $M_{e,v} = 0$. In the resulting matrix game, the edge player chooses a row e and the vertex player chooses a column v, and the outcome is $M_{e,v}$. In playing the game repeatedly, each player uses a strategy that is a probability distribution over the options, and then the expected outcome is the probability that the chosen vertex is covered by the chosen edge. The edge or "covering" player wants to maximize this probability; the vertex player wants to minimize it.

Using the probability distribution x over the rows guarantees outcome at least the smallest entry in $x^T M$, no matter what the vertex player does. Hence the edge player seeks a probability distribution x to maximize t such that $\sum_{e \in E_v} x_e \geq t$ for all $v \in V(H)$. Dividing by t turns this into the linear programming formulation for $\tau_f(H)$, with the resulting optimum being $1/t$. This yields the following relationship.

Proposition 2.1. *(Theorem 1.4.1 of [9]) If M is the covering matrix for a hypergraph H, then $\tau_f(H) = 1/t$, where t is the value of the matrix game given by M.*

Just as any strategy x for the edge player establishes $\min x^T M$ as a lower bound on the value, so any strategy y for the vertex player establishes $\max My$ as an upper bound. The value is established by providing strategies x and y so that these bounds are equal. As noted in [9], such strategies always exist.

For fractional separation dimension, we thus obtain the *separation game*. The rows correspond to vertex orderings and the columns to pairs of nonincident edges. The players are the *ordering player* and the *pair player*, respectively. To prove $\pi_f(G) \le 1/t$, it suffices to find a distribution for the ordering player such that each nonincident pair is separated with probability at least t. To prove $\pi_f(G) \ge 1/t$, it suffices to find a distribution for the pair player such that for each ordering the probability that the chosen pair is separated is at most t.

The proof of Theorem 1.1 can be phrased in this language. By making all vertex orderings equally likely, the ordering player achieves separation probability exactly $\frac{1}{3}$ for each pair, yielding $\pi_f(G) \le 3$. By playing the three nonincident pairs in a single copy of K_4 equally likely and ignoring all other pairs, the pair player achieves separation probability exactly $1/3$ against any ordering, yielding $\pi_f(G) \ge 3$.

Another standard result about these games will be useful to us. Let $\mathcal{P}$ denote the set of pairs of nonincident edges in a graph G. Symmetry in G greatly simplifies the task of finding an optimal strategy for the pair player.

Proposition 2.2. *(Exercise 1.7.3 of [9]) If, for any two pairs of nonincident edges in a graph G, some automorphism of G maps one to the other, then there is an optimal strategy for the pair player in which all pairs in $\mathcal{P}$ are made equally likely. In general, there is an optimal strategy that is constant on orbits of the pairs under the automorphism group of G.*

Proof. Consider an optimal strategy y, yielding $\max My = t$. Automorphisms of G induce permutations of the coordinates of y. The entries in My' for any resulting strategy y' are the same as in My. Summing these vectors over all permutations and dividing by the number of permutations yields a strategy y^* that is constant over orbits and satisfies $\max My^* \le t$. □

When there is an optimal strategy in which the pair player plays all pairs in $\mathcal{P}$ equally, the value of the separation game is just the largest fraction of $\mathcal{P}$ separated by any ordering. For $\tau_f(H)$ in general, Proposition 1.3.4 in [9] states this by saying that for a vertex-transitive hypergraph H, always $\tau_f(H) = |V(H)|/r$, where r is the maximum size of an edge. For separation dimension, this yields the following:

Corollary 2.3. *Let G be a graph. If for any two pairs of nonincident edges in G, there is an automorphism of G mapping one pair of edges to the other, then $\tau_f(G) = q/r$, where q is the number of nonincident pairs of edges in G and r is the maximum number of pairs separated by any vertex ordering.*

3 Characterizing the Extremal Graphs

When $K_4 \not\subseteq G$, we can separate $\pi_f(G)$ from 3 by a function of n.

Theorem 3.1. *If G is an n-vertex graph and $K_4 \not\subseteq G$, then $\pi_f(G) \le 3\left(1 - \frac{12}{n^4} + O(\frac{1}{n^5})\right)$.*

Proof. Let $p = \frac{1}{3} + \frac{4(n-4)!}{n!}$; note that $1/p$ has the form $3\left(1 - \frac{12}{n^4} + O(\frac{1}{n^5})\right)$. It suffices to give a probability distribution on the orderings of $V(G)$ such that each nonincident pair of edges is separated with probability at least p. We do this by modifying the list of all orderings.

Choose any four vertices $a, b, c, d \in V(G)$. For each ordering ρ of the remaining $n-4$ vertices, 24 orderings begin with $\{a, b, c, d\}$ and end with ρ. By symmetry, we may assume $ac \notin E(G)$. Thus the possible pairs of nonincident edges induced by $\{a, b, c, d\}$ are $\{ab, cd\}$ and $\{ad, bc\}$. We increase the separation probability for these pairs, even though these four edges need not all exist.

The pairs $\{ab, cd\}$ and $\{ad, bc\}$ are each separated eight times in the list of 24 orderings. We replace these 24 with another list of 24 (that is, the same total weight) that separate $\{ab, cd\}$ and $\{ad, bc\}$ each at least twelve times, while any other pair of disjoint vertex pairs not involving $\{a, c\}$ is separated at least eight times. Since $\{a, b, c, d\}$ is arbitrary and we do this for each 4-set, the pairs $\{ab, cd\}$ and $\{ad, bc\}$ remain separated at least eight times in all other groups of 24 orderings. Thus the separation probability increases from $\frac{1}{3}$ to at least p for all pairs of nonincident edges.

Use four orderings each that start with $abcd$ or $bcad$ and eight each that start with $cdba$ or $adbc$, always followed by ρ. By inspection, each of $\{ab, cd\}$ and $\{ad, bc\}$ is separated twelve times in the list. The number of orderings that separate any pair of nonincident edges having at most two vertices in $\{a, b, c, d\}$ does not change.

It remains only to check pairs with three vertices in this set, consisting of one edge induced by this set and another edge with one endpoint in the set. The induced edge is one of $\{ab, cd, bc, ad, bd\}$ (never ac), and the other edge uses one of the remaining two vertices in $\{a, b, c, d\}$. In each case, the endpoints of the induced edge appear before the third vertex in at least eight of the orderings in the new list of 24; this completes the proof. □

For n-vertex graphs not containing K_4, Theorem 3.1 separates $\pi_f(G)$ from 3 by a small amount. We believe that a much larger separation also holds (Conjecture 1.5). Nevertheless, we show next that even when G is bipartite there is no upper bound less than 3.

Theorem 3.2. $\pi_f(K_{m,m}) = \frac{3m}{m+1}$ *for $m \ge 2$.*

Proof. The pairs in $\mathcal{P}$ all lie in the same orbit under automorphisms of $K_{m,m}$, so Corollary 2.3 applies. There are $2\binom{m}{2}^2$ pairs in $\mathcal{P}$ (played equally by the pair player). It suffices to show that the maximum number of pairs separated by any ordering is $\frac{m+1}{3m} 2\binom{m}{2}^2$.

Let the parts of $K_{m,m}$ be X and Y. Let σ be an ordering $v_1, \ldots, v_{2m}$ such that each pair $\{v_{2i-1}, v_{2i}\}$ consists of one vertex of X and one vertex of Y. The ordering player will in fact make all such orderings equally likely. It suffices to show that σ separates $\frac{m+1}{3m} 2\binom{m}{2}^2$ pairs and that no ordering separates more.

By symmetry, we may index X as $x_1, \ldots, x_m$ and Y as $y_1, \ldots, y_m$ in order in σ, so that $\{v_{2i-1}, v_{2i}\} = \{x_i, y_i\}$ for $1 \le i \le m$, though x_i and y_i may appear in either order. Consider an element of $\mathcal{P}$ separated by σ. The vertices involved in the separation may use two, three, or four indices among 1 through m.

Pairs hitting i, j, k, l with $i < j < k < l$ must be separating $x_i y_j$ or $y_i x_j$ from $x_k y_l$ or $y_k x_l$. Hence there are $4\binom{m}{4}$ such pairs.

Pairs hitting only i, j, k with $i < j < k$ involve two vertices with the same index. If that index is i or k, then there are two ways to complete the edge pair. However, if x_j and y_j are both used, then there is only one way to choose from $\{x_i, y_i\}$ and from $\{x_k, y_k\}$ to complete a separated pair, determined by the order of x_j and y_j. Hence there are $5\binom{m}{3}$ such pairs.

A separated pair hitting only i and j must be $\{x_i y_i, x_j y_j\}$. Hence in total σ separates $4\binom{m}{4} + 5\binom{m}{3} + \binom{m}{2}$ pairs in $\mathcal{P}$. In fact, this sum equals $\frac{m+1}{3m} 2\binom{m}{2}^2$.

Now let σ be an ordering not of the specified form. By symmetry we may again index X as $x_1, \ldots, x_m$ and Y as $y_1, \ldots, y_m$ in order in σ. However, now some vertex precedes another vertex with a lesser index. That is, by symmetry we may assume that y_j appears immediately before x_i for some i and j with $j > i$.

Form σ' from σ by interchanging the positions of y_j and x_i. Any pair separated by exactly one of σ and σ' has x_i and y_j as endpoints of the two distinct edges. There are $(i-1)(m-j)$ such pairs in σ and $(j-1)(m-i)$ such pairs in σ'. Since $m \ge 2$ and $j > i$, comparing these quantities shows that σ' separates strictly more pairs than σ. $\square$

To prove that always $\pi_f(K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}) = \frac{3m}{m+1}$, where $m = \lfloor n/2 \rfloor$, we need also to compute $\pi_f(K_{m+1,m})$. We postpone this to Section 6. Note that the simple final expression arises when we cancellation common factors in the numerator and denominator. We would hope that such a simple formula has a simple direct proof, but we have not found one.

4 Graphs with Larger Girth

Among sparser graphs, it is natural to think first about cycles.

Proposition 4.1. *$\pi_f(C_n) = \frac{n}{n-2}$, for $n \ge 4$.*

Proof. The ordering player uses the n rotations of an n-vertex path along the cycle, equally likely. Nonincident edges e and e' are separated unless e or e' consists of the first and last vertex. Hence any pair in $\mathcal{P}$ is separated with probability $\frac{n-2}{n}$.

Letting the vertices be $v_1, \ldots, v_n$ in order along the cycle, the pair player makes the pairs $\{v_{i-1}v_i, v_{i+1}v_{i+2}\}$ (modulo n) equally likely. It suffices to show that any ordering separates at most $n-2$ of these pairs. Otherwise, by symmetry some ordering σ separates the $n-1$ of them satisfying $2 \le i \le n$. By symmetry $\{v_1, v_2\}$ precedes $\{v_3, v_4\}$ in σ. If v_i precedes v_{i+2}, then separating $v_i v_{i+1}$ from $v_{i+2}v_{i+3}$ requires v_{i+1} to precede v_{i+3}. Iterating this argument yields v_{n-2} before v_n and v_{n-1} before v_1 in σ. Since v_1 precedes both v_3 and v_4, choosing the right one by parity leads to v_1 preceding v_1, a contradiction. □

Proposition 4.1 suggests that π_f decreases as girth increases. We compute π_f for the smallest 3-regular graphs of girth 5 (the Petersen graph) and girth 6 (the Heawood graph).

Proposition 4.2. *The fractional separation dimension of the Petersen graph is* $\frac{30}{17}$.

Proof. (Sketch) The vertices of the Petersen graph are the 2-element subsets of $\{1, 2, 3, 4, 5\}$; vertices are adjacent if and only if the corresponding 2-element sets are disjoint.

The 75 pairs of nonincident edges fall into two orbits: the 15 pairs that occur as opposite edges on a 6-cycle (Type 1) and the 60 pairs that do not (Type 2). Thus some optimal strategy for the pair player will make Type 1 pairs equally likely and make Type 2 pairs equally likely. There exist orderings that separate nine Type 1 pairs and 34 Type 2 pairs. Such an ordering σ is

$$(12, 34, 51, 23, 45, 13, 42, 35, 41, 25)$$

Since $\frac{9}{15} > \frac{34}{60}$, making all 120 orderings generated from σ by permutations of $\{1, 2, 3, 4, 5\}$ equally likely establishes $\frac{30}{17}$ as an upper bound on the fractional separation dimension.

Since $\frac{9}{15} > \frac{34}{60}$, the pair player establishes a matching lower bound by playing only Type 2 pairs, equally likely, if no ordering separates more than 34 Type 2 pairs. By checking the number of Type 2 pairs separated by each permutation, computer search confirms this. □

Proposition 4.3. *The fractional separation dimension of the Heawood graph is* $\frac{28}{17}$.

Proof. (Sketch) The vertices of the Heawood graph are the points and lines of the Fano plane; the points are the integers modulo 7, and the lines are the triples of the form $\{i, i+1, i+3\}$. The edges are the pairs consisting of a point and a line containing it.

The 168 pairs of nonincident edges fall into two orbits: 84 pairs are opposite on a a 6-cycle (Type 1), and 84 pairs have a common incident edge (Type 2). Some optimal strategy for the pair player makes Type 1 pairs equally likely and makes Type 2 pairs equally likely. There exist orderings separating 51 Type 2 pairs and 54 Type 1 pairs. Such an ordering is

$$1, 124, 4, 457, 5, 561, 6, 346, 3, 235, 2, 672, 7, 713$$

Making the 336 orderings generated by automorphisms equally likely establishes $\frac{28}{17}$ as an upper bound on the fractional separation dimension.

The pair player establishes a matching lower bound by playing only Type 2 pairs, equally likely, if no ordering separates more than 51 Type 2 pairs. By checking the number of Type 2 pairs separated by each permutation (up to symmetry under automorphisms), computer search confirms this. □

These small graphs suggested that perhaps $\pi_f(G) < 2$ when G has girth at least 5. However, Alon [1] observed using the Expander Mixing Lemma that expander graphs with large girth (such as Ramanujan graphs) still have π_f arbitrarily close to 3.

Lubotzky, Phillips, and Sarnak [8] introduced *Ramanujan graphs* as d-regular graphs in which every eigenvalue with magnitude less than d has magnitude at most $2\sqrt{d-1}$. For $d-1$ being prime, they further introduced an infinite family of such graphs whose girth is at least $\frac{2}{3}\log_{d-1} n$ when n is the number of vertices.

Let G be a d-regular n-vertex graph whose eigenvalues other than d have magnitude at most λ. The Expander Mixing Lemma of Alon and Chung [3] states that whenever A and B are two vertex sets in G, the number of edges of G joining A and B differs from $|A|\,|B|(d/n)$ by at most $\lambda\sqrt{|A|\,|B|}$ (edges with both endpoints in $A \cap B$ are counted twice).

Alon applied this lemma to an arbitrary vertex ordering σ of G, breaking σ into k blocks of consecutive vertices, each with length at most $\lceil n/k \rceil$. Intuitively, by the Expander Mixing Lemma the vast majority of the edges can be viewed as forming a blowup of a complete graph with k vertices. With k chosen to be about $d^{1/3}$, Alon shows that asymptotically only $\frac{d^2n^2}{24}$ pairs of nonincident edges can be separated by σ . However, there are asymptotically $\frac{d^2n^2}{8}$ pairs of nonincident edges. Thus every ordering can separate only about a third of the pairs. As noted, this graph G can be chosen to have arbitrarily large girth.

Alon extended the question in our Conjecture 1.5 by asking how small ϵ can be made so that there is an n-vertex graph G with girth at least g such that $\pi_f(G) \geq 3-\epsilon$. His detailed computations [1] with the error terms yield $\epsilon < n^{-c/g}$ for some positive constant c.

Graphs with good expansion properties are not planar. The original paper [4] proved $\pi(G) \leq 2$ for every outerplanar graph G, and hence also $\pi_f(G) \leq 2$. Equality holds for outerplanar graphs with 4-cycles. We suggest seeking sharp upper bounds for the family of outerplanar graphs with girth at least g, and similarly for planar graphs with girth at least g. For the latter question, we suggest first studying grids (cartesian products of two paths).

5 Trees

Although $\lim_{g\to\infty} \frac{g}{g-2} = 1$, it is not true that $\pi_f(G) = 1$ whenever G is a tree. The graphs G with $\pi_f(G) = 1$ are just the graphs with $\pi(G) = 1$, as holds for every hypergraph covering parameter. These graphs were characterized in BCGMR [4]. Each component is obtained from a path P by adding independent vertices that have one neighbor or two consecutive neighbors on P, but for any two consecutive vertices on P at most one common neighbor can be added.

This implies that the trees with fractional separation dimension 1 are the caterpillars. We seek the sharpest general upper bound for trees.

Theorem 5.1. *$\pi_f(G) < \sqrt{2}$ when G is a tree.*

Proof. We construct a strategy for the ordering player to show that the separation game has value at least $\frac{1}{\sqrt{2}}$. Since $\pi_f(G)$ is rational, the inequality is strict.

Root T at a vertex v. For a vertex u other than v, let u' be the parent of u. We describe the strategy for the ordering player by an iterative probabilistic algorithm that generates an ordering. Starting with v, we iteratively add the children of previously placed vertices according to the following rules, where β is a probability to be specified later and the choices for placing the vertices relative to their parents are made independently.

(R1) Each child of v is assigned to precede or follow v, with probability $\frac{1}{2}$ for each option.

(R2) Each child of a non-root vertex u is assigned to be between u and its parent u' with probability $1-\beta$ and assigned to be on the side of u away from u' with probability β.

(R3) For each vertex, the children assigned to precede it are placed immediately next to it in a (uniformly selected) random order. The same is true for the children assigned to follow it.

Note that the resulting ordering has the following property:

($*$) Any vertex between a vertex u and a child of u is a descendant of u.

We must prove that the separation probability is at least $\frac{1}{\sqrt{2}}$ for each pair of nonincident edges. Given nonincident edges ab and cd, let w denote the common ancestor of these vertices that is farthest from the root. We may assume $a = b'$ and $c = d'$. Without loss of generality, there are three types of pairs, as shown in Figure 1.

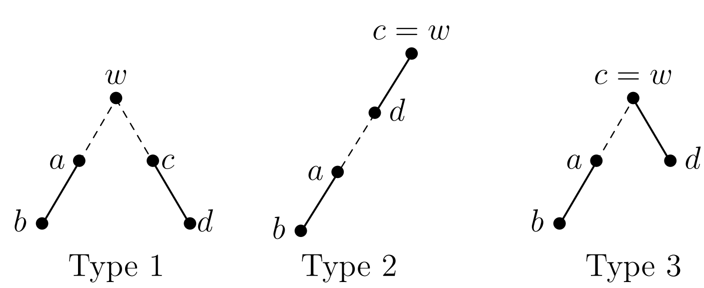

Figure 1: The three possible types of pairs.

In Type 1, neither edge contains an ancestor of a vertex in the other edge. Hence (∗) implies that no vertex of one edge can lie between the vertices of the other edge. Thus Type 1 pairs are separated with probability 1.

In Type 2, both vertices in one edge are descendants of the vertices in the other edge, say a and b below d. By (∗), the pair fails to be separated if and only if a is between c and d. This occurs if and only if the child of d on the path from d to a is placed between d and its parent, c. This occurs with probability $1 - \beta$, so the separation probability is β.

In Type 3, the vertices in ab are below c but not d. Again separation fails if and only if a is between c and d. This requires d and the child of c on the path to a to be placed on the same side of c, after which the probability of having a between c and d is $\frac{1}{2}$. The probability of having two specified children of c on the same side of c is $(1-\beta)^2 + \beta^2$ if $c \neq v$; it is $\frac{1}{2}$ if $c = v$. If $c = v$, then the separation probability is $\frac{3}{4}$, greater than $\frac{1}{\sqrt{2}}$. If $c \neq v$, then the separation probability is $1 - \frac{1}{2}(1-\beta)^2 - \frac{1}{2}\beta^2$.

We optimize by solving $\beta = 1 - \frac{1}{2}(1-\beta)^2 - \frac{1}{2}\beta^2$ and setting $\beta = \frac{1}{\sqrt{2}}$. Now each pair of nonincident edges is separated with probability at least $\frac{1}{\sqrt{2}}$. □

If a root v can be chosen in a tree G so that the all pairs of Type 3 involve v, then in the proof of Theorem 5.1 setting $\beta = \frac{3}{4}$ yields $\pi_f(G) \leq \frac{4}{3}$. This proves the following corollary.

Corollary 5.2. *$\pi_f(G) \leq \frac{4}{3}$ for any tree G produced from a subdivision of a star by adding any number of pendent vertices to each leaf.*

The bound in Corollary 5.2 cannot be improved.

Proposition 5.3. *$\pi_f(K'_{1,n}) = \frac{4m-2}{3m-1}$, where $m = \lceil n/2 \rceil$ and $K'_{1,n}$ is the graph obtained from $K_{1,n}$ by subdividing every edge once.*

Proof. Form $K'_{1,n}$ from the star with center v and leaves $y_1, \ldots, y_n$ by introducing x_i to subdivide vy_i, for $1 \leq i \leq n$. Let $X = x_1, \ldots, x_n$.

If in some ordering a vertex of degree 1 does not appear next to its neighbor, then moving it next to its neighbor does not make any separated pair unseparated. Hence the ordering player should play only orderings in which every vertex of degree 1 appears next to its neighbor; it does not matter on which side of its neighbor the vertex is placed.

Nonincident edges of the form x_iy_i and x_jy_j are always separated by any ordering that puts y_i next to x_i for all i; the pair player will not play these. The remaining $n(n-1)$ pairs of nonincident edges have the form $\{vx_i, x_jy_j\}$ and lie in a single orbit. By Corollary 2.3, some optimal strategy for the pair player makes them equally likely.

An optimal strategy for the ordering player will thus make equally likely all orderings obtained by permuting the positions of the pairs x_ry_r within an ordering that maximizes the number of separated pairs of the nontrivial form $\{vx_i, x_jy_j\}$. Such a pair is separated when x_i and x_j lie on opposite sides of v and when x_i is between v and x_j.

To count such pairs, it matters only how many vertices of X appear to the left of v, since y_i appears next to x_i for all i. If k vertices of X appear to the left of v, then the count of separated nontrivial pairs is $2k(n-k)+\binom{k}{2}+\binom{n-k}{2}$. This formula simplifies to $\binom{n}{2}+k(n-k)$, which is maximized only when $k \in \{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor\}$.

Thus the ordering player puts v in the middle, $\lfloor \frac{n}{2} \rfloor$ vertices of X on one side, and $\lceil \frac{n}{2} \rceil$ vertices of X on the other side. Whether n is $2m$ or $2m-1$, the ratio of $\binom{n}{2}+\lfloor \frac{n^2}{4} \rfloor$ to $n(n-1)$ simplifies to $\frac{3m-1}{4m-2}$, as desired. □

6 Complete Multipartite Graphs

To prove that always $\pi_f(K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}) = \frac{3m}{m+1}$, where $m = \lfloor n/2 \rfloor$, we need also to compute $\pi_f(K_{m+1,m})$. This is the special case $q = 1$ of our next theorem. We postponed it because the counting argument for the generalization is more technical than our earlier arguments.

Theorem 6.1. $\pi_f(K_{m+1,qm}) = 3\left(1 - \frac{(q+1)m-2}{(2m+1)mq-m-2}\right)$ *for* $m, q \in \mathbb{N}$ *with* $mq > 1$.

Proof. Note that $3\left(1 - \frac{(q+1)m-2}{(2m+1)mq-m-2}\right) = \frac{6m(mq-1)}{(2m+1)mq-m-2}$. Let $p = \frac{(2m+1)mq-m-2}{6m(mq-1)}$. The pairs in $\mathcal{P}$ all lie in the same orbit, so Corollary 2.3 applies, and the pair player can make all $2\binom{m+1}{2}\binom{mq}{2}$ pairs in $\mathcal{P}$ equally likely. It suffices to show that the maximum number of pairs separated by any ordering is $2p\binom{m+1}{2}\binom{mq}{2}$. The proof is similar to that of Theorem 3.2.

Let the parts of $K_{m+1,qm}$ be X and Y, with $|X| = m+1$ and $|Y| = qm$. Let σ be an ordering $v_0, \ldots, v_{(q+1)m}$ such that $v_i \in X$ if and only if $i \equiv 0 \mod (q+1)$. The ordering player will in fact make all such orderings equally likely. We show that σ separates $2p\binom{m+1}{2}\binom{mq}{2}$ pairs and that no ordering separates more.

Let $X = \{x_0, \ldots, x_m\}$, indexed in order of appearance in σ, and similarly let $Y = \{y_1, \ldots, y_{qm}\}$. Let $B_0 = \{x_0\}$, and for $1 \le i \le m$ let B_i consist of $\{y_{q(i-1)+1}, \ldots, y_{qi}, x_i\}$. To count pairs in $\mathcal{P}$ separated by σ, we consider which blocks contain the vertices used.

If the indices are i, j, k, l with $1 \le i < j < k < l \le m$, then one edge consists of x_i or x_j and a Y-vertex from the other block among $\{B_i, B_j\}$, and similarly for $\{B_k, B_l\}$. Hence there are $4q^2\binom{m}{4}$ such pairs. If $i = 0$, then we must use x_0, and there are $2q^2\binom{m}{3}$ such pairs.

If the indices are i, j, k with $1 \le i < j < k \le m$, then we use two vertices from one block. If we use two in B_i, then other edge uses x_j or x_k and a Y-vertex from the remaining block, yielding $2q^2$ separated pairs. Similarly, $2q^2$ separated pairs use two vertices in B_k. Two vertices used from B_j may both be from Y or may include x_j. In the first case x_i and x_k are used, while in the second case x_j and x_i are used; thus the vertices from Y can be chosen in $\binom{q}{2} + q^2$ ways. Hence for such index choices a total of $(\binom{q}{2} + 5q^2)\binom{m}{3}$ pairs are separated.

If the indices are $0, j, k$ with $1 \le j < k \le m$, then x_i is used. If x_j is used, then there are q^2 ways to complete the pair of separated edges, and if x_k is used then there are $q^2 + \binom{q}{2}$ ways to complete it. Hence this case contributes $(\binom{q}{2} + 2q^2)\binom{m}{2}$ pairs.

If the indices are i and j with $1 \le i < j \le m$, then either we use two vertices from each of B_i and B_j (with one edge within each block) or we use three vertices from B_j and only the vertex x_i from B_i. This yields $(q^2 + \binom{q}{2})\binom{m}{2}$ separated pairs. If $i = 0$, then we must use x_0 and three vertices from B_j, for a total of $\binom{q}{2}m$ pairs.

Thus σ separates $[4q^2]\binom{m}{4} + [7q^2 + \binom{q}{2}]\binom{m}{3} + [3q^2 + 2\binom{q}{2}]\binom{m}{2} + \binom{q}{2}m$ pairs. Direct computation shows that this equals $2p\binom{m+1}{2}\binom{mq}{2}$. In particular, since $p = \frac{(2m+1)mq-m-2}{6m(mq-1)}$, the formula $2p\binom{m+1}{2}\binom{mq}{2}$ simplifies to $\frac{1}{12}[(2m+1)mq - m - 2](m+1)mq$, and indeed factoring $\frac{1}{12}mq$ out of the number of pairs separated leaves $[(2m+1)mq - m - 2](m+1)$.

It remains to show that no ordering separates more pairs than the orderings of this type. Let σ be an ordering not of this type. Index X and Y as before. If σ does not start with x_0, then let y be the vertex immediately preceeding x_0. Form σ' from σ by exchanging the positions of y and x_0. Since no pair of the form $x'y, x_0y'$ is separated by σ, every pair separated by σ is also separated by σ'.

Hence we may assume by symmetry that σ starts with x_0 and ends with x_m. If σ does not have the desired form, then by symmetry there is a least index j such that more than qj vertices of Y precede x_j, while fewer than $q(m-j)$ follow x_j. Form σ' by exchanging the positions of x_j and the vertex y immediately preceding it in σ. Let r be the number of vertices of Y preceding x_j The number of pairs separated by σ but not σ' is $j(mq-r)$, while the number separated by σ' but not σ is $(r-1)(m-j)$. The difference is $m(r-jq)-(m-j)$. Since $r > qj$ and $j < m$, the difference is positive, and σ' separates more pairs than σ. □

Since $\frac{1}{p} = \frac{6(mq-1)}{(2m+1)q-m-2/m} \le \frac{6m}{2m+1} = 3(1 - \frac{1}{2m+1})$, with equality only when $m = 1$, always $\pi_f(K_{m,r})$ is bounded away from 3 by a function of m. In particular, having π_f tend to 3 on a sequence of bipartite graphs requires the sizes of both parts to grow.

We expect that $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ maximizes π_f among n-vertex bipartite graphs. By monotonicity, the maximum occurs at $K_{k,n-k}$ for some k. We have $\pi_f(K_{m,m}) = \pi_f(K_{m+1,m}) = \frac{3m}{m+1}$. For unbalanced instances with $2m+1$ vertices (assuming integrality of ratios for simplicity), Theorem 6.1 yields $\pi_f(K_{\frac{2m}{q+1}+1,\frac{2qm}{q+1}})$. The value is highest for the balanced case.

It would be desirable to have a direct argument showing that moving a vertex from the larger part to the smaller part in $K_{k,n-k}$ increases π_f when $k \le n/2 - 1$. This would prove Conjecture 1.4. However, the statement surprisingly is not true in general for complete tripartite graphs. Computation has shown $\pi_f(K_{m+2,m,m}) > \pi_f(K_{m+1,m+1,m})$ when $2 \le m \le 4$. Even more surprising, by computing the values of π_f for $K_{m,m,m}$ and $K_{1,m,m}$, we obtain $\pi_f(K_{1,(n-1)/2,(n-1)/2}) > \pi_f(K_{n/3,n/3,n/3})$ when n is an odd multiple of 3. This follows from the remaining results in this section and motivates our Conjecture 1.5.

Theorem 6.2. $\pi_f(K_{m,m,m}) = \frac{6m}{2m+1}$ *for* $m \ge 2$.

Proof. There are two types of pairs in $\mathcal{P}$: those with endpoints in two parts, called *double-pairs* or *D-pairs*, and those with endpoints in all three parts, called *triple-pairs* or *T-pairs.*

Within these two types of pairs in $\mathcal{P}$, any pair can be mapped to any other pair via an automorphism, so by Corollary 2.3 some optimal strategy for the pair player makes D-pairs equally likely and makes T-pairs equally likely.

Let the parts of $K_{m,m,m}$ be X, Y, and Z. Let σ be an ordering $v_1, \dots, v_{3m}$ such that each triple $\{v_{3i-2}, v_{3i-1}, v_{3i}\}$ consists of one vertex from each part, for $1 \le i \le m$. The ordering player will make all such orderings equally likely. The restrictions of such orderings to two parts are the orderings used in Theorem 3.2, which separate the fraction $\frac{m+1}{3m}$ of the D-pairs.

We show that each such ordering separates the fraction $\frac{2m+1}{6m}$ of the T-pairs. This fraction is smaller than $\frac{m+1}{3m}$. Hence this strategy shows that the separation game has value at least $\frac{2m+1}{6m}$. By making the T-pairs equally likely, the pair player establishes equality if also no other ordering separates more T-pairs.

For use in the next theorem, we distinguish each T-pair as a W-pair, for $W \in \{X, Y, Z\}$, when W is the part contributing two vertices to the pair. Furthermore, with $w \in \{x, y, z\}$, we index W as $w_1, \dots, w_m$ in order of appearance in σ. Let the *block* B_i be $\{v_{3i-2}, v_{3i-1}, v_{3i}\}$, so $B_i = \{x_i, y_i, z_i\}$ for $1 \le i \le m$, though $\{x_i, y_i, z_i\}$ may appear in any order in σ. The vertices of a T-pair separated by σ may use two, three, or four indices in $\{1, \dots, m\}$. In each case, let W be the part contributing a vertex to each edge of the pair.

T-pairs hitting B_i, B_j, B_k, B_l with $i < j < k < l$ consist of one edge in $B_i \cup B_j$ and the other in $B_k \cup B_l$. We can choose the blocks for the two vertices of W in four ways (B_i or B_j, and B_j or B_k), and then we just choose which of the other two parts finishes the first edge. Hence there are $8\binom{m}{4}$ such W-pairs for each W, together $24\binom{m}{4}$ such T-pairs.

Pairs hitting only B_i, B_j, B_k with $i < j < k$ consist of one edge in $B_i \cup B_j$ and the other in $B_j \cup B_k$, with care in making a separated pair needed when B_j contributes two vertices. If the repeated part W contributes w_i and w_k, then there are five ways to complete the pair, one with two vertices from B_j and two each having an edge within B_i or B_j. If w_i and w_j are used, then there are two W-pairs having an edge within B_i. The number of W-pairs using a second vertex from B_j is $t-1$ when w_j is the tth vertex of B_j in σ, and none having two vertices from B_k. When w_j and w_k are used, the contributions depending on the position of w_j are reversed. Hence in each case 11 W-pairs are separated, for a total of $33\binom{m}{3}$ T-pairs.

A separated T-pair hitting only B_i and B_j uses w_i and w_j. Picking one additional vertex from each block yields two separated W-pairs. There is also one separated W-pair with three vertices in B_i if w_i is not the last vertex of B_i, and one with three vertices in B_j if w_j is not the first vertex of B_j. Summing over W yields $10\binom{m}{2}$ T-pairs of this type.

In total σ separates $24\binom{m}{4}+33\binom{m}{3}+10\binom{m}{2}$ T-pairs. This sum equals $\frac{2m+1}{m}\binom{m}{2}$. Altogether there are $6m^2\binom{m}{2}$ T-pairs, so the fraction of them separated is $\frac{2m+1}{6m}$, as desired.

For an ordering σ not of the specified form, index the vertices of each part in increasing order in σ. Avoiding the specified form means that some vertex precedes another vertex with a lesser index. By symmetry, we may assume that y_j appears immediately before x_i in σ for some i and j with $j > i$. Let k be the number of vertices of Z appearing before y_j.

Form σ' from σ by interchanging the positions of y_j and x_i. Any T-pair separated by exactly one of σ and σ' has x_i and y_j as endpoints of the two distinct edges. Considering whether a vertex of Z is used to complete the first, second, or both edges, there are $k(m-j)+(i-1)(m-k)+k(m-k)$ such T-pairs in σ and $k(m-i)+(j-1)(m-k)+k(m-k)$ such T-pairs in σ'. The difference is $m(j-i)$. Since $m \geq 2$ and $j > i$, the comparison shows that σ' separates strictly more T-pairs than σ. $\square$

We also compute the fractional separation dimension of $K_{m+1,m,m}$. As with $K_{m+1,m}$, the extra vertex imposes no extra cost.

Theorem 6.3. *$\pi_f(K_{m+1,m,m}) = \frac{6m}{2m+1}$ for $m \geq 2$.*

Proof. Let the parts of $K_{m+1,m,m}$ be X, Y, and Z with $|X| = m+1$. By monotonicity, $\pi_f(K_{m+1,m,m}) \geq \pi_f(K_{m,m,m}) = \frac{6m}{2m+1}$. To prove equality, it suffices to give a strategy for the ordering player that separates any pair in $\mathcal{P}$ with probability at least $\frac{2m+1}{6m}$. Given an ordering σ as $v_1, \ldots, v_{3m+1}$, let $B_i = \{v_{3i-2}, v_{3i-1}, v_{3i}\}$ for $1 \leq i \leq m$ as in Theorem 6.2. Use $W \in \{X, Y, Z\}$ and $W = \{w_1, \ldots, w_t\}$ as before, indexed as ordered in σ. The ordering player makes equally likely all orderings such that $(v_{3i-2}, v_{3i-1}, v_{3i}) = (x_i, y_i, z_i)$ in order, with x_{3m+1} at the end, and all those that switch Y and Z. By Corollary 2.3, it suffices to show that σ separates at least the fraction $\frac{2m+1}{6m}$ of the pairs in each orbit.

For the pairs in $\mathcal{P}$ with endpoints in only two parts, the number of pairs separated by σ depends only on the restriction of σ to those parts. The restriction is precisely an ordering used in Theorem 3.2 or Theorem 6.1. There we showed that the fraction of such pairs separated is $\frac{m+1}{3m}$, which is larger than $\frac{2m+1}{6m}$.

In remains to consider the T-pairs. As in Theorem 6.2, classify these as W-pairs for $W \in \{X, Y, Z\}$. The Y-pairs and Z-pairs are in one orbit, the X-pairs in another.

Deleting x_{3m+1} (the last vertex) leaves an ordering considered in Theorem 6.2. There we counted W-pairs within that ordering. There were the same number of separated T-pairs of each type, except for those hitting only two blocks. Since each block B_k appears in the order (x_k, y_k, z_k), each pair of blocks yields three separated X-pairs, four Y-pairs, and three Z-pairs among the 10 T-pairs counted earlier.

We conclude that the ordering separates $8\binom{m}{4} + 11\binom{m}{3} + 3\binom{m}{2}$ X-pairs and a total of $16\binom{m}{4} + 22\binom{m}{3} + 7\binom{m}{2}$ Y-pairs and Z-pairs not involving x_{3m+1}.

Separated T-pairs involving x_{3m+1} hit at most three earlier blocks. Using one vertex each from B_i, B_j, and B_k with $i < j < k$, we obtain $4\binom{m}{3}$ X-pairs and a total of $4\binom{m}{3}$ Y-pairs and Z-pairs. Using x_{3m+1} and vertices from B_i and B_j, there are $5\binom{m}{2}$ X-pairs, $2\binom{m}{2}$ Y-pairs and $\binom{m}{2}$ Z-pairs. Using x_{3m+1} and all three vertices of B_i, we obtain one X-pair, since x_i comes first, and no Y-pairs or Z-pairs.

Summing these possibilities, we find that σ separates $8\binom{m}{4} + 15\binom{m}{3} + 8\binom{m}{2} + m$ of the $m^3(m+1)$ X-pairs and $16\binom{m}{4} + 26\binom{m}{3} + 10\binom{m}{2}$ of the $2m^2(m^2-1)$ Y-pairs and Z-pairs. Remarkably, each ratio is exactly $\frac{2m+1}{6m}$. $\square$

In Theorem 6.2 we used more general orderings to simplify the optimality argument. Not needing that proof, here we used more restricted orderings to simplify counting T-pairs.

Theorem 6.4. *$\pi_f(K_{1,m,m}) = \frac{24m}{8m+5+3/(2\lceil m/2\rceil-1)}$ for $m \ge 1$.*

Proof. Let the parts be X, Y, and Z with $X = \{x\}$. Again we have D-pairs and T-pairs, but the D-pairs all lie in $Y \cup Z$, and the T-pairs all use x and are Y-pairs or Z-pairs, designated by the part contributing a vertex to each edge. The D-pairs lie in one orbit, as do the T-pairs, so by Corollary 2.3 some optimal strategy for the pair player makes D-pairs equally likely and makes T-pairs equally likely.

Let σ be a vertex ordering of the form $v_1, \ldots, v_{2k}, x, v_{2k+1}, \ldots, v_{2m}$ such that each pair of the form $\{v_{2i-1}, v_{2i}\}$ consists of one vertex from each of Y and Z, for $1 \le i \le m$. We count the pairs separated by σ. After optimizing over k, the ordering player will make all orderings with that k equally likely.

For all k, the restrictions of such orderings to $Y \cup Z$ are the orderings used in Theorem 3.2, which separate the fraction $\frac{m+1}{3m}$ of the D-pairs, and no ordering separates more such pairs.

Index Y as $y_1, \ldots, y_m$ and Z as $z_1, \ldots, z_m$ in order in σ, so that $\{v_{2i-1}, v_{2i}\} = \{y_i, z_i\}$ for $1 \le i \le m$. Each T-pair separated by σ involves x. For the edge xw, an edge separated from xw by the ordering is obtained by picking one vertex each from Y and Z that are both on the opposite side of x from w or both on the opposite side of w from x. When $w \in \{y_j, z_j\}$ with $1 \le j \le k$, taking the two cases of y_j and z_j together yields $(j-1)(j-1+j) + 2(m-k)^2$ pairs. Summing over j yields $2k(m-k)^2 + \sum_{j=1}^{k} \binom{2j-1}{2}$ pairs. Similarly, summing over $k+1 \le j \le m$ yields $2(m-k)k^2 + \sum_{j=1}^{m-k} \binom{2j-1}{2}$ pairs.

Let $f(k)$ be the sum of these two quantities, the total number of T-pairs separated. Note that $f(k) = 2mk(m-k) + \sum_{j=1}^{k} \binom{2j-1}{2} + \sum_{j=1}^{m-k} \binom{2j-1}{2}$. Letting $g(k) = f(k) - f(k-1)$, we have $g(k) = 2m(m-2k+1) + \binom{2k-1}{2} - \binom{2(m-k+1)-1}{2}$, which simplifies to $m - 2k + 1$. Thus $g(k)$ is a decreasing function of k. Also, $g(\frac{m}{2}) > 0$ and $g(\frac{m+1}{2}) = 0$. Hence the number of T-pairs is maximized by choosing k as the integer closest to $m/2$.

By induction on k, it is easily verified that $\sum_{j=1}^{k} \binom{2j-1}{2} = \frac{1}{6}(4k+1)k(k-1)$. Hence when m is even and $k = m/2$, our orderings separate $\frac{m}{12}(8m^2 - 3m - 2)$ pairs. When m is odd, they separate $\frac{m-1}{12}(8m^2 + 5m + 3)$. With altogether $2m^2(m-1)$ T-pairs, the ratio is $\frac{8m^2-3m-2}{24m(m-1)}$ when m is even and $\frac{8m^2+5m+3}{24m^2}$ when m is odd. Dividing numerator and denominator by $m-1$ or m yields the unified formula $\frac{8m+5+3/(2\lceil m/2\rceil-1)}{24m}$ for the fraction separated.

Note that the fraction of T-pairs separated is smaller than the fraction of D-pairs separated. It suffices to show that no ordering that does not pair vertices of Y and Z and place x between two pairs separates the maximum number of T-pairs. The pair player achieves equality in the game by making the T-pairs equally likely.

Since we have considered all k, avoiding the specified form means that some vertex in $Y \cup Z$ precedes another vertex with a lesser index or that x occurs between y_i and z_i for some i. In the first case, we may assume that y_j appears before z_i with $j > i$ and no vertex of $Y \cup Z$ between y_j and z_i. In the second case, we may assume by symmetry that $i < m$ and y_i is before z_i. In either case, form σ' from σ by moving z_i one position earlier; this exchanges z_i with y_j or with x.

If x appears before y_j in σ, then $m - j$ T-pairs are separated in σ but not σ', and $m - i$ T-pairs are separated in σ' but not σ. If x appears after z_i, then $i - 1$ T-pairs are separated by σ but not σ', and $j - 1$ T-pairs are separated by σ' but not σ. Since $j > i$, in each case σ' separates more T-pairs.

In the remaining case, x appears between y_j and z_i with $j \geq i$ and $i < m$. Now $(i+j-1)(m-j)$ T-pairs are separated by σ but not σ', and $(2m-i-j)j$ T-pairs separated by σ' but not σ. We have $(2m - i - j)j > (i + j - 1)(m - j)$ when $j < m(j - i + 1)$, which is true when $i < j \leq m$ and $i < m$. □

7 Circular Separation Dimension

Instead of considering linear orderings of the $V(G)$, we may consider circular orderings of $V(G)$. A pair of nonincident edges $\{xy, zw\}$ is *separated* by a circular ordering σ if the endpoints of the two edges do not alternate. The *circular separation dimension* is the minimum number of circular orderings needed to separate all pairs of nonincident edges in this way. The *circular t-separation dimension* $\pi_t^\circ(G)$ is the minimum size of a multiset of circular orderings needed to separate all the pairs at least t times. The *fractional circular separation dimension* $\pi_f^\circ(G)$ is $\liminf_{t\to\infty} \pi_f^\circ(G)/t$.

Like $\pi(G)$, also $\pi^\circ(G)$ is a hypergraph covering problem. The vertex set $\mathcal{P}$ of the hypergraph H is the same, but the edges corresponding to vertex orderings of G are larger. Thus $\pi^\circ(G) \leq \pi(G)$ and $\pi_f^\circ(G) \leq \pi_f(G)$.

Before discussing the fractional problem, one should first determine the graphs G such that $\pi^\circ(G)$ (and hence also $\pi_f^\circ(G)$) equals 1. Surprisingly, this characterization is quite easy. Unfortunately, it does not generalize to geometrically characterize graphs with $\pi^\circ(G) = t$ like the boxicity result in [4, 5].

Proposition 7.1. *$\pi^\circ(G) = 1$ if and only if G is outerplanar.*

Proof. When $\pi^\circ(G) = 1$, the ordering provides an outerplanar embedding of G by drawing all edges as chords. Chords cross if and only if their endpoints alternate in the ordering.

For sufficiency, it suffices to consider a maximal outerplanar graph, since the parameter is monotone. The outer boundary in an embedding is a spanning cycle; use that as the vertex order. All pairs in $\mathcal{P}$ are separated, since alternating endpoints yield crossing chords. □

The lower bound $\pi(K_{m,n}) \geq \log_2(\min\{m, n\})$ relies on the fact that when two vertices of one part precede two vertices of the other, both nonincident pairs induced by these four vertices fail to be separated. In an circular ordering, always at last one of the two pairs is separated. This leads to the surprising result that $\pi^\circ(G) \in \{1, 2\}$ when G is bipartite.

Proposition 7.2. *$\pi^\circ(K_{m,n}) = 2$ when $m, n \geq 2$ with $mn > 4$.*

Proof. The exceptions are the cases where $K_{m,n}$ is outerplanar and Proposition 7.1 applies. Let σ be an circular ordering in which each partite set occurs as a consecutive segment of vertices. Obtain σ' from σ by reversing one of the partite sets. A nonincident pair of edges alternates endpoints in σ if and only if it does not alternate endpoints in σ'. Hence it is separated in exactly one of the two orderings. □

Nevertheless, π° is unbounded. It suffices to consider K_n, where a classical result provides the lower bound. A list of d-tuples is *monotone* if in each coordinate the list is strictly increasing or weakly decreasing. The multidimensional generalization of the Erdős–Szekeres Theorem by de Bruijn states that any list of more than l^{2^d} vectors in $\mathbb{R}^d$ contains a monotone sublist of more than l vectors. The result is sharp, but this does not yield equality in the lower bound on $\pi^\circ(K_n)$. Our best upper bound is logarithmic, from $\pi(K_n) \leq 4 \log_{3/2} n$ [5].

Theorem 7.3. $\pi^\circ(G) > \log_2 \log_3(\omega(G) - 1)$.

Proof. Note first that a set of circular orderings separates all pairs of nonincident edges in K_n if and only if every 4-set appears cyclically ordered in more than one way (not counting reversal). This follows because each cyclic ordering of K_4 alternates endpoints of exactly one pair of nonincident edges, and for the three cyclic orderings (unchanged under reversal) the pairs that alternate are distinct.

Consider d circular orderings of $\{v_1, \ldots, v_n\}$. Write them linearly by starting with v_1. Associate with each v_i a vector w_i in $\mathbb{R}^d$ whose jth coordinate is the position of v_i in the jth linear ordering. If $n > 3^{2^d}$, then by the multidimensional generalization of the Erdős–Szekeres Theorem $w_1, \ldots, w_n$ has a monotone sublist of four elements. The four corresponding vertices x_1, x_2, x_3, x_4 appear in increasing order or in decreasing order in each linear order. Hence they appear in the same cyclic order or its reverse in each of the original circular orderings. In particular, x_1x_3 and x_2x_4 are not separated by these circular orderings. Since we considered any d circular orderings, $\pi^\circ(K_n) > d$ when $n = 3^{2^d} + 1$. □

We next turn to the fractional context. Since $\pi^\circ(G)$ is a hypergraph covering problem, again and π_f° is computed from a matrix game, with each row provided by the set of pairs in $\mathcal{P}$ separated by a circular orderinge

Our earlier results have analogues in the circular context. A circular ordering of four vertices separates two of the three pairs instead of one, which improves some bounds by a factor of 2. The characterization of the extremal graphs then mirrors the proof of Theorem 3.1.

Theorem 7.4. *$\pi_f^\circ(G) \le \frac{3}{2}$, with equality if and only if $K_4 \subseteq G$. Furthermore, if G has n vertices and $K_4 \subseteq G$, then $\pi_f^\circ(G) \le \frac{3}{2}\left(1 - \frac{6}{n^4} + O(\frac{1}{n^5})\right)$.*

Proof. A circular ordering separates two of the three pairs in each set of four vertices, so making all circular orderings of n vertices equally likely yields $\pi_f^\circ(G) \le \frac{3}{2}$. Equality holds when $K_4 \subseteq G$, since the pair player can give probability $\frac{1}{3}$ to each pair of nonincident edges in a copy of K_4.

Now suppose $K_4 \not\subseteq G$. Let $p = \frac{2}{3} + \frac{4(n-4)!}{n!}$. We provide a distribution on the circular orderings of $V(G)$ such that each nonincident pair of edges is separated with probability at least p. We create a list of $n!$ linear orderings of $V(G)$, which we view as $n!$ circular orderings.

Consider $S = \{a, b, c, d\} \subseteq V(G)$. For each ordering ρ of the remaining $n-4$ vertices, 24 orderings begin with S and end with ρ. By symmetry, we may assume $ac \notin E(G)$. Thus the possible pairs of nonincident edges induced by S are $\{ab, cd\}$ and $\{ad, bc\}$. We increase the separation probability for these vertex pairs.

Circular separation includes nesting when written linearly; only alternation of endpoints fails. The pairs $\{ab, cd\}$ and $\{ad, bc\}$ are each separated 16 times in the 24 orderings of S followed by ρ. The new 24 orderings will separate $\{ab, cd\}$ and $\{ad, bc\}$ each at least 20 times and any other pair (not including $\{a, c\}$) at least 16 times.

The 24 new orderings are two copies each where the first four vertices are (in order) *abdc*, *badc*, *dcba*, *cbad*, *adbc*, *adcb*, *acbd*, or *dbac*, and four copies each using *cdab* or *bcda*, always followed by ρ. By inspection, each of $\{ab, cd\}$ and $\{ad, bc\}$ is separated 20 times in the list.

The number of orderings that separate any pair of nonincident edges having at most two vertices in S is the same as before. Hence we need only check pairs with three vertices in S, consisting of one edge in $\{ab, cd, bc, ad, bd\}$ (never ac) and another edge with one endpoint among the remaining two vertices in S. In each case, the endpoints of the induced edge appear before or after the third vertex in at least 16 of the orderings in the new list of 24.

Since $\{a, b, c, d\}$ is arbitrary and we do this for each 4-set, the pairs $\{ab, cd\}$ and $\{ad, bc\}$ are separated with probability at least $\frac{5}{6}$ by the 24 orderings that start with $\{a, b, c, d\}$ and then are made circular, and with probability at least $\frac{2}{3}$ among the remaining orderings. Thus the separation probability increases from $\frac{2}{3}$ to at least p for each pair. □

Again there is no sharper bound for bipartite graphs or graphs with girth 4: $\pi_f^\circ(K_{m,m}) \to \frac{3}{2}$. The orderings used to give the optimal upper bound for $\pi_f^\circ(K_{m,qm})$ are in some sense the farthest possible from those giving the optimal upper bound for $\pi^\circ(K_{m,qm})$ in Proposition 7.2.

Theorem 7.5. *$\pi_f^\circ(K_{m,qm}) = \frac{6(qm-1)}{4mq+q-3}$. In particular, $\pi_f^\circ(K_{m,m}) = \frac{3m-3}{2m-1}$.*

Proof. Again Corollary 2.3 (for the circular separation game) applies. The $2\binom{m}{2}\binom{qm}{2}$ pairs of nonincident edges lie in one orbit, so it suffices to make circular orderings that separate $\frac{4mq+q-3}{6(qm-1)}2\binom{m}{2}\binom{qm}{2}$ pairs equally likely and show that no ordering separates more.

Let X and Y be the parts of the bipartition, with $|X| = m$. Let σ be a circular ordering in which the vertices of X are equally spaced, with q vertices of Y between any two successive vertices of X.

There are two types of pairs separated by σ. In one, the parts for the four vertices alternate as $XYXY$; in the other, they occur as $XYYX$, cyclically. Choose the first member of X in m ways. Let k be the number of steps within X taken to get from there to the other member of X used. In the first case, there are $kq(m-k)q$ ways to choose the vertices from Y and two ways to group the chosen vertices to form a separated nonincident pair, but either of the vertices of X could have been called the first vertex. In the second case, there are $\binom{kq}{2}$ ways to choose from Y, one way to group, and only one choice for the first vertex of X.

Thus to count the separated pairs, we sum over k and use $\sum_{k=-n}^{m} \binom{n+k}{r}\binom{m-k}{s} = \binom{n+m+1}{r+s+1}$ and $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ to compute

$$m\sum_{k=1}^{m-1} kq(m-k)q + m\sum_{k=1}^{m-1}\binom{kq}{2} = m\sum_{k=0}^{m} q^2\binom{0+k}{1}\binom{m-k}{1} + \frac{mq}{2}\sum_{k=1}^{m-1}(k^2q-k)$$
$$= mq^2\binom{m+1}{3} + \frac{mq^2}{2}\frac{(m-1)m(2m-1)}{6} - \frac{mq}{2}\binom{m}{2}.$$

Factoring out $2\binom{m}{2}\frac{qm}{2}$ leaves $\frac{1}{6}(4mq+q-3)$, as desired.

It remains to show that no other circular ordering separates as many pairs of nonincident edges. We do this by finding, for every circular ordering σ other than those discussed above, an ordering $\hat{\sigma}$ that separates more pairs.

With $X = \{x_1, \ldots, x_m\}$ in cyclic order, the ordering σ is described by a list $q_1, \ldots, q_m$ of nonnegative integers summing to qm, where q_i is the number of vertices of Y between x_{i-1} and x_i (indexed modulo m). Index so that $q_1 = \max_i q_i$; we may assume $q_1 \geq q+1$.

Let σ' be the ordering obtained by interchanging x_m with the vertex y immediately following it (note that $y \in Y$, since $q_1 > q$). The pairs in $\mathcal{P}$ separated by σ or σ' but not both are those consisting of an edge yx_k for some k with $1 \leq k \leq m-1$ and an edge x_my'. For those separated by σ but not σ' there are $\sum_{j=k+1}^{m} q_j$ choices for y'. For those separated by σ' but not σ there are $(\sum_{i=1}^{k} q_i) - 1$ choices for y'.

After isolating the terms involving q_1, the net gain in switching from σ to σ' is thus

$$\sum_{k=1}^{m-1}\left(q_1 - 1 + \sum_{i=2}^{k} q_i - \sum_{j=k+1}^{m} q_j\right).$$

Consider instead the ordering σ'' obtained from σ by interchanging x_1 with the vertex y immediately preceding it (again $y \in Y$, since $q_1 > q$. The net change in the number of separated pairs follows the same computation, except that $q_2, \ldots, q_m$ are indexed in the reverse order. More precisely, the change in moving from σ to σ'' is

$$\sum_{k=2}^{m}\left(q_1 - 1 + \sum_{j=k+1}^{m} q_j - \sum_{i=2}^{k} q_i\right).$$

In summing the two net changes, the summations in the terms for $2 \le k \le m-1$ cancel. The sum is thus

$$2(q_1-1)(m-1) - \sum_{j=2}^{m} q_j - \sum_{i=2}^{m} q_i.$$

Since $\sum_{j=2}^{m} q_j = qm - q_1$, the net sum simplifies to $2q_1m - 2qm - 2(m-1)$. Since $q_1 \ge q+1$, the value is at least 2. Since the sum of the two net changes is positive, at least one of them is positive, and σ does not separate the most pairs. $\square$

Note that $K_{2,r}$ is planar with girth 4, for $r \ge 2$. Theorem 7.5 yields $\pi_f^\circ(K_{2,2q}) = \frac{4q-4}{3q-1} \to \frac{4}{3}$. It remains open how large π_f° can be for planar graphs with girth 4, and for graphs (planar or not) with larger girth. For girth 5, computer computation shows that the fractional circular separation dimension of the Petersen graph is $\frac{8}{7}$.

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