Longest Cycles in $k$-connected Graphs with Given Independence Number

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http://www.math.uiuc.edu/~west (click "Preprints")
Spanning Cycles

Sufficient cond. for spanning cycles in $n$-vertex graphs:

**Thm.** (Dirac [1952]) $\delta(G) \geq n/2$.

**Thm.** (Ore [1960]) $d(x)+d(y) \geq n$ whenever $xy \notin E(G)$.

**Thm.** (Chvátal–Erdős [1972]) $\kappa(G) \geq \alpha(G)$. 
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Long-cycle versions for 2-connected graphs ($c(G) =$ circumference $=$ length of longest cycle):

Thm. (Dirac [1952]) $c(G) \geq \min\{n, 2\delta(G)\}$.

Thm. (Bondy [1971]; also Bermond [1976], Linial [1976]) $c(G) \geq \min\{n,s\}$ if $d(x) + d(y) \geq s$ whenever $xy \not\in E(G)$. 
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Long-cycle version of Chvátal–Erdős Theorem?
Conjecture

**Conj.** (Fouquet–Jolivet [1976]) If \( \kappa(G) \leq \alpha(G) \), then
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c(G) \geq \frac{k(n+a-k)}{a},
\]
where \( k = \kappa(G) \) and \( a = \alpha(G) \).
Conjecture (Fouquet–Jolivet [1976]) If $\kappa(G) \leq \alpha(G)$, then $c(G) \geq \frac{k(n+a-k)}{a}$, where $k = \kappa(G)$ and $a = \alpha(G)$.

- Equality holds infinitely often: $K_k \vee aK_m$ for $m \geq k \geq 2$. 

![Diagram](image-url)
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n = k + am, \quad \alpha(G) = a, \quad \kappa(G) = k, \quad c(G) = k(1+m) = \frac{k(n+a-k)}{a}.
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- Known true for $a \in \{k + 1, k + 2\}$ (Fournier [1982]), $k = 2$ (Fournier [1984]), $k = 3$ (Manoussakis [2009]), $k = 4$ & $a < 2k - 1$ (Chen–Hu–Y.Wu [2010(a&b)+])
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Stronger Conjecture

**Conj.** (Chen–Chen–Liu) If $C$ and $C'$ are distinct cycles in a $k$-connected graph, then there are cycles $D$ and $D'$ with $V(C) \cup V(C') \subseteq V(D) \cup V(D')$ and $|V(D) \cap V(D')| \geq k$. 
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We prove the F–J Conjecture without proving the C–C–L Conjecture.
**Thm.** (Kouider [1994]) If $H$ is a subgraph of a $k$-connected graph $G$, then $G$ has a cycle $C$ such that $\alpha(H - V(C)) \leq \max\{0, \alpha(H) - k\}$. 
A Tool

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**Cor.** $\kappa(G) \geq \alpha(G) \Rightarrow$ spanning cycle.

**Pf.** Set $H = G$.  

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**Cor.** $c(G) \geq n/\lceil \alpha(G)/\kappa(G) \rceil$ when $\kappa(G) \leq \alpha(G)$.

**Pf.** Apply Kouider’s Theorem at most $\lceil \alpha(G)/\kappa(G) \rceil$ times to obtain cycles together covering $V(G)$. □
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- The F–J Conjecture requires $c(G) \geq nk/\alpha + k(1 - k/\alpha)$. 
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**Thm.** (Path Lemma) If $H$ is a subgraph of a $k$-connected graph $G$, and $u, v \in V(G)$, then $G$ has a $u, v$-path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$.
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**Lem.** (Multi-Cycle Lemma) If $G$ is a $k$-connected graph with independence number $\alpha$, and $0 \leq \ell \leq \alpha - k$, then there exist cycles $C_0, \ldots, C_\ell$ satisfying the following:

1. $\alpha(G - \bigcup_{i=0}^{\ell} V(C_i)) \leq \alpha - k - \ell$,
2. $|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq \ell$. 
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1. $\alpha(G - \bigcup_{i=0}^l V(C_i)) \leq \alpha - k - l$ \([l = 0 \text{ is Kouider w. } H = G] \)
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Lem. (Multi-Cycle) For $0 \leq \ell \leq a - k$, $\exists C_0, \ldots, C_\ell$ with

1. $\alpha(G - \bigcup_{i=0}^{\ell} V(C_i)) \leq a - k - \ell$

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Multi-Cycle $\Rightarrow$ Fouquet–Jolivet
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Cor. $G$ has a cycle of length at least $\frac{k(n+a-k)}{a}$.
Lem. (Multi-Cycle) For \(0 \leq \ell \leq \alpha - k\), \(\exists C_0, \ldots, C_\ell\) with

(1) \(\alpha(G - \bigcup_{i=0}^\ell V(C_i)) \leq \alpha - k - \ell\)

(2) \(|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| \leq \frac{|V(C_0)|}{k} - 1\) for \(1 \leq i \leq \ell\).

Cor. \(G\) has a cycle of length at least \(\frac{k(n + \alpha - k)}{\alpha}\).

Pf. Set \(\ell = \alpha - k\), so \(C_0, \ldots, C_\ell\) cover \(V(G)\), by (1).
Multi-Cycle ⇒ Fouquet–Jolivet

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**Cor.** $G$ has a cycle of length at least $\frac{k(n+a-k)}{a}$.

**Pf.** Set $\ell = a - k$, so $C_0, \ldots, C_\ell$ cover $V(G)$, by (1). Every vertex appears first in some $C_i$. 
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**Lem.** (Multi-Cycle) For $0 \leq l \leq a - k$, $\exists C_0, \ldots, C_l$ with

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**Pf.** Set $l = a - k$, so $C_0, \ldots, C_l$ cover $V(G)$, by (1).

Every vertex appears first in some $C_i$.

Summing $|V(C_0)|$ and (2) for $1 \leq i \leq l$ yields

\[
n = |V(C_0)| + \sum_{i=1}^{l} \left| V(C_i) - \bigcup_{j=0}^{i-1} V(C_j) \right| \\
\leq |V(C_0)| + (a - k)\left(\frac{|V(C_0)|}{k} - 1\right).
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Multi-Cycle ⇒ Fouquet–Jolivet

**Lem.** (Multi-Cycle) For $0 \leq \ell \leq a - k$, $\exists C_0, \ldots, C_\ell$ with

\begin{align*}
(1) & \quad \alpha(G - \bigcup_{i=0}^{\ell} V(C_i)) \leq a - k - \ell \\
(2) & \quad |V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| \leq \frac{|V(C_0)|}{k} - 1 \quad \text{for } 1 \leq i \leq \ell.
\end{align*}

**Cor.** $G$ has a cycle of length at least $\frac{k(n+a-k)}{a}$. 

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Every vertex appears first in some $C_i$.
Summing $|V(C_0)|$ and (2) for $1 \leq i \leq \ell$ yields

\begin{align*}
n &= |V(C_0)| + \sum_{i=1}^{\ell} \left| V(C_i) - \bigcup_{j=0}^{i-1} V(C_j) \right| \\
&\leq |V(C_0)| + (a - k) \left( \frac{|V(C_0)|}{k} - 1 \right).
\end{align*}

The inequality simplifies to $|V(C_0)| \geq \frac{k(n+a-k)}{a}$. ■
**Lem. (Cycle)** Given disjoint $H$ and cycle $C$ (length $\geq k$), $\exists \ C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

**Cor. (Multi-Cycle)** For $0 \leq l \leq \alpha - k$, $\exists C_0, \ldots, C_l$ with

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Cycle Lemma \implies Multi-Cycle Lemma

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**Pf.** Use induction on $l$. For $l = 0$, it is Kouider’s Thm.
**Cycle Lemma** \[\Rightarrow\] **Multi-Cycle Lemma**

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**Pf.** Use induction on $\ell$. For $\ell = 0$, it is Kouider’s Thm. For $\ell > 0$, we are given $C_0, \ldots, C_{\ell-1}$. 
Cycle Lemma ⇒ Multi-Cycle Lemma

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**Pf.** Use induction on $l$. For $l = 0$, it is Kouider’s Thm. For $l > 0$, we are given $C_0, \ldots, C_{l-1}$. From (1) for $l = 0$, we have $|V(C_0)| \geq k$. 
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For $l > 0$, we are given $C_0, \ldots, C_{l-1}$.

From (1) for $l = 0$, we have $|V(C_0)| \geq k$.

Let $H = G - \bigcup_{i=0}^{l-1} V(C_i)$; we are given $\alpha(H) \leq \alpha - k - (l-1)$. 

Lem. (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), $\exists C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H \setminus V(C')) \leq \alpha(H) - 1$.

Cor. (Multi-Cycle) For $0 \leq l \leq a - k$, $\exists C_0, \ldots, C_l$ with

1. $|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq l$.
2. $\alpha(G - \bigcup_{i=0}^{l} V(C_i)) \leq \alpha(G) - (a - k - l)$

Pf. Use induction on $l$. For $l = 0$, it is Kouider’s Thm.

For $l > 0$, we are given $C_0, \ldots, C_{l-1}$.

From (1) for $l = 0$, we have $|V(C_0)| \geq k$.

Let $H = G - \bigcup_{i=0}^{l-1} V(C_i)$; we are given $\alpha(H) \leq \alpha(G) - (a - k - (l - 1))$.

If $\alpha(H) > 0$, apply Cycle Lem. with $C_0$ as $C$ to get $C'$: $\alpha(H \setminus V(C')) \leq \alpha(H) - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.  


Cycle Lemma $\Rightarrow$ Multi-Cycle Lemma

**Lem. (Cycle)** Given disjoint $H$ and cycle $C$ (length $\geq k$), $\exists C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

**Cor. (Multi-Cycle)** For $0 \leq l \leq a - k$, $\exists C_0, \ldots, C_l$ with

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**Pf.** Use induction on $l$. For $l = 0$, it is Kouider’s Thm.
For $l > 0$, we are given $C_0, \ldots, C_{l-1}$.
From (1) for $l = 0$, we have $|V(C_0)| \geq k$.
Let $H = G - \bigcup_{i=0}^{l-1} V(C_i)$; we are given $\alpha(H) \leq a - k - (l-1)$.
If $\alpha(H) > 0$, apply Cycle Lem. with $C_0$ as $C$ to get $C'$:
$\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.
Including $C'$ in the list yields (1), but we also need (2).
Lem. (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), $\exists$ $C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

Cor. (Multi-Cycle) For $0 \leq l \leq a - k$, $\exists C_0, \ldots, C_l$ with

1. $\alpha(G - \bigcup_{i=0}^{l} V(C_i)) \leq a - k - l$
2. $|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq l$.

Pf. Apply Cycle Lem. with $C_0$ as $C$ to get $C'$:
$\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$. 
Lem. (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), $\exists$ $C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

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Pf. Apply Cycle Lem. with $C_0$ as $C$ to get $C'$: $\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.

Case 1: $|V(C')| \leq |V(C_0)|$.
Complete the desired list by setting $C_l = C'$; okay since

$$|V(C') - \bigcup_{j=0}^{l-1} V(C_i)| \leq |V(C') - V(C_0)| \leq |V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1.$$
**Lem.** (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), $\exists \ C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

**Cor.** (Multi-Cycle) For $0 \leq l \leq a - k$, $\exists C_0, \ldots, C_l$ with

1. $\alpha(G - \bigcup_{i=0}^{l} V(C_i)) \leq a - k - l$
2. $|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq l$.

**Pf.** Apply Cycle Lem. with $C_0$ as $C$ to get $C'$:

$\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.

**Case 2:** $|V(C')| > |V(C_0)|$.
New list: Set $C'_0 = C'$, with $C'_i = C_{i-1}$ for $1 \leq i \leq l$. 
Lem. (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), $\exists C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

Cor. (Multi-Cycle) For $0 \leq l \leq a - k$, $\exists C_0, \ldots, C_l$ with
(1) $\alpha(G - \bigcup_{i=0}^{l} V(C_i)) \leq a - k - l$
(2) $|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq l$.

Pf. Apply Cycle Lem. with $C_0$ as $C$ to get $C'$: $\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.

Case 2: $|V(C')| > |V(C_0)|$.
New list: Set $C'_0 = C'$, with $C'_i = C_{i-1}$ for $1 \leq i \leq l$.

$i = 1$: $V(C'_1) - V(C'_0) = V(C_0) - V(C')$.

$i > 1$: $V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) \subseteq V(C_{i-1}) - \bigcup_{j=0}^{i-2} V(C_j)$. 

Case 2: $|V(C')| > |V(C_0)|$. 
New list: Set $C'_0 = C'$, with $C'_i = C_{i-1}$ for $1 \leq i \leq l$.

$i = 1$: $V(C'_1) - V(C'_0) = V(C_0) - V(C')$.

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New list: Set $C'_0 = C'$, with $C'_i = C_{i-1}$ for $1 \leq i \leq l$.

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$i > 1$: $V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) \subseteq V(C_{i-1}) - \bigcup_{j=0}^{i-2} V(C_j)$. 

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Case 2: $|V(C')| > |V(C_0)|$. 
New list: Set $C'_0 = C'$, with $C'_i = C_{i-1}$ for $1 \leq i \leq l$.

$i = 1$: $V(C'_1) - V(C'_0) = V(C_0) - V(C')$.

$i > 1$: $V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) \subseteq V(C_{i-1}) - \bigcup_{j=0}^{i-2} V(C_j)$.
**Cycle Lemma** ⇒ **Multi-Cycle Lemma (cont.)**

**Lem.** (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), $\exists C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

**Cor.** (Multi-Cycle) For $0 \leq \ell \leq a - k$, $\exists C_0, \ldots, C_\ell$ with

1. $\alpha(G - \bigcup_{i=0}^{\ell} V(C_i)) \leq a - k - \ell$
2. $|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq \ell$.

**Pf.** Apply Cycle Lem. with $C_0$ as $C$ to get $C'$: $\alpha(H - V(C')) \leq a - k - \ell$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.

**Case 2:** $|V(C')| > |V(C_0)|$.

New list: Set $C'_0 = C'$, with $C'_i = C_{i-1}$ for $1 \leq i \leq \ell$.

$i = 1$: $V(C'_1) - V(C'_0) = V(C_0) - V(C')$.

$i > 1$: $V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) \subseteq V(C_{i-1}) - \bigcup_{j=0}^{i-2} V(C_j)$.

So, $|V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j)| \leq \frac{|V(C_0)|}{k} - 1 \leq \frac{|V(C'_0)|}{k} - 1$.  ■
Path Lemma  ⇒  Cycle Lemma

**Lem. (Path)** Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists a path $P$ such that $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$.

**Cor. (Cycle)** Given disjoint $H$ and cycle $C$ (length $\geq k$), there exists a cycle $C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$. 
Path Lemma ⇒ Cycle Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists a path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k-1)\}$.

**Cor.** (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), there exists $C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

**Pf.** If $H$ is disconnected or has a cut-vertex, then find $C'$ using an induced subgraph of $H$. Otherwise, . . .
Path Lemma \implies Cycle Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, \exists $u, v$-path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$.

**Cor.** (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), \exists $C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

**Pf.** If $H$ is disconnected or has a cut-vertex, then find $C'$ using an induced subgraph of $H$. Otherwise, . . .
Path Lemma ⇒ Cycle Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists a $u, v$-path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k-1)\}$.

**Cor.** (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), there exists a cycle $C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

**Pf.** If $H$ is disconnected or has a cut-vertex, then find $C'$ using an induced subgraph of $H$. Otherwise, . . .

If $d_C(v_1, v_2) \leq L/k$, build cycle $C'$ using $P$ from Path Lem.
Path Lemma $\Rightarrow$ Cycle Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists a $u, v$-path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$.

**Cor.** (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), there exists a cycle $C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

**Pf.** If $H$ is disconnected or has a cut-vertex, then find $C'$ using an induced subgraph of $H$. Otherwise, . . .

\[ L = |V(C)| \]

\[ B = \text{block of } G - V(C) \]

If $d_C(v_1, v_2) \leq L/k$, build cycle $C'$ using $P$ from Path Lem. Let $\{c_1, \ldots, c_m\}$ be arrival points on $C$ of paths from $B$. 
Path Lemma  ⇒  Cycle Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists an $u, v$-path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$.

**Cor.** (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), there exists $C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

**Pf.**
If $H$ is disconnected or has a cut-vertex, then find $C'$ using an induced subgraph of $H$. Otherwise, . . .

If $d_C(v_1, v_2) \leq L/k$, build cycle $C'$ using $P$ from Path Lem. Let $\{c_1, \ldots, c_m\}$ be arrival points on $C$ of paths from $B$. Strict paths from $b, b' \in B$ reaching $c_i, c_{i+1}$ are disjoint.
Path Lemma \implies Cycle Lemma

**Lem.** (Path) Given \( H \subseteq G, \kappa(G) \geq k \), and \( u, v \in V(G) \), there exists a \( u, v \)-path \( P \) such that \( \alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\} \).

**Cor.** (Cycle) Given disjoint \( H \) and cycle \( C \) (length \( \geq k \)), there exists a cycle \( C' \) such that \( |V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1 \) and \( \alpha(H - V(C')) \leq \alpha(H) - 1 \).

**Pf.** \( t = |\{i: c_i \& c_{i+1} \text{ reached from diff verts of } B\}| < k \).
Path Lemma ⇒ Cycle Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, 
\[ \exists u, v\text{-path } P \text{ with } \alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k-1)\}. \]

Cor. (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), \[ \exists C' \text{ with } |V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1 \text{ and } \alpha(H - V(C')) \leq \alpha(H) - 1. \]

Pf. \[ t = |\{ i: c_i \& c_{i+1} \text{ reached from diff verts of } B \}| < k. \]
But, Menger ⇒ $k$ paths from $b \in B$ to $C$. 
Path Lemma \implies Cycle Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, 
\exists $u, v$-path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$.

**Cor.** (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), \exists $C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

**Pf.** \therefore $t = |\{i : c_i \& c_{i+1} \text{ reached from diff verts of } B\}| < k$.
But, Menger \implies $k$ paths from $b \in B$ to $C$.
At least $k - t$ “$b$-segments” of $C$ meet paths from no other $b' \in B$.  

Path Lemma $\Rightarrow$ Cycle Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists a $u, v$-path $P$ with $\alpha(H-V(P)) \leq \max\{0, \alpha(H) - (k-1)\}$.

**Cor.** (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), there exists $C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H-V(C')) \leq \alpha(H) - 1$.

**Pf.** $t = |\{i: c_i \& c_{i+1} \text{ reached from diff verts of } B\}| < k$. But, Menger $\Rightarrow k$ paths from $b \in B$ to $C$. At least $k - t$ “$b$-segments” of $C$ meet paths from no other $b' \in B$. Over all $b \in B$, total length $< L - t(L/k) = L(k - t)/k$. 
**Path Lemma \Rightarrow Cycle Lemma**

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists a $u, v$-path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$.

**Cor.** (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), there exists a cycle $C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

**Pf.** \( t = |\{i: c_i \& c_{i+1} \text{ reached from diff verts of } B\}| < k. \)

But, Menger \( \Rightarrow \) $k$ paths from $b \in B$ to $C$.

At least $k - t$ “$b$-segments” of $C$ meet paths from no other $b' \in B$.

Over all $b \in B$, total length $< L - t(L/k) = L(k - t)/k$.

Pick a shortest $\forall b \in B$; total length $< L/k$. 

![Diagram](image)
Path Lemma \implies Cycle Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists a path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$.

**Cor.** (Cycle) Given disjoint $H$ and cycle $C$ (length $\geq k$), there exists a cycle $C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

**Pf.** $t = |\{i: c_i \& c_{i+1} \text{ reached from diff verts of } B\}| < k$. But, Menger $\implies k$ paths from $b \in B$ to $C$. At least $k - t$ “$b$-segments” of $C$ meet paths from no other $b' \in B$.

Over all $b \in B$, total length $< L - t(L/k) = L(k - t)/k$. Pick a shortest $\forall b \in B$; total length $< L/k$. Form $C'$.
Proof of Path Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists a $u, v$-path $P$ such that $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$. 
Proof of Path Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists a $u, v$-path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$.

Pf. For each $u, v$-path $P$, let $F_P$ be a smallest component of $G - V(P)$ that intersects $H$. Choose $P$ so that:
(i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), $F_P$ has the fewest vertices.
Proof of Path Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists a $u, v$-path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$.

**Pf.** For each $u, v$-path $P$, let $F_P$ be a smallest component of $G - V(P)$ that intersects $H$. Choose $P$ so that:
(i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), $F_P$ has the fewest vertices.

If $V(H) \not\subseteq V(P)$, then let $p_1, \ldots, p_m$ have neighbors in $F_P$. 

![Diagram of a path P with vertices u, v, p1, ..., pm and a component F_P intersecting H]
Proof of Path Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, \(\exists\ u, v\)-path $P$ with $\alpha(H-V(P)) \leq \max\{0, \alpha(H)-(k-1)\}$.

Pf. For each $u, v$-path $P$, let $F_P$ be a smallest component of $G-V(P)$ that intersects $H$. Choose $P$ so that:
(i) $\alpha(H-V(P))$ is smallest;
(ii) subject to (i), $F_P$ has the fewest vertices.

If $V(H) \not\subseteq V(P)$, then let $p_1, \ldots, p_m$ have neighbors in $F_P$.
Proof of Path Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists a $u, v$-path $P$ with $\alpha(H-V(P)) \leq \max\{0, \alpha(H)-(k-1)\}$.

**Pf.** For each $u, v$-path $P$, let $F_P$ be a smallest component of $G - V(P)$ that intersects $H$. Choose $P$ so that:

(i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), $F_P$ has the fewest vertices.

If $V(H) \not\subseteq V(P)$, then let $p_1, \ldots, p_m$ have neighbors in $F_P$.

- $V(F_P) \cap V(H) \not\subseteq V(P')$, else $\alpha(H-V(P')) < \alpha(H-V(P))$. 

![Diagram](image)
Proof of Path Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, 
$\exists u, v$-path $P$ with $\alpha(H - V(P)) \leq \max \{0, \alpha(H) - (k - 1)\}$.

**Pf.** For each $u, v$-path $P$, let $F_P$ be a smallest component of $G - V(P)$ that intersects $H$. Choose $P$ so that:
(i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), $F_P$ has the fewest vertices.

If $V(H) \not\subseteq V(P)$, then let $p_1, \ldots, p_m$ have neighbors in $F_P$.

$V(F_P) \cap V(H) \not\subseteq V(P')$, else $\alpha(H - V(P')) < \alpha(H - V(P))$.
$\therefore F_P - V(P')$ has a component intersecting $H$. 

\begin{center}
\begin{tikzpicture}
\node (u) at (0,0) {$u$};
\node (p1) at (1,0) {$p_1$};
\node (pi) at (2,0) {$p_i$};
\node (pi+1) at (3,0) {$p_{i+1}$};
\node (pm) at (4,0) {$p_m$};
\node (v) at (5,0) {$v$};
\node (fp) at (2,-1) {$F_P$};
\node (qi) at (2,-2) {$Q_i$};
\node (ui) at (2,-3) {$U_i$};
\node (p) at (-1,0) {$P'$};
\node (p') at (6,0) {$P'$};
\draw (u) -- (p1);
\draw (p1) -- (pi);
\draw (pi) -- (pi+1);
\draw (pi+1) -- (pm);
\draw (pm) -- (v);
\draw (p1) -- (qi) -- (ui) -- (pm);
\end{tikzpicture}
\end{center}
Proof of Path Lemma

**Lem.** (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, there exists a $u, v$-path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$.

**Pf.** For each $u, v$-path $P$, let $F_P$ be a smallest component of $G - V(P)$ that intersects $H$. Choose $P$ so that:

(i) $\alpha(H - V(P))$ is smallest;

(ii) subject to (i), $F_P$ has the fewest vertices.

If $V(H) \not\subseteq V(P)$, then let $p_1, \ldots, p_m$ have neighbors in $F_P$.

- $V(F_P) \cap V(H) \not\subseteq V(P')$, else $\alpha(H - V(P')) < \alpha(H - V(P))$.

$\therefore F_P - V(P')$ has a component intersecting $H$.

Thus $\alpha(H - V(P - U_i)) \geq \alpha(H - V(P')) > \alpha(H - V(P))$. 
Proof of Path Lemma, cont.

**Recall:** $P$ is chosen so that:  
(i) $\alpha(H - V(P))$ is smallest;  
(ii) subject to (i), $F_P$ has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all $i$. 

![Diagram showing a path $P$ with vertices $u, p_i, p_{i+1}, p_j, p_{j+1}$ and the set $U_i, U_j$ with $F_P$ highlighted.]
Proof of Path Lemma, cont.

**Recall:** $P$ is chosen so that: (i) $\alpha(H - V(P))$ is smallest; (ii) subject to (i), $F_P$ has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all $i$.

 Restore $U_i$ from the left; $q_i$ is where $\alpha$ first increases.
Proof of Path Lemma, cont.

**Recall:** $P$ is chosen so that: (i) $\alpha(H - V(P))$ is smallest; (ii) subject to (i), $F_P$ has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all $i$.

Restore $U_i$ from the left; $q_i$ is where $\alpha$ first increases.

A path $\hat{P}$ outside $P$ joining $U'_i$ and $U'_j$ can’t visit $F_P$. 
Proof of Path Lemma, cont.

**Recall:** $P$ is chosen so that: (i) $\alpha(H - V(P))$ is smallest; (ii) subject to (i), $F_P$ has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all $i$.

Restore $U_i$ from the left; $q_i$ is where $\alpha$ first increases.

A path $\hat{P}$ outside $P$ joining $U_i'$ and $U_j'$ can’t visit $F_P$.

Pick leftmost such $r_i$ and form new path $P'$. 

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Proof of Path Lemma, cont.

**Recall:** $P$ is chosen so that: (i) $\alpha(H - V(P))$ is smallest; (ii) subject to (i), $F_P$ has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all $i$.

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![Diagram](image)

Restore $U_i$ from the left; $q_i$ is where $\alpha$ first increases.

A path $\hat{P}$ outside $P$ joining $U'_i$ and $U'_j$ can’t visit $F_P$.

Pick leftmost such $r_i$ and form new path $P'$.

Restoring can’t increase $\alpha$: again $\alpha(H - V(P')) \leq \alpha(H - V(P))$, contradicting choice of $P$.
Proof of Path Lemma, cont.

**Recall:** $P$ is chosen so that: (i) $\alpha(H - V(P))$ is smallest; (ii) subject to (i), $F_P$ has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all $i$.

![Diagram](image)

Restore $U_i$ from the left; $q_i$ is where $\alpha$ first increases.

Sets $U'_1, \ldots, U'_{m-1}$ are in diff comps of $G - V(P - U)$. 
Proof of Path Lemma, cont.

**Recall:** $P$ is chosen so that: (i) $\alpha(H - V(P))$ is smallest; (ii) subject to (i), $F_P$ has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all $i$.

![Diagram](image)

Restore $U_i$ from the left; $q_i$ is where $\alpha$ first increases.

Sets $U'_1, \ldots, U'_{m-1}$ are in diff comps of $G - V(P - U)$.

$\therefore \alpha(H - V(P - U)) \geq \alpha(H - V(P)) + m - 1 \geq \alpha(H - V(P)) + k - 1.$
Proof of Path Lemma, cont.

**Recall:** $P$ is chosen so that: (i) $\alpha(H - V(P))$ is smallest; (ii) subject to (i), $F_P$ has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all $i$.

![Diagram of graphs](image)

Restore $U_i$ from the left; $q_i$ is where $\alpha$ first increases.

Sets $U'_1, \ldots, U'_{m-1}$ are in diff comps of $G - V(P - U)$.

\[ \therefore \alpha(H - V(P - U)) \geq \alpha(H - V(P)) + m - 1 \geq \alpha(H - V(P)) + k - 1. \]

\[ \therefore \alpha(H) \geq \alpha(H - V(P)) + k - 1. \]

\[ \Box \]