Longest Cycles in k-connected Graphs with Given Independence Number

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Joint work with

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slides and paper at

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Spanning Cycles

Sufficient cond. for spanning cycles in n-vertex graphs:

Thm. (Dirac [1952]) $\delta(G) \geq n/2$.

Thm. (Ore [1960]) $d(x)+d(y) \geq n$ whenever $xy \notin E(G)$.

Thm. (Chvátal–Erdős [1972]) $\kappa(G) \geq \alpha(G)$.

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Long-cycle versions for 2-connected graphs
($c(G)$ = circumference = length of longest cycle):

Thm. (Dirac [1952]) $c(G) \ge \min\{n, 2\delta(G)\}$.

Thm. (Bondy [1971]; also Bermond [1976], Linial [1976])
$c(G) \ge \min\{n,s\}$ if $d(x)+d(y) \ge s$ whenever $xy \notin E(G)$.

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Long-cycle version of Chvátal–Erdős Theorem?

Conjecture

Conj. (Fouquet–Jolivet [1976]) If $\kappa(G) \leq \alpha(G)$, then $c(G) \geq \frac{k(n+\alpha-k)}{\alpha}$, where $k = \kappa(G)$ and $\alpha = \alpha(G)$.

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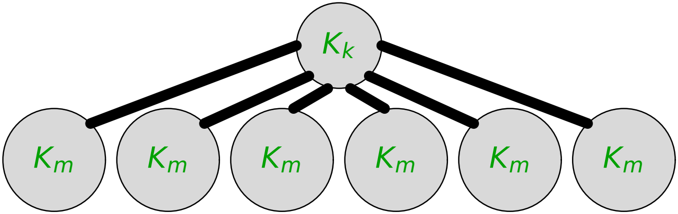

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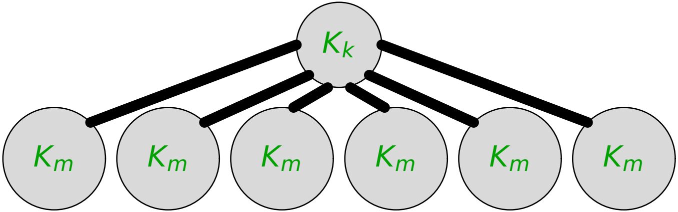

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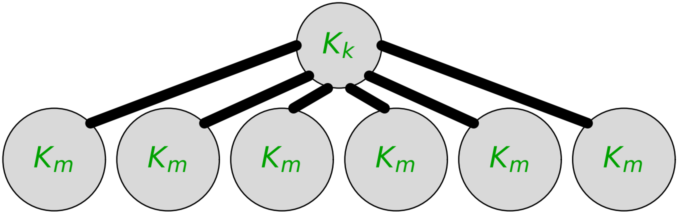

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- Known true for $a \in \{k+1, k+2\}$ (Fournier [1982]), $k = 2$ (Fournier [1984]), $k = 3$ (Manoussakis [2009]), $k = 4$ & $a < 2k-1$ (Chen–Hu–Y.Wu [2010(a&b)+])

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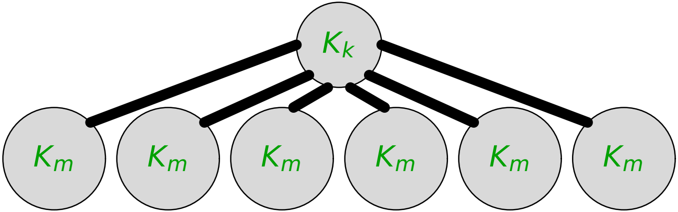

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Thm. (W.-O.-W.) The F–J Conjecture is true.

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Conj. (Chen–Chen–Liu) If C and C' are distinct cycles in a k-connected graph, then there are cycles D and D' with $V(C) \cup V(C') \subseteq V(D) \cup V(D')$ and $|V(D) \cap V(D')| \geq k$.

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We prove the F–J Conjecture without proving the C–C–L Conjecture.

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Cor. $c(G) \ge n/\lceil \alpha(G)/\kappa(G) \rceil$ when $\kappa(G) \le \alpha(G)$.

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Lem. (Multi-Cycle Lemma) If G is a k-connected graph with independence number a, and $0 \leq \ell \leq a - k$, then there exist cycles $C_0, \ldots, C_\ell$ satisfying the following:

(1) $\alpha(G - \bigcup_{i=0}^{\ell} V(C_i)) \leq a - k - \ell$,

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Path Lem. $\Rightarrow$ Cycle Lem. $\Rightarrow$ Multi-Cycle Lem. $\Rightarrow$ F–J Conj.

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$$n = |V(C_0)| + \sum_{i=1}^{l} \left| V(C_i) - \bigcup_{j=0}^{i-1} V(C_j) \right|$$
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The inequality simplifies to $|V(C_0)| \ge \frac{k(n+a-k)}{a}$. ∎

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Lem. (Cycle) Given disjoint H and cycle C (length $\geq k$), $\exists\ C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

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If $\alpha(H) > 0$, apply Cycle Lem. with C_0 as C to get C':
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Let $H = G - \bigcup_{i=0}^{l-1} V(C_i)$; we are given $\alpha(H) \leq a-k-(l-1)$.

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$\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.

Including C' in the list yields (1), but we also need (2).

Cycle Lemma $\Rightarrow$ Multi-Cycle Lemma (cont.)

Lem. (Cycle) Given disjoint H and cycle C (length $\geq k$), $\exists\ C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

Cor. (Multi-Cycle) For $0 \leq \ell \leq a - k$, $\exists\ C_0, \ldots, C_\ell$ with
(1) $\alpha(G - \bigcup_{i=0}^{\ell} V(C_i)) \leq a - k - \ell$
(2) $\left|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)\right| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq \ell$.

Pf. Apply Cycle Lem. with C_0 as C to get C':
$\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.

Cycle Lemma $\Rightarrow$ Multi-Cycle Lemma (cont.)

Lem. (Cycle) Given disjoint H and cycle C (length $\geq k$), $\exists$ C' with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

Cor. (Multi-Cycle) For $0 \leq \ell \leq a - k$, $\exists$ $C_0, \ldots, C_\ell$ with
(1) $\alpha(G - \bigcup_{i=0}^{\ell} V(C_i)) \leq a - k - \ell$
(2) $\left|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)\right| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq \ell$.

Pf. Apply Cycle Lem. with C_0 as C to get C':
$\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.

Case 1: $|V(C')| \leq |V(C_0)|$.
Complete the desired list by setting $C_\ell = C'$; okay since

$$\left|V(C') - \bigcup_{j=0}^{l-1} V(C_i)\right| \leq \left|V(C') - V(C_0)\right| \leq \left|V(C_0) - V(C')\right| \leq \frac{|V(C_0)|}{k} - 1.$$

Cycle Lemma $\Rightarrow$ Multi-Cycle Lemma (cont.)

Lem. (Cycle) Given disjoint H and cycle C (length $\geq k$), $\exists$ C' with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

Cor. (Multi-Cycle) For $0 \leq \ell \leq a - k$, $\exists$ $C_0, \ldots, C_\ell$ with
(1) $\alpha(G - \bigcup_{i=0}^{\ell} V(C_i)) \leq a - k - \ell$
(2) $\left|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)\right| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq \ell$.

Pf. Apply Cycle Lem. with C_0 as C to get C':
$\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.

Case 2: $|V(C')| > |V(C_0)|$.
New list: Set $C'_0 = C'$, with $C'_i = C_{i-1}$ for $1 \leq i \leq \ell$.

Cycle Lemma $\Rightarrow$ Multi-Cycle Lemma (cont.)

Lem. (Cycle) Given disjoint H and cycle C (length $\geq k$), $\exists$ C' with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

Cor. (Multi-Cycle) For $0 \leq \ell \leq \alpha - k$, $\exists$ $C_0, \ldots, C_\ell$ with
(1) $\alpha(G - \bigcup_{i=0}^{\ell} V(C_i)) \leq \alpha - k - \ell$
(2) $\left|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)\right| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq \ell$.

Pf. Apply Cycle Lem. with C_0 as C to get C':
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Case 2: $|V(C')| > |V(C_0)|$.
New list: Set $C'_0 = C'$, with $C'_i = C_{i-1}$ for $1 \leq i \leq \ell$.
$i = 1$: $V(C'_1) - V(C'_0) = V(C_0) - V(C')$.
$i > 1$: $V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) \subseteq V(C_{i-1}) - \bigcup_{j=0}^{i-2} V(C_j)$.

Cycle Lemma $\Rightarrow$ Multi-Cycle Lemma (cont.)

Lem. (Cycle) Given disjoint H and cycle C (length $\geq k$), $\exists\ C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

Cor. (Multi-Cycle) For $0 \leq \ell \leq a - k$, $\exists\ C_0, \ldots, C_\ell$ with
(1) $\alpha(G - \bigcup_{i=0}^{\ell} V(C_i)) \leq a - k - \ell$
(2) $\left|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)\right| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq \ell$.

Pf. Apply Cycle Lem. with C_0 as C to get C':
$\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.

Case 2: $|V(C')| > |V(C_0)|$.
New list: Set $C_0' = C'$, with $C_i' = C_{i-1}$ for $1 \leq i \leq \ell$.
$i = 1$: $V(C_1') - V(C_0') = V(C_0) - V(C')$.
$i > 1$: $V(C_i') - \bigcup_{j=0}^{i-1} V(C_j') \subseteq V(C_{i-1}) - \bigcup_{j=0}^{i-2} V(C_j)$.
So, $\left|V(C_i') - \bigcup_{j=0}^{i-1} V(C_j')\right| \leq \frac{|V(C_0)|}{k} - 1 \leq \frac{|V(C_0')|}{k} - 1$. ■

Path Lemma $\Rightarrow$ Cycle Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, $\exists$ u, v-path P with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k-1)\}$.

Cor. (Cycle) Given disjoint H and cycle C (length $\geq k$), $\exists$ C' with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

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Pf. If H is disconnected or has a cut-vertex, then find C' using an induced subgraph of H. Otherwise, . . .

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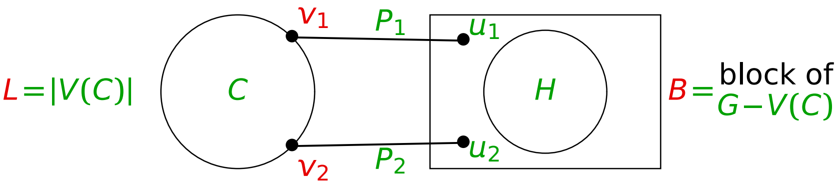

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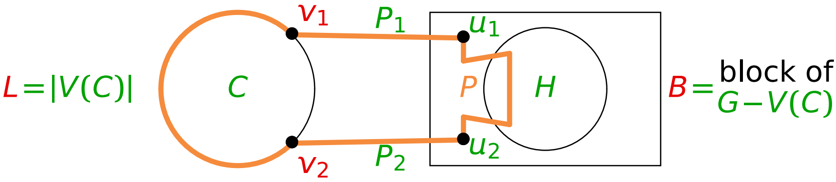

If $d_C(v_1, v_2) \leq L/k$, build cycle C' using P from Path Lem.

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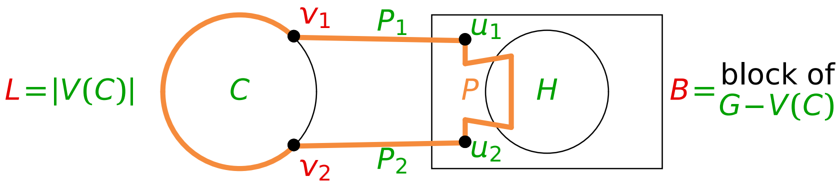

If $d_C(v_1, v_2) \leq L/k$, build cycle C' using P from Path Lem.
Let $\{c_1, \ldots, c_m\}$ be arrival points on C of paths from B.

Path Lemma $\Rightarrow$ Cycle Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, $\exists$ u, v-path P with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k-1)\}$.

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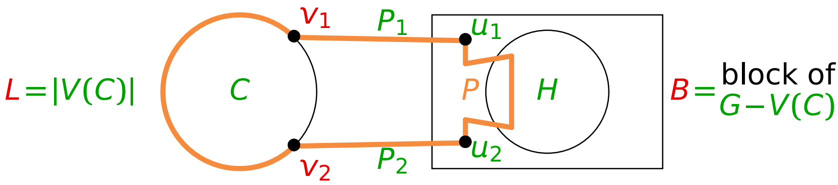

If $d_C(v_1, v_2) \leq L/k$, build cycle C' using P from Path Lem.

Let $\{c_1, \ldots, c_m\}$ be arrival points on C of paths from B.

Strict paths from $b, b' \in B$ reaching c_i, c_{i+1} are disjoint.

Path Lemma $\Rightarrow$ Cycle Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, $\exists$ u, v-path P with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k-1)\}$.

Cor. (Cycle) Given disjoint H and cycle C (length $\geq k$), $\exists$ C' with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

Pf. $\therefore$ $t = |\{i : c_i \;\&\; c_{i+1}$ reached from diff verts of $B\}| < k$.

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Cor. (Cycle) Given disjoint H and cycle C (length $\geq k$), $\exists$ C'
with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

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But, Menger $\Rightarrow$ k paths from $b \in B$ to C.

Path Lemma $\Rightarrow$ Cycle Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, $\exists$ u, v-path P with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k-1)\}$.

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At least $k - t$ "b-segments" of C meet paths from no other $b' \in B$.

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Over all $b \in B$, total length $< L - t(L/k) = L(k - t)/k$.

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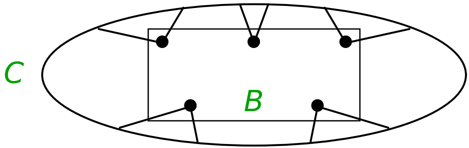
Pick a shortest $\forall$ $b \in B$; total length $< L/k$.



Path Lemma $\Rightarrow$ Cycle Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, $\exists\, u, v$-path P with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k-1)\}$.

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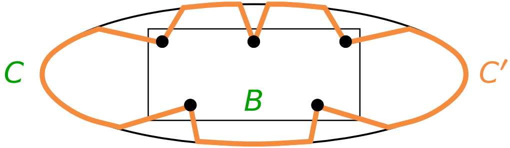
Pf. $\therefore\ t = |\{i\colon c_i\ \&\ c_{i+1}$ reached from diff verts of $B\}| < k$.

But, Menger $\Rightarrow$ k paths from $b \in B$ to C.

At least $k - t$ "b-segments" of C meet paths from no other $b' \in B$.

Over all $b \in B$, total length $< L - t(L/k) = L(k-t)/k$.

Pick a shortest $\forall\, b \in B$; total length $< L/k$. Form C'.



Proof of Path Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, $\exists$ u,v-path P with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k-1)\}$.

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Pf. For each u, v-path P, let F_P be a smallest component of $G - V(P)$ that intersects H. Choose P so that:

(i) $\alpha(H - V(P))$ is smallest;

(ii) subject to (i), F_P has the fewest vertices.

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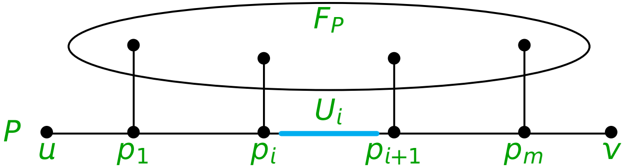
If $V(H) \not\subseteq V(P)$, then let $p_1, \ldots, p_m$ have neighbors in F_P.



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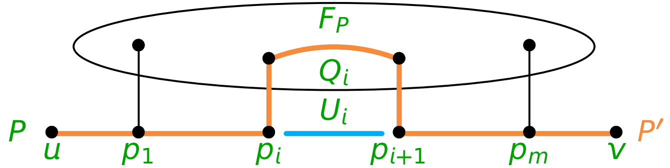

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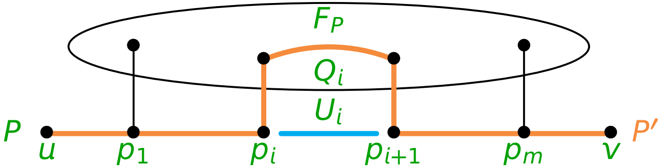

- $V(F_P) \cap V(H) \not\subseteq V(P')$, else $\alpha(H - V(P')) < \alpha(H - V(P))$.

Proof of Path Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$, $\exists$ u,v-path P with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k-1)\}$.

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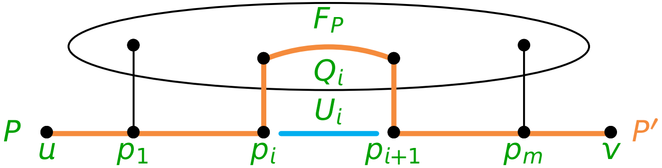

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$\therefore$ $F_P - V(P')$ has a component intersecting H.

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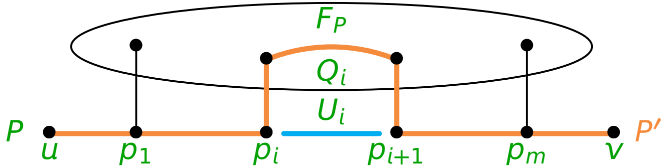


- $V(F_P) \cap V(H) \not\subseteq V(P')$, else $\alpha(H - V(P')) < \alpha(H - V(P))$.

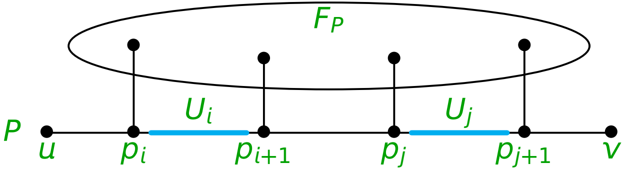
$\therefore$ $F_P - V(P')$ has a component intersecting H.

Thus $\alpha(H - V(P - U_i)) \geq \alpha(H - V(P')) > \alpha(H - V(P))$.

Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), F_P has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all i.



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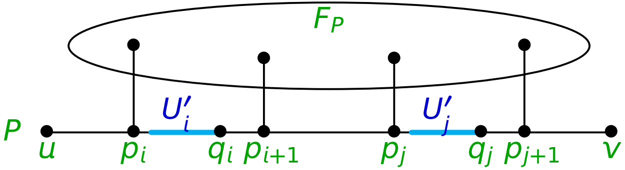

Restore U_i from the left; q_i is where α first increases.

Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
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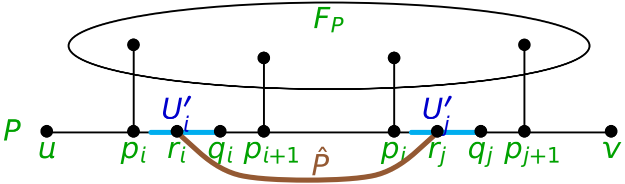


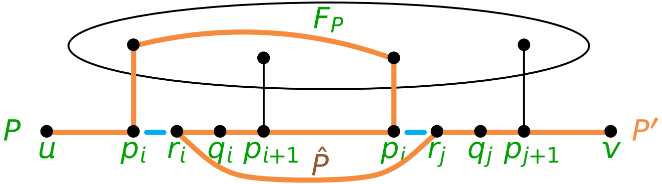
Restore U_i from the left; q_i is where α first increases.

A path $\hat{P}$ outside P joining U_i' and U_j' can't visit F_P.

Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), F_P has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all i.



Restore U_i from the left; q_i is where α first increases.

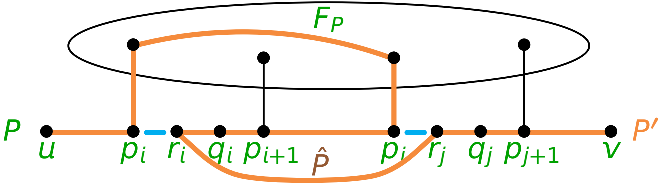
A path $\hat{P}$ outside P joining U'_i and U'_j can't visit F_P.

Pick leftmost such r_i and form new path P'.

Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), F_P has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all i.



Restore U_i from the left; q_i is where α first increases.

A path $\hat{P}$ outside P joining U'_i and U'_j can't visit F_P.

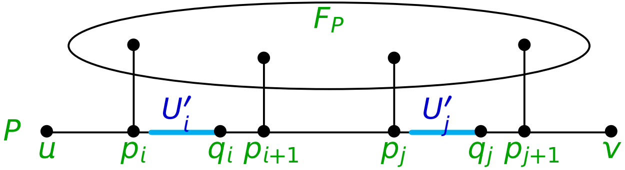
Pick leftmost such r_i and form new path P'.

Restoring ▬ can't increase α: again
$\alpha(H - V(P')) \le \alpha(H - V(P))$, contradicting choice of P.

Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), F_P has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all i.



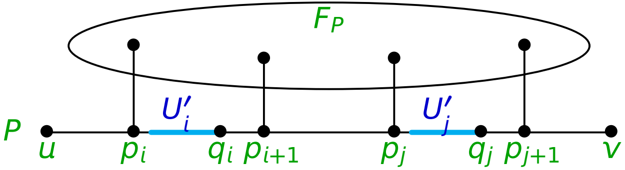
Restore U_i from the left; q_i is where α first increases.

Sets $U'_1, \ldots, U'_{m-1}$ are in diff comps of $G - V(P - U)$.

Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), F_P has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all i.



Restore U_i from the left; q_i is where α first increases.

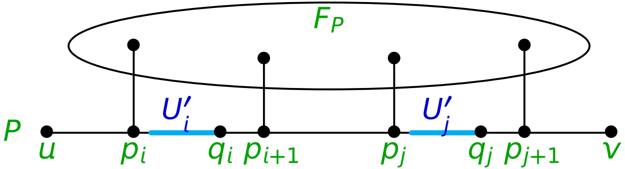
Sets $U_1', \ldots, U_{m-1}'$ are in diff comps of $G - V(P - U)$.

$\therefore\ \alpha(H - V(P - U)) \geq \alpha(H - V(P)) + m - 1 \geq \alpha(H - V(P)) + k - 1.$

Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), F_P has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all i.



Restore U_i from the left; q_i is where α first increases.

Sets $U_1', \ldots, U_{m-1}'$ are in diff comps of $G - V(P - U)$.

$\therefore \alpha(H - V(P - U)) \geq \alpha(H - V(P)) + m - 1 \geq \alpha(H - V(P)) + k - 1.$

$\therefore \alpha(H) \geq \alpha(H - V(P)) + k - 1.$ ■