

Longest Cycles in k -connected Graphs with Given Independence Number

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Abstract

The Chvátal–Erdős Theorem states that every graph whose connectivity is at least its independence number has a spanning cycle. In 1976, Fouquet and Jolivet conjectured an extension: If G is an n -vertex k -connected graph with independence number a , and $a \geq k$, then G has a cycle with length at least $\frac{k(n+a-k)}{a}$. We prove this conjecture.

1 Introduction

The famous theorem of Chvátal and Erdős [4], published in 1972, relates connectivity, independence number, and circumference of graphs. A graph G is k -connected if it has more than k vertices and every subgraph obtained by deleting fewer than k vertices is connected; the *connectivity* of G , written $\kappa(G)$, is the maximum k such that G is k -connected. An *independent set* is a set of pairwise nonadjacent vertices, and the *independence number* of G , written $\alpha(G)$, is the maximum size of such a set. The *circumference* is the maximum length of a cycle in G .

Theorem 1.1 (Chvátal–Erdős [4]) *If G is a graph such that $\kappa(G) \geq \alpha(G)$, then G has a cycle through all its vertices.*

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It is natural to ask what can be said when the condition $\kappa(G) \geq \alpha(G)$ is weakened: given a k -connected n -vertex graph with independence number a , where $a \geq k$, what is the best lower bound on the circumference? In 1976, Fouquet and Jolivet conjectured an answer.

Conjecture 1.2 (Fouquet–Jolivet [5]) *If G is a k -connected n -vertex graph with independence number a , and $a \geq k$, then G has a cycle with length at least $\frac{k(n+a-k)}{a}$.*

The case $k = a$ simplifies to the Chvátal–Erdős Theorem. The conjecture is sharp; infinitely often the circumference of G equals $k(n + a - k)/a$. For $a \geq k \geq 2$ and $m \in \mathbb{N}$, we construct such a graph: form G from one copy of the complete graph K_k and a disjoint copies of K_m by making every vertex in the k -clique adjacent to all the other vertices. Now G has $k + am$ vertices, $\alpha(G) = a$, $\kappa(G) = k$, and the maximum cycle length is $k(1 + m)$. Letting $n = k + am$, we see that $k(1 + m) = k(n + a - k)/a$.

In 1982, Fournier [6] proved Conjecture 1.2 for $a \in \{k + 1, k + 2\}$. Two years later, he also proved it for $k = 2$ [7], using the fact that if C_1 and C_2 are distinct cycles in a 2-connected graph G , then there are distinct cycles C'_1 and C'_2 in G such that $V(C_1) \cup V(C_2) \subseteq V(C'_1) \cup V(C'_2)$ and $|V(C'_1) \cap V(C'_2)| \geq 2$. In 2009, Manoussakis [9] proved the case $k = 3$ using a similar fact. This leads to a general conjecture.

Conjecture 1.3 (Chen–Chen–Liu) *If C_1 and C_2 are distinct cycles in a k -connected graph G , then there are distinct cycles C'_1 and C'_2 in G such that $V(C_1) \cup V(C_2) \subseteq V(C'_1) \cup V(C'_2)$ and $|V(C'_1) \cap V(C'_2)| \geq k$.*

Recently, Chen, Hu, and Wu [2] proved Conjecture 1.2 for $k = 4$. In another paper [3], they proved that Conjecture 1.3 implies Conjecture 1.2, and they also proved Conjecture 1.2 for $a < 2k - 1$. In this paper, without proving the stronger Conjecture 1.3, we prove Conjecture 1.2 in full. Akira Saito asked whether the sharpness construction above is essentially unique; possibly our techniques could be used to prove that.

2 The Path Lemma

We need notation for induced subgraphs. Given $S \subseteq V(G)$, let $\overline{S} = V(G) - S$. The subgraph of G induced by S is the subgraph obtained by deleting the vertices of \overline{S} ; this may be written as $G[S]$ or $G - \overline{S}$. When $\overline{S} = \{v\}$, we write $G - v$ instead of $G - \{v\}$. We also write $G - e$ for the (non-induced) subgraph obtained by deleting an edge e .

The following result of Kouider [8] has been used in partial results toward Conjecture 1.2.

Theorem 2.1 (Kouider [8]) *If H is a subgraph of a k -connected graph G , then either $V(H)$ can be covered by a cycle in G , or there is a cycle C in G such that $\alpha(H - V(C)) \leq \alpha(H) - k$.*

A single application of Theorem 2.1 with $H = G$ implies Theorem 1.1 of Chvátal and Erdős when $\kappa(G) \geq \alpha(G)$; a spanning cycle is guaranteed. When $\kappa(G) < \alpha(G)$, repeatedly applying Theorem 2.1 with H being the subgraph left by deleting the vertices of earlier cycles shows that the vertices of a graph G can be covered by at most $\lceil \frac{\alpha(G)}{\kappa(G)} \rceil$ cycles. For a k -connected n -vertex graph with independence number a , among these cycles is one of length at least $n/\lceil a/k \rceil$, which is close to the conjectured threshold of $nk/a + k(1 - k/a)$.

Inspired by Kouider's result and her proof, we prove an analogous theorem about paths joining two specified vertices. We actually only need Theorem 2.2 for $k = 2$, but the proof of the general statement is the same. The proof is a slight modification of Kouider's proof of Theorem 2.1. We will need notation for subpaths of a path. Let u and v be distinct vertices in a graph G . A u, v -path is a path with first vertex u and last vertex v . Given a path P and vertices $a, b \in V(P)$, let $P[a, b]$ be the a, b -path contained in P . Similarly, let $P(a, b) = P[a, b] - \{a, b\}$, let $P[a, b) = P[a, b] - b$, and let $P(a, b) = P[a, b] - a$.

Theorem 2.2 *Let G be a k -connected graph. If $H \subseteq G$, and u and v are distinct vertices in G , then G contains a u, v -path P such that $V(H) \subseteq V(P)$ or $\alpha(H - V(P)) \leq \alpha(H) - (k - 1)$.*

Proof. We may assume that no u, v -path P contains $V(H)$. For each u, v -path P , let F_P be a smallest component of $G - V(P)$ that intersects H . Choose a u, v -path P such that:

- (i) $\alpha(H - V(P))$ is smallest;
- (ii) subject to (i), F_P has the fewest vertices.

Let p_1, \dots, p_m be the vertices of P (in order) with neighbors in F_P . Let $U_i = V(P(p_i, p_{i+1}))$.

Claim 1: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for $1 \leq i < m$. Otherwise, $\alpha(H - V(P - U_i)) = \alpha(H - V(P))$; assume this. Let P' be a u, v -path obtained from P by deleting U_i and adding a p_i, p_{i+1} -path whose set of internal vertices is nonempty and lies in F_P . If $V(F_P) \cap V(H) \subseteq V(P')$, then $\alpha(H - V(P')) < \alpha(H - V(P))$, contradicting (i). Hence P' does not cover $V(F_P) \cap V(H)$. Since $V(P - U_i) \subseteq V(P')$, we have $\alpha(H - V(P')) \leq \alpha(H - V(P - U_i)) = \alpha(H - V(P))$. Since there remains a vertex of $F_P \cap H$ outside P' , we have $|V(F_{P'})| < |V(F_P)|$, contradicting (ii). This proves the claim.

By Claim 1, restoring U_i to the induced subgraph $H - V(P)$ increases the independence number. Since U_i is nonempty, p_1, \dots, p_m is a separating set, and hence $m \geq k$. Restoring the vertices of U_i in order, starting from p_i , let q_i be the first vertex at which the independence number increases (see Figure 1). That is, with $U'_i = V(P(p_i, q_i))$, we have $\alpha(H - V(P - U'_i)) = \alpha(H - V(P)) + 1$, but $\alpha(H - V(P - U'_i) - q_i) = \alpha(H - V(P))$.

Claim 2: For $1 \leq i < j < m$, no path whose internal vertices all lie outside P joins U'_i and U'_j . Otherwise, let $r_i \in U'_i$ and $r_j \in U'_j$ be the endpoints of such a path \hat{P} , chosen so that r_i is as close to p_i along P as possible. Since F_P is a component of $G - V(P)$, and vertices

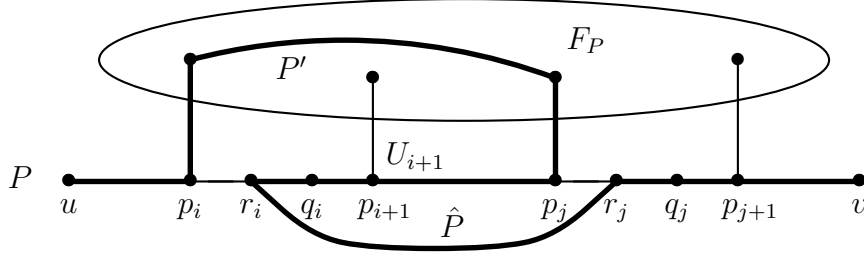


Figure 1: Finding a better path (Claim 2)

of U'_i and U'_j have no neighbors in F_P , the path \hat{P} does not visit F_P . Form P' from P by deleting $V(P(p_i, r_i))$ and $V(P(p_j, r_j))$ and adding \hat{P} and a p_i, p_j -path through F_P .

Since $r_j \in U'_j$, restoring the vertices in $P(p_j, r_j)$ to $H - V(P)$ does not produce a larger independent set than exists in $H - V(P)$, and the same is true of $P(p_i, r_i)$. Furthermore, the choice of r_i forbids paths from $V(P(p_i, r_i))$ to $V(P(p_j, r_j))$ in $H - V(P)$, so restoring both sets adds them to different components of $H - V(P)$, and hence restoring both does not increase the independence number.

We conclude that $\alpha(H - V(P')) \leq \alpha(H - V(P))$. As in the proof of Claim 1, $V(F_P) \cap V(H) \subseteq V(P')$ yields strict inequality and violates (i), while $V(F_P) \cap V(H) \not\subseteq V(P')$ and equality imply that P' violates (ii). This proves the claim.

By the choice of q_i , we have $\alpha(H - V(P - U'_i)) \geq \alpha(H - V(P)) + 1$. Let $U = \bigcup_{i=1}^{m-1} U'_i$. By Claim 2, the sets U'_1, \dots, U'_{m-1} lie in different components of $G - V(P - U)$. Hence $\alpha(H - V(P - U)) \geq \alpha(H - V(P)) + m - 1$. Since $\alpha(H) \geq \alpha(H - V(P - U))$ and $m \geq k$, we have $\alpha(H - V(P)) \leq \alpha(H) - k + 1$ for the chosen path P . \square

Theorem 2.2 implies a conjecture posed in Chen, Hu, and Wu [2].

Corollary 2.3 *If a graph G admits no vertex partition (V_1, V_2) such that $\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2])$, then G is connected and has no cut-vertex, and any distinct vertices $u, v \in V(G)$ are the endpoints of a path P such that $\alpha(G - V(P)) < \alpha(G)$.*

Proof. If G is disconnected, then such a partition exists. Suppose that G is connected and has a cut-vertex x . Let A be a component of $G - x$, and let $B = G - x - V(A)$. Let $A' = G - V(B)$ and $B' = G - V(A)$. If $\alpha(A) = \alpha(A')$, then

$$\alpha(G) \leq \alpha(A') + \alpha(B) = \alpha(A) + \alpha(B) \leq \alpha(G).$$

Equality holds throughout, and $(V(A'), V(B))$ is the required partition.

The remaining alternative is $\alpha(A) = \alpha(A') - 1$. Now there is an independent set S of size $\alpha(A)$ that contains no neighbor of x . We compute

$$\alpha(G) \leq \alpha(A) + \alpha(B') = |S| + \alpha(B') \leq \alpha(G),$$

and $(V(A), V(B'))$ is the required partition.

The final statement holds trivially if $G \in \{K_1, K_2\}$. Otherwise, Theorem 2.2 now applies with $k = 2$ and $H = G$. \square

The sufficient condition given is not a necessary condition, as shown by the union of two complete graphs sharing one vertex. Examples where the conclusion fails include graphs consisting of two disjoint complete graphs plus one edge joining them.

3 Finding a Good Cycle

Given disjoint subgraphs F and H of a graph G , let an F, H -path in G be a path with endpoints in $V(F)$ and $V(H)$ and no internal vertex in $V(F) \cup V(H)$; this generalizes “ u, v -path”. Given a specified orientation of a cycle C and vertices $a, b \in V(C)$, let $C[a, b]$ be the a, b -path on C in the given orientation. Similarly, let $C(a, b) = C[a, b] - \{a, b\}$. A *block* in a graph is a maximal subgraph having no cut-vertex; a graph is the union of its blocks.

Theorem 3.1 *Let k be an integer greater than 1. If C is a cycle with length at least k in a k -connected graph G , then for any nonempty subgraph H of $G - V(C)$, there exists a cycle C' in G such that $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.*

Proof. Consider a minimal counterexample H for some graph G and cycle C . Let $L = |V(C)|$. If H is disconnected or has a cut-vertex, then $\alpha(H) = \alpha(H[V_1]) + \alpha(H[V_2])$ for some partition (V_1, V_2) of $V(H)$, by Corollary 2.3. By the minimality of H , there is a cycle C' in $H[V_1]$ such that $|V(C) - V(C')| \leq (L/k) - 1$ and $\alpha(H[V_1 - V(C')]) \leq \alpha(H[V_1]) - 1$. Now $\alpha(H - V(C')) \leq \alpha(H[V_1 - V(C')]) + \alpha(H[V_2]) \leq \alpha(H[V_1]) - 1 + \alpha(H[V_2]) = \alpha(H) - 1$.

We may therefore assume that H is 2-connected or $H \in \{K_1, K_2\}$. Let B be the block of $G - V(C)$ that contains H . For B, C -paths P_1 and P_2 , define the C -distance between P_1 and P_2 to be the distance in C between the endpoints of P_1 and P_2 in C .

For $b \in V(B)$, a standard consequence of Menger’s Theorem yields k paths from b to C that pairwise share only b ; call this a b, C -fan. By the pigeonhole principle, the C -distance between some two paths in a b, C -fan is at most L/k . If b is the only vertex of B (and hence $H = B$), then using those two paths to replace the part of C between their endpoints yields the desired cycle C' . Hence we may assume $|V(B)| > 1$.

Let P_1 and P_2 be two disjoint B, C -paths, with P_i having endpoints $u_i \in B$ and $v_i \in C$. Since B is connected and has no cut-vertex, Theorem 2.2 guarantees a u_1, u_2 -path P in B such that $\alpha(H - V(P)) \leq \alpha(H) - 1$. If $|C(v_1, v_2)| \leq L/k - 1$, then $(C - C(v_1, v_2)) \cup P_1 \cup P \cup P_2$ is the desired cycle C' (see Figure 2). Hence we may assume (*) *the C -distance between any two disjoint B, C -paths is more than L/k* . Note also that B, C -paths with distinct endpoints in B are internally disjoint, since B is a block in $G - V(C)$.

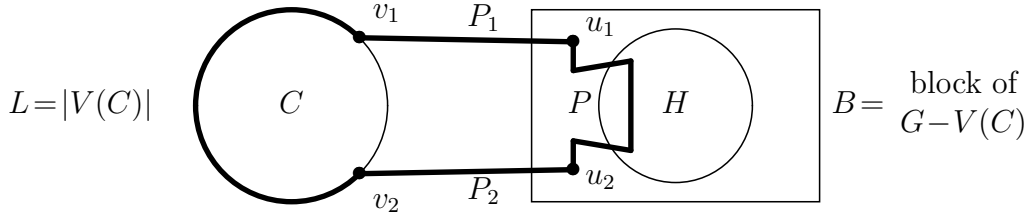


Figure 2: A detour to reduce $\alpha(H)$

Let c_1, \dots, c_m be the endpoints in C of B, C -paths, indexed so that c_1, \dots, c_m appear in that order along a fixed orientation of C . Let $P_i = C[c_i, c_{i+1}]$ (indices modulo m); call P_i the i th segment of C . Let t be the number of indices i (modulo m) such that c_i and c_{i+1} are the endpoints of B, C -paths from distinct vertices of B . By $(*)$, each such segment has length more than L/k , and hence $t < k$.

For $b \in V(B)$, a b, C -fan has k endpoints in C . Some $k - t$ of the paths along C joining consecutive endpoints of the fan must not contain endpoints of B, C -paths from other vertices of B . Hence these paths are distinct for distinct vertices of B . Consider a segment within each such path.

Since these segments avoid the t excluded segments, their total length is less than $L - t(L/k)$, which equals $L(k - t)/k$. For each vertex of B , choose a shortest among these $k - t$ segments. The total length of the union of the chosen segments is less than L/k .

Form C' from C by deleting the chosen segments and adding, for each $b \in B$, the two paths in the b, C -fan whose endpoints are the ends of the segment chosen for b (see Figure 3). The subgraph C' is a cycle, because B, C -paths from distinct vertices of B are internally disjoint. Since the total length of the chosen segments is less than L/k and $V(H) \subseteq V(B) \subseteq V(C')$, the cycle C' has the desired properties. \square

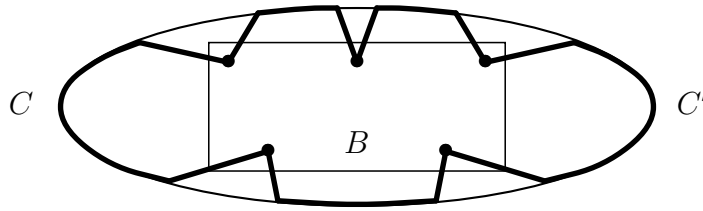


Figure 3: Skipping the chosen segments

Lemma 3.2 *If G is a k -connected graph with independence number a , and $0 \leq l \leq a - k$, then there exist cycles C_0, \dots, C_l satisfying the following conditions:*

- (1) $\alpha(G - \bigcup_{i=0}^l V(C_i)) \leq a - k - l$,
(2) $\left| V(C_i) - \bigcup_{j=0}^{i-1} V(C_j) \right| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq l$.

Proof. We prove the claim by induction on l . For $l = 0$, Theorem 2.1 with $H = G$ provides a cycle C_0 such that $\alpha(G - V(C_0)) \leq a - k$. For the induction step, consider l with $0 < l \leq a - k$, and suppose that cycles C_0, \dots, C_{l-1} exist satisfying the claim for $l - 1$. We observe first that $|V(C_0)| \geq k$; when $l = 1$ this holds because the case $l = 0$ of (1) states that $\alpha(G - V(C_0)) \leq a - k$, and when $l > 1$ it holds because the left side of (2) is nonnegative.

Let $H = G - \bigcup_{i=0}^{l-1} V(C_i)$; by hypothesis, $\alpha(H) \leq a - k - (l - 1)$. We may assume $\alpha(H) \geq 1$; otherwise, just let $C_l = C_0$. Since $|V(C_0)| \geq k$, we can apply Theorem 3.1 using C_0 as C to obtain a cycle C' in G such that $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1 \leq a - k - l$. Now adding C' to the list as C_l satisfies (1), but we must also satisfy (2).

Case 1: $|V(C')| \leq |V(C_0)|$. Note that

$$\left| V(C') - \bigcup_{j=0}^{l-1} V(C_j) \right| \leq |V(C') - V(C_0)| \leq |V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1.$$

In this case it suffices to add C' as C_l .

Case 2: $|V(C')| > |V(C_0)|$. Define a new list C'_0, \dots, C'_l of cycles by letting $C'_0 = C'$ and letting $C'_i = C_{i-1}$ for $1 \leq i \leq l$. Now $\alpha(G - \bigcup_{i=0}^l V(C'_i)) = \alpha(H - V(C')) \leq a - k - l$, satisfying (1). Also, for $i = 1$ we have $V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) = V(C'_1) - V(C'_0) = V(C_0) - V(C')$, and for $2 \leq i \leq l$ we have $V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) \subseteq V(C_{i-1}) - \bigcup_{j=0}^{i-2} V(C_j)$. In both cases,

$$\left| V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) \right| \leq \frac{|V(C_0)|}{k} - 1 \leq \frac{|V(C'_0)|}{k} - 1.$$

Hence C'_0, \dots, C'_l satisfies the required conditions. \square

We can now prove Conjecture 1.2, the conjecture of Fouquet and Jolivet.

Corollary 3.3 *If G is a k -connected n -vertex graph with independence number a , and $a \geq k$, then G has a cycle of length at least $\frac{k(n+a-k)}{a}$.*

Proof. Consider $l = a - k$ in Lemma 3.2. By (1), the resulting cycles C_0, \dots, C_l cover $V(G)$. Using this and then summing the inequalities in (2), we obtain

$$n = |V(C_0)| + \sum_{i=1}^l \left| V(C_i) - \bigcup_{j=0}^{i-1} V(C_j) \right| \leq |V(C_0)| + (a - k) \left(\frac{|V(C_0)|}{k} - 1 \right).$$

The inequality simplifies to $|V(C_0)| \geq \frac{k(n+a-k)}{a}$. \square

References

- [1] G. Chen, R. J. Faudree, and R. J. Gould, Intersections of longest cycles in k -connected graphs. *J. Combin. Theory (B)* **72** (1998), 143–149.
- [2] G. Chen, Z. Hu, and Y. Wu, Circumferences of k -connected graphs involving independence numbers. Manuscript, 2010.
- [3] G. Chen, Z. Hu, and Y. Wu, Circumferences of k -connected graphs involving independence numbers, II. Manuscript, 2010.
- [4] V. Chvátal and P. Erdős, A note on hamiltonian circuits. *Discr. Math.* **2** (1972) 111–113.
- [5] J.L. Fouquet and J.L. Jolivet, Problème 438, in *Problèmes combinatoires et théorie des graphes, Orsay*, (1976).
- [6] I. Fournier, Thesis, University Parix-XI, Orsay (1982).
- [7] I. Fournier, Longest cycles in 2-connected graphs of stability α , in *Cycles in Graphs* (B. Alspach and C.D. Godsil, eds.), (Elsevier/North-Holland, 1984).
- [8] M. Kouider, Cycles in graphs with prescribed stability number and connectivity, *J. Combin. Th (B)* **60** (1994), 315–318.
- [9] Y. Manoussakis, Longest cycles in 3-connected graphs with given indepdence number. *Graphs & Combin.* (2009), 377–384.