INTERVAL DIGRAPHS THAT ARE INDIFFERENCE DIGRAPHS

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Abstract

A digraph is an *interval digraph* if each vertex can be assigned a source interval and a sink interval such that there is an edge from u to v if and only if the source interval for u intersects the sink interval for v. It is an *indifference digraph* or *unit interval digraph* if it has such a representation in which every interval has unit length. We prove that an interval digraph is a unit interval digraph if and only if its adjacency matrix is free of six forbidden submatrices: three 3 by 4 matrices and their transposes.

1 Introduction

An *intersection representation* of a graph G with vertex set V is a family $\{S_v : v \in V\}$ of sets such that u, v are adjacent if and only $S_u \cap S_v \neq \emptyset$. An *interval graph* has such a representation in which the sets are real intervals; it is a *unit interval graph* if the intervals may have the same length. Analogous models exist for directed graphs [1,9] or general binary relations [2]. An *intersection representation* of a digraph D with vertex set V is a family $\{(S_v, T_v) : v \in V\}$ of ordered pairs of sets such that uv is an edge if and only if $S_u \cap T_v \neq \emptyset$. Here S_u is the *source set* of u, T_v is the *sink set* of v, and D is the *intersection digraph* of the family.

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An *interval digraph* is the intersection digraph of a family of pairs of real intervals; it is a *unit interval digraph* if the intervals may have the same length. The former were characterized in [9] and in [10], the latter in [7]. These characterizations generalize results for interval graphs and unit interval graphs, because (see [8]) a graph G is an interval graph if and only if a corresponding digraph $D(G)$ is an interval digraph (similarly for "unit interval"). The adjacency matrix of $D(G)$ is obtained from the adjacency matrix of G by adding 1's on the diagonal; this is called the *augmented adjacency matrix* $A^*(G)$.

Roberts [5] proved unit interval graphs equivalent to *proper interval graphs* (having a representation with no interval containing another) and to *indifference graphs* (admitting a vertex-weighting f such that u, v are adjacent if and only if $|f(u) - f(v)| \leq 1$.) Sen and Sanyal [7] generalized this by proving the equivalence of unit interval, proper interval, and indifference digraphs, where an *indifference digraph* admits a pair of weightings f, g such that uv is an edge if and only if $|f(u) - g(v)| \leq 1$. They also characterized adjacency matrices of digraphs in these families (generalizing Roberts' characterization [4] of $A^*(G)$ for proper interval graphs), and they used this characterization to generalize the Scott-Suppes theorem [6].

Roberts [5] also characterized unit interval graphs as interval graphs not having the complete bipartite graph $K_{1,3}$ as an induced subgraph. We prove an analogous characterization for unit interval digraphs, characterizing unit interval digraphs as interval digraphs whose adjacency matrices have no submatrix in a set $\mathbf{F}$ of six matrices. We use the adjacency matrix characterizations for interval digraphs and unit interval digraphs, stated below. First, a digraph D is a *Ferrers digraph* (introduced in [3]) if its adjacency matrix $A(D)$ has no 2 by 2 permutation submatrix. Equivalently, the rows and columns of $A(D)$ can be permuted independently so the 1's form a Ferrers diagram. We use $\overline{D}$ to denote the digraph whose adjacency matrix is the difference between $A(D)$ and the all-1 matrix.

Theorem A (Sen *et al* [9,10]). The following are equivalent:

1) D is an interval digraph.

2) $\overline{D}$ is the union of two disjoint Ferrers digraphs.

3) The rows and columns of $A(D)$ can be permuted independently so each 0 can be labeled R or C in such a way that every position to the right of an R is an R and every position below a C is a C. □

A matrix satisfying (3) above has the *partitionable zeros property*, and we say such a matrix is *zero-partitionable*. The corresponding permutation and partition exhibiting the property is a *zero-partition* of the matrix. The matrix characterization of unit interval digraphs is a further restriction of the partitionable zeros property. A 0,1-matrix has a *monotone consecutive arrangement* if there exist independent row and column permutations exhibiting the following structure: the 0's of the resulting matrix can be labeled R or C such that every position above or to the right of an R is a 0 labeled R, and every position below or to the left of a C is a 0 labeled C. We say that the resulting matrix and labeling *is* a monotone consecutive arrangement, which we abbreviate as "MCA".

Theorem B (Sen and Sanyal [7]). D is a unit interval digraph if and only if $A(D)$ has a monotone consecutive arrangement. □

The adjective "monotone" arises from an equivalent description. Given an m by n zero-partition, record the sequences $r_1, \ldots, r_m$ and $c_1, \ldots, c_n$ of the number of R's in each row and C's in each column. The zero-partition is an MCA if and only if $r_1 \geq \ldots \geq r_m$ and $c_1 \geq \ldots \geq c_n$. We use m, n because the definitions of zero-partition and MCA generalize to non-square 0,1-matrices.

If an intersection model specifies no relationship between the source set and sink set assigned to a given vertex of D, then rows and columns of $A(D)$ can be permuted independently without affecting representability. This holds for interval and unit interval digraphs. Harary, Kabell, and McMorris [2] introduced an intersection model for bipartite graphs that is relevant here. Given bipartition A, B of the vertices, assign each $u \in A$ a set S_u and each $v \in B$ a set T_v, and include edge uv when $S_u \cap T_v \neq \emptyset$. This model differs from ordinary intersection representations by explicitly ignoring intersections between sets for vertices of the same partite set. It generalizes the digraph model by allowing non-square matrices to encode the adjacency relation, but the distinction is unimportant when the null set is allowable in the digraph intersection representations.

Hence our results can be phrased as statements about 0,1-matrices and binary relations, and the context of "intersection bigraphs" is apt. The adjacency characterizations of interval digraphs and interval bigraphs are essentially the same, because adding a row or column of 0's does not affect zero-partitionability or the existence of MCA's. We therefore characterize interval digraphs that are unit interval digraphs by obtaining the list of minimal forbidden submatrices characterizing the zero-partitionable matrices that have MCA's. We consider adjacency matrices instead of forbidden subgraphs because a submatrix of the adjacency matrix can be realized in many ways as a subdigraph, just as there are several forbidden induced subdigraphs for Ferrers digraphs but only one minimal forbidden submatrix for the adjacency matrix.

Deleting a row or column from an MCA leaves an MCA (similarly for zero-partitions); hence the properties are hereditary and a forbidden submatrix characterization must exist. Sen and Sanyal [7] found a 4-vertex interval digraph that is not a unit interval digraph. Its adjacency matrix has a row of 0's; the remaining rows form one of our minimal forbidden submatrices. Let F_1, F_2, F_3 be the matrices below, and let $\mathbf{F} = \{F_1, F_2, F_3, F_1^T, F_2^T, F_3^T\}$. We prove that a zero-partitionable matrix has an MCA if and only if it does not have a submatrix in $\mathbf{F}$, i.e. is $\mathbf{F}$-*free*. (Note: arbitrary row and column permutations of these are also forbidden.) In the bigraph model, our result becomes a list of three forbidden 7-vertex subgraphs.

$$\begin{matrix} 1\,1\,1\,0 \\ 0\,1\,1\,1 \\ 0\,0\,1\,0 \\ F_1 \end{matrix} \qquad \begin{matrix} 1\,1\,0\,0 \\ 1\,1\,1\,1 \\ 0\,1\,0\,1 \\ F_2 \end{matrix} \qquad \begin{matrix} 1\,1\,0\,0 \\ 1\,0\,1\,0 \\ 1\,0\,0\,1 \\ F_3 \end{matrix}$$

Since these submatrices have bounded size, this yields a polynomial-time algorithm (naively $O(m^3n^4 + m^4n^3)$) to test whether a zero-partitioned matrix has an MCA. Under more careful examination, our inductive proof can be converted to an algorithm running in time $O((m+n)^2)$; we describe this in the final section. This is not the same as testing whether an interval digraph is a unit interval digraph, because the algorithm assumes we are given a zero-partition, which by the results of [9] or [10] is equivalent to having an interval representation. No polynomial-time algorithm is yet known to produce such a representation or to recognize interval digraphs. We hope such an algorithm exists but do not expect a finite list of forbidden submatrices, just as interval graphs have a polynomial-time recognition algorithm but an infinite list of minimal forbidden induced subgraphs.

2 Forbidden Configurations

In this section we verify that the matrices in **F** are minimal forbidden submatrices.

Lemma 1 *Zero-partitionable matrices with at most two rows or at most two columns have MCAs.*

Proof: Consider a zero-partition with two columns. List the rows in order as follows: first every 10 as 1R, then every 11, then every 01 as C1, then every 00 as CC. □

Lemma 2 *A 3 by n zero-partitionable matrix has an MCA if and only if it is* **F***-free.*

Proof: Since identical columns can be placed consecutively, and a column of zeros can be added leftmost as C's or rightmost as R's in any MCA, it suffices to consider a 3-row matrix M with distinct nonzero columns. For $i \in \{1, 2, 3\}$, a column with i 1's is an *i-column.* If M contains all three 2-columns, then their 2 by 2 submatrices prevent a zero-partition. If M does not have all three 2-columns and has no 3-column, then M can be permuted to (a submatrix of) the MCA T_1 below:

1 1 0 0 0	1 1 R	1 1 R R
0 1 1 1 0	C 1 R	1 1 1 R
0 0 0 1 1	C 1 1	C 1 1 1
T_1	T_2	T_3

Hence we may assume M has a 3-column. If M also has all three 1-columns, then M contains F_3. In a zero-partition, these three columns must appear as an identity submatrix, but then there is no way to insert the column of 1's to produce an MCA. Hence F_3 is forbidden.

We may now assume M has at most two distinct 1-columns. If two, then they contain a 2 by 2 permutation submatrix, and an MCA must place the 3-column between them. The zeros in the leftmost of these columns then must be C's and in the rightmost must be R's, and we have fixed the row permutation and zero-partition of this submatrix as in T_2 above. Of the 2-columns, we can add 110^T and/or 011^T, but we cannot add 101^T to produce an MCA. Adding 101^T yields a permutation of F_1, which is therefore forbidden.

Finally, we may assume M has the 3-column, at most two 2-columns, and at most one 1-column. If there is no 1-column, then the 3-column can be placed between the 2-columns to obtain an MCA, so suppose there is a 1-column, v. If M also has both 2-columns whose 0 is in the same row as the 0's in v, then we obtain F_2 as a submatrix. In F_2, the 2 by 2 permutation submatrix in the 2-columns forces one 0 to be C and one to be R. This forces the column of 1's between them in any MCA, but then there is no way to add v and still have an MCA, so F_2 is forbidden. Hence the two 2-columns having a 0 in the same row as v cannot both occur. With any other set of two 2-columns (one of these two and the complement of v), we obtain an MCA, as indicated by T_3 above. □

We have proved that a 3-row zero-partitionable matrix has an MCA if and only if it is **F**-free. This and Lemma 1 imply that the matrices of **F** are minimal forbidden submatrices.

3 Structure of Counterexamples

Since the matrices of **F** are minimal forbidden submatrices, it suffices to prove that any **F**-free zero-partitionable matrix has an MCA. Let us explore the properties of a minimal counterexample.

Lemma 3 *A minimal zero-partitioned matrix M having no MCA has the following properties:*

1) M has no row or column of zeros.

2) M has no block partition $\binom{A\ 0}{0\ B}$.

3) M has no pair of identical rows or identical columns.

4) M has some $r_i = 0$ and $c_j = 0$, where these are not both the last values.

Proof: For (1),(2),(3), it is easy to construct a monotone consecutive arrangement for M from that obtained from appropriate submatrices. For (4), (1) implies that the last row and column have a 1, but if the only such 1 is in the lower right corner, then M violates (2). □

A minimal zero-partitioned matrix M having no MCA has an ascent in one of the sequences $\{r_i\}, \{c_j\}$. Since we desire nonincreasing sequences, we define an *inversion* to be a pair of (not necessarily consecutive) entries in r or c for which the later entry is larger than the earlier entry. Also, as parameters of a zero-partition, we let k, l denote the number of rows with no C and the number of columns with no R, respectively. Note that the rows with no C must be the initial rows, and the columns with no R must be the initial columns. As additional parameters, we let k', l' denote the (maximum) number of initial rows and initial columns that form an MCA, respectively. The initial k' rows form an MCA if and only if $r_1 \geq \ldots \geq r_{k'}$ and $c_1 \geq \ldots \geq c_{j'} \geq m - k' \geq \max\{c_j : j > j'\}$ for some j'. We define a *critical* matrix to be a minimal zero-partitioned **F**-free matrix with no MCA, such that among all possible zero-partitions, this zero-partition is chosen to maximize $k+l$, and among those that maximize $k+l$, this zero-partition maximizes $k'+l'$.

Lemma 4 *A critical matrix M (with k C-free rows and l R-free columns) has the following properties:*

1) $k = m - \max\{c_j\}$, $l = n - \max\{r_i\}$, and $k, l \geq 1$.

2) $r_1 = n - l$ or $c_1 = m - k$.

2) $c_1 > \ldots > c_l$ and $r_1 > \ldots > r_k$.

4) $l \leq c_1 - c_l + 1$ and $k \leq r_1 - r_k + 1$.

5) $k' \geq k$ and $l' \geq l$.

Proof: (1): Because M is a zero-partition and has no row or column of 0's. (2): If $r_1 \neq \max\{r_i\} = r_{i'}$ and $c_1 \neq \max\{c_j\} = c_{j'}$, then $r_{i'} \geq n - j'$ and $c_{j'} \geq m - i'$ is impossible, as it specifies that position (i', j') is both R and C. On the other hand, $c_{j'} < m - i'$ permits the first i' rows to be placed in monotone order, while $r_{i'} < n - j'$ permits the first j' columns to be placed in monotone order. Hence $r_1 = \max\{r_i\}$ or $c_1 = \max\{c_j\}$, and (1) completes the proof. (3): By the distinctness of rows and columns and the maximality of $k'+l'$. (4): By (3). (5): By (3) and the definition of k, l. □

Lemma 4 (parts 1 and 2) and symmetry allow us to assume $r_1 = \max\{r_i\}$, and we define another parameter q to be the smallest index such that $c_q = \max\{c_j\}$. We define submatrices $J, A, B, C, R, D, \hat{D}, E, F, \hat{F}, G$ as illustrated below. The ranges of columns are 1 to l, $l+1$ to $n - r_k$, and $n - r_k + 1$ to n. Except for $\hat{D}, \hat{F}$, the ranges of rows are 1 to k, $k+1$ to $m - c_l$, and $m - c_l + 1$ to m. The matrices $\hat{D}, \hat{F}$ are the submatrices of D, F consisting of rows $m - c_1 + 1$ to $m - c_l$; they are proper submatrices of D, F if $q > 1$ and are empty if $l = 1$.

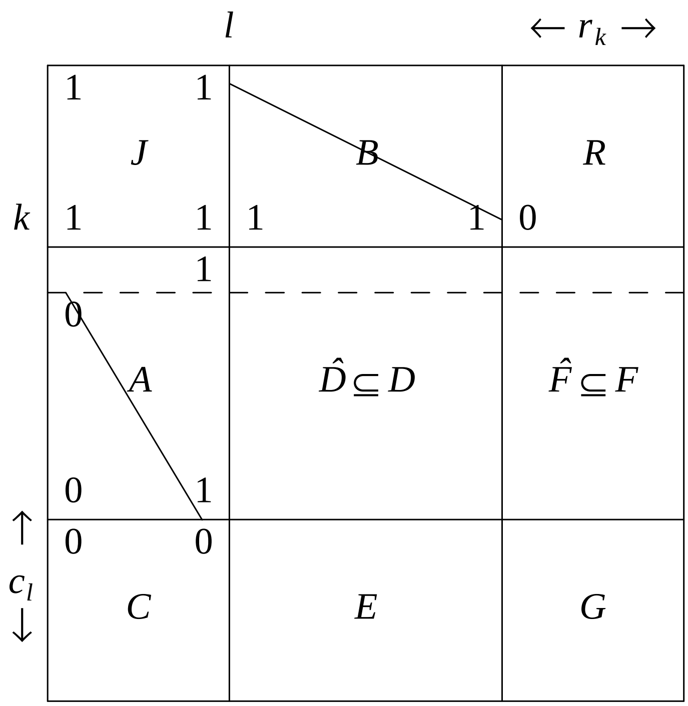

Fig. 1. Block decomposition of critical M.

Note that C is a matrix of 0's labeled C and R is a matrix of 0's labeled R. Also, J is a matrix of 1's, and the last row of B and last column of A are all 1's. Hence $k = 1$ if and only if $r_k = n - l$, in which case B, D, E are empty. Similarly, if $q = 1$, then $l = 1$ if and only if $c_l = m - k$, in which case A, D, F are empty. Since a critical matrix has no zero column, $q > 1$ implies $k > 1$. In particular, if $k = 1$ and $l = 1$, then we have a block decomposition forbidden by Lemma 3.2. If $c_l = 0$, then C, E, G are empty, and if $r_k = 0$, then R, F, G are empty.

We denote rows and columns in submatrices by their indices in M. To specify a submatrix with rows $i_1, \ldots, i_s$ and columns $j_1, \ldots, j_t$, we write $[i_1, \ldots, i_s : j_1, \ldots, j_t]$.

Lemma 5 *There is no critical M with nonempty D such that $D = 0$.*

Proof: Suppose D is nonempty and $D = 0$, and suppose the lower-right corner element of D is a C. This implies $E = 0$, since the bottom row of D can have no R. If we reverse the order of rows $1, \ldots, k$ and change all R's in B to C's, we obtain a zero-partition having $n - r_k$ columns with no R and at least one row with no C. Because Lemma 4.3-4 implies that B has more columns than rows, this contradicts the maximality of $k + l$. This argument makes no assumptions about the values of k, l, q, and the symmetric argument holds when the corner element of D is labeled R. □

Lemma 6 *If M is a critical zero-partition, then $\hat{D}$ is constant (or empty).*

Proof: Recall that $\hat{D}$ is the matrix with rows $\{m - c_1 + 1, \ldots, m - c_l\}$ and columns $\{l + 1, \ldots, n - r_k\}$. Since $\hat{D}$ is empty if $l = 1$, and D is empty if $k = 1$, we may assume $k, l > 1$. If in $\hat{D}$ row i has a 0 in column j and a 1 in column j', then $[1, k, i : 1, l, j, j'] = F_2$. If in $\hat{D}$ column j has a 0 in row i and a 1 in row i', then $[1, k, i, i' : 1, l, j] = F_2^T$. □

Lemma 7 *There is no critical M with $q > 1$.*

Proof: Consider such a critical M. Since $q > n - r_k$ would imply that column q is all 0, we have $k > 1$ and $1 \leq l < q \leq n - r_k$. Hence D is nonempty and has an entry labeled C, which by Lemma 6 implies that $\hat{D}$ has no nonzero entry. Lemma 5 implies that row $k + 1$ in D is not all C, though since $c_q = m - k$ (Lemma 4.1) it has some C. Let p denote the highest index of a row in $D - \hat{D}$ that is not completely labeled C; we have $k < p \leq m - c_1$. Let H denote the first p rows of M; note that $M_{p,q} = 0$.

We first claim $H \cap F = 0$. If $H \cap F$ has a 1 in position (i, j), then row i has no R in D. Since $H \cap D$ has no row that is all labeled C, row

i must have a 1 in D, say in column j'. But now $[i, k, 1 : j, j', l, q] = F_1$. Hence $H \cap F = 0$.

We next claim $H \cap (B \cup D)$ is a Ferrers matrix. If this contains a 2 by 2 permutation submatrix, in rows $i < i'$ and columns $j < j'$, then i, i' must be distinct from $1, k$, since those rows are all 0 and all 1, respectively. Since M is a zero-partition, we have C in position (i', j) and R in position (i, j'), but now $[i, k, 1, i' : j, l, j'] = F_2^T$.

Since $H \cap F$ is all 0 and $H \cap A$ is all 1, we have shown that H is a Ferrers matrix. By permuting columns of H between $l+1$ and $n - r_k$, we can arrange that every 0 in H has only 0's to its right. If we can apply such a permutation and still have a zero-partition, then the zeros of H will be relabeled to R, and we will have increased k without decreasing l, which contradicts criticality.

We claim that any column j that has a 1 in $H \cap D$ (say in row i) has no 1 below H. Since $\hat{D}$ has no nonzero entry, the choice of p implies that any such 1 must be in E, say in position (i', j), but now $[i', i, k, 1 : j, l, q] = F_1^T$. Hence if there is any 1 below H, all of $H \cap D$ above it is already labeled R, and we need not disturb these columns to right-justify all 0's in H. We will only permute columns $l+1, \ldots, t$ for some t, where t is the rightmost column of $H \cap D$ with a 1, and all of these columns are 0 below row p. Hence we will still have a zero-partition if we label every $M_{i,j}$ with $i > p$ and $l < j \leq t$ as a C. □

4 The Main Case

By Lemma 7, we may henceforth assume the symmetric situation $r_1 = \max\{r_i\}$ and $c_1 = \max\{c_j\}$, which implies $\hat{D} = D$ and $\hat{F} = F$. This together with Lemmas 5 and 6 implies that D is empty or is all 1's. The argument that we need for this case will be applied repeatedly to a more general situation, so we phrase part of it as a technical lemma.

Lemma 8 *Suppose that M is critical and there are rows $s \leq t \leq u$ and columns $x \leq y \leq z$ with the following properties: $k' \geq t$, $l' \geq y$, $r_s = n - y$, $r_t = n - z$, $c_x = m - t$, $c_y = m - u$, and $M_{i,j} = 1$ if $t < i \leq u$ and $y < j \leq z$. Then $k' \geq u$ and $l' \geq z$.*

Proof: Let M' denote the submatrix of M consisting of rows $s, \ldots, m$ and columns $x, \ldots, n$, indexed as usual by their indices in M. For ease of reference, we designate blocks of M' by analogy with those of M in Figure 1. These blocks appear in Figure 2 as $J', B', R', A', D', F', C', E', G'$, with the ranges of rows being s to t, $t+1$ to u, and $u+1$ to m, and the ranges of columns being x to y, $y+1$ to z, and $z+1$ to n. Note that A' has first column 0 labeled C and last column 1, and that B' has first row 0 labeled R and last column 1. Also, J' and D' are matrices of 1's. If $x = y$, then $t = u$, and the assumption $k' \geq t$ implies $k' \geq u$ (A', D', F' are empty). Similarly, if $s = t$, then $y = z$, and the assumption $l' \geq y$ implies $l' \geq z$ (B', D', E' are empty). By symmetry, it thus suffices to prove that $x < y$ implies $k' \geq u$.

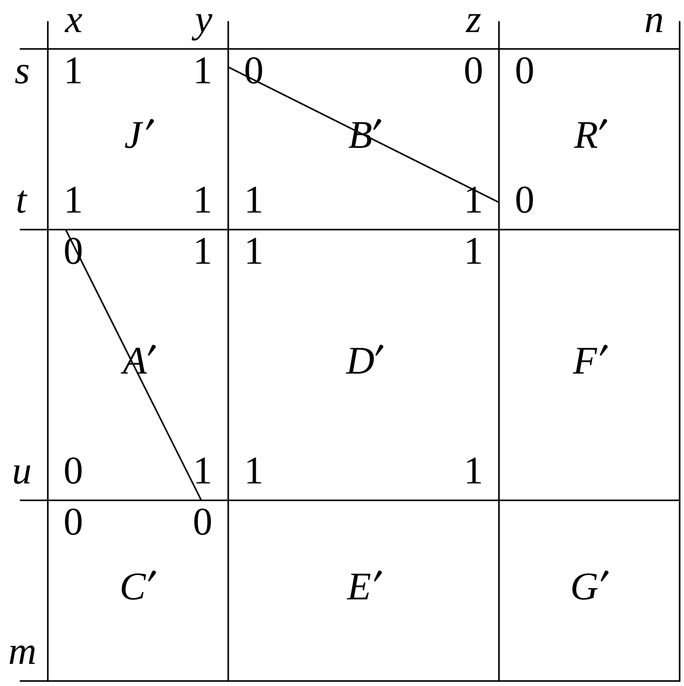

Fig. 2. Block decomposition of M' in critical M.

Given the structure assumed for M', the meaning of $k' \geq u$ is that $r_{t+1} \geq \ldots \geq r_u$ and that $c_j \leq m - u$ for $j > z$, the latter condition of which means that F' has no 0 labeled C. If $t = u$, then $k' \geq u$ by hypothesis, so we may assume $t < u$ and F' nonempty. Note that if D' is all 1, then $k' > t$, since otherwise there is a column of 0's.

Fact 1: there cannot be a 1 above 0 in A' (column j) and a 1 above 0 in F' (column j') in the same rows ($i < i'$). If such i, i', j, j'

exist, then j cannot be the first or last column of A', and we find $[t, i, i' : x, j, y, j'] = F_1$.

Fact 2: F' has no 2 by 2 permutation submatrix. If such exists in rows $i < i'$ and columns $j < j'$, then the fact that M is a zero-partition requires $M_{i',j} = M_{i,j'} = 0$, and we have $[t, i, i' : y, x, j, j'] = F_3$.

Fact 1 implies that any two rows in F' corresponding to an inversion in r are identical to the left of F'. If F' has no 0 labeled C, then we can switch such rows to reduce the number of inversions without reducing k' or l' until $r_{t+1} \geq \ldots \geq r_u$, at which point $k' \geq u$. If F' has a 0 labeled C, then $k' < u$. It suffices to show that we can obtain another zero-partition with larger $k' + l'$, contradicting criticality Note that k' is in fact the row just above the highest C in F'.

Let g be the index of the row i achieving $\max_{k'<i\leq u} r_i$, and let $h = n - r_g$. Note that h is the rightmost column that has no R in F' below row k'. Let j be the column of the leftmost C in F' in row g, so $j \leq h$. Note also that $M_{k',j} = 1$, since otherwise the column of M having the highest C in F' is all 0. Let i be the row of the top 1 in column j, so $i \leq k'$. If $M_{g,j'} = 1$ for some $j' > j$, then Fact 2 implies $M_{i,j'} = 1$. Now we can conclude $M_{i',j'} = 0$ for all $i' > u$, since $M_{i',j'} = 1$ would imply $[t, g, i, i' : y, j', j] = F_1^T$.

Since there is no R below row u and to the left of column j, we still have a zero-partition if we convert every 0 below F' in columns $j, \ldots, h$ to label C. Since $l' < j$, this does not change l' or k'. Fact 2 now permits us to switch any pair of columns between j and j' that have a 1 to the right of a C in row g, still having a zero-partition with the same k', l'. Finally, Fact 1 implies that rows $i, \ldots, g$ are identical to the left of F', so we can now convert every C in row g to label R and move row g up above row k' to its proper place so that in the new zero-partition, $r_1 \geq \ldots \geq r_{k'+1}$. In particular, we have increased k' without decreasing l'. □

Now, we can complete the proof of the main theorem.

Theorem 1 *A zero-partitionable matrix has an MCA if and only if it is* **F***-free.*

Proof: If the claim fails, we may choose a critical M with parameters k, l, k', l' as defined above. We prove that fixed values of k', l' that do not equal m, n lead to a contradiction. The results of Section 3 imply that the hypotheses of Lemma 8 are satisfied when we choose $(s, t, u) = (1, k, m - c_l)$ and $(x, y, z) = (1, l, n - r_k)$. Since Lemma 8 guarantees $k' + l' \geq u + z$, it suffices to show that satisfaction of the hypotheses of Lemma 8 for a given (s, t, u, x, y, z) implies that these hypotheses are also satisfied by some (s', t', u', x', y', z') with $u' + z' > u + z$.

We may assume $x < y$ by symmetry, because $s = t$ and $x = y$ produces the block decomposition forbidden by Lemma 3.2. We claim that $(t, u, m - c_z, y, z, n - r_u)$ is the desired sextuple. Since $k' \geq u$ and $l' \geq z$, we have $r_u \leq n - z$ and $c_z \leq m - u$. Furthermore, we cannot have equality in both, since this would again yield the forbidden block decomposition, so $u' + z' = m - c_z + n - r_u > u + z$. We also have $k' \geq u = t'$, $l' \geq z = y'$, $r_{s'} = r_t = n - z = n - y'$, $r_{t'} = r_u = n - z'$, $c_{x'} = c_y = m - u = m - t'$, and $c_{y'} = c_z = m - u'$.

It remains only to prove that $M_{i,j} = 1$ if $t' < i \leq u'$ and $y' < j \leq z'$. Translating this back to the original parameters, the condition is vacuous except for i, j such that $u < i \leq m - c_z$ and $z < j \leq n - r_u$. Note that the condition $u < m - c_z$ implies $y \neq z$. For any such i, j, $M_{i,j} = 0$ and the condition $x < y$ implies $[t, u, i; x, y, z, j] = F_1$, which is forbidden. The contradiction completes the proof. □

5 The Recognition Algorithm

Finally, we comment briefly on the conversion of this proof to a recognition algorithm. Given an initial zero-partition, the algorithm produces an MCA or an **F**-submatrix. From the values of $r_1, \ldots, r_m$ and $c_1, \ldots, c_n$ in the initial zero-partition, we immediately compute the initial k, l ($k = m - \max\{c_j\}$ and $l = n - \max\{r_i\}$). This induces a structure $J, B, R, A, D, F, C, E, G$ as in Section 3. The first phase of the algorithm examines D; we will find an **F**-submatrix or reach a zero-partition in which D is all 1. If D is empty or is all 1, the first phase ends successfully; note also that $\hat{D}$ cannot be empty if D is not empty. The second phase applies the results of Section 4. We need to be able to test certain conditions quickly.

Lemma 9 *If N is an m by n matrix with a zero-partition described by r, c, then the upper left a by b submatrix can be tested for being 0 in time linear in $a + b$, returning the position of a 1 if it is not all 0.*

Proof: A 0 in position $1,1$ requires $r_1 = n$ or $c_1 = m$. If one of these holds, then that row or column is 0, and we can ignore it and proceed by induction on $a + b$. $\square$

Lemma 10 *If N is an m by n matrix with a zero-partition described by r, c, then N can be tested for being a Ferrers matrix in time linear in $m + n$, returning the positions i, i', j, j' of a 2 by 2 permutation submatrix if N is not a Ferrers matrix.*

Proof: Since r, c is a zero-partition, the only possible 2 by 2 permutation submatrices have $N_{i',j} = N_{j',i} = 0$, where $i < i'$ and $j < j'$. Consider first the possibility $i = 1$. This requires $j < n - r_1$. Let $h = m - \max\{c_j : 1 \le j < n - r_1\}$. The matrix has a 2 by 2 permutation submatrix with $i = 1$ if and only if the submatrix N' consisting of rows $h+1, \ldots, m$ and columns $n - r_1 + 1, \ldots, n$ has a 1. By Lemma 9, we can test for this in time $c \cdot (m - h + r_1)$. If N' has a 1, we find it and are finished. If N' is all 0, then we have found a block decomposition of the form $\binom{1\ A}{B\ 0}$, and in addition B has a row of 0's and A a column of 0's that we can discard. We now apply the search recursively to these submatrices. The overall contributions from searching for 1's are linear in $m + n$, as are the maximizations. $\square$

By applying Lemma 9 to the matrix formed by D, F, E, G, we can determine whether D and $\hat{D}$ are all 0 or all 1 (testing for all 1 is simply a maximization over the appropriate values in r and c). By Lemma 6, we obtain an **F**-submatrix unless $\hat{D}$ is constant. If D is all zero, then we find a rearrangement that increases $k + l$. If D is not all zero, then we have $q > 1$ or $q = 1$. If $q > 1$, we examine the entries as in the proof of Lemma 7. First we determine whether $H \cap F = 0$, again by using Lemma 9; if not, we have our **F**-matrix. Next, we check whether $H \cap (B \cup D - \hat{D})$ is a Ferrers matrix, by using Lemma 10. If it fails, we have an **F**-matrix. Next, the columns j that have 1's in $H \cap D$ are those such that $c_j < m - p$ and $\max\{r_i : k < i \le p\} < n - j$. These columns can all be found in linear time using an extension of Lemma 9. We can also check quickly for 1's below H in these columns; if any occurs, we have an **F**-matrix. Otherwise, the proof of Lemma 7 yields a reordering that increases $k + l$. We then reenter Phase 1. Since $k + l$ grows, the total time for Phase 1 is at most quadratic in $m + n$.

If Phase 1 concludes successfully, then we have $q = 1$ and $D = 1$. As in the proof of the Theorem, we can initialize suitable values for s, t, u, x, y, z from k, l, r_k, c_l to apply Lemma 8. In time linear in $(u-t)+(n-z)$, we can verify Facts 1 and 2 or find an **F**-matrix. Close examination of Lemma 8 shows that again we produce a better k' or an **F**-matrix. We do the same with l' and E'. As in the proof of the Theorem, we now must verify that the new candidate for D' is all 1's, but again it is easy to check whether submatrix of a zero-partition is all 1's. Having increased $k'+l'$ in linear time, we again get a quadratic contribution from Phase 2, and the algorithm runs in time quadratic in $m+n$.

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