Decomposition of Sparse Graphs

Douglas B. West

Department of Mathematics
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

preprint & slides at
http://www.math.uiuc.edu/~west/pubs/publink.html

Joint work with

Seog-Jin Kim, Alexandr V. Kostochka,
Hehui Wu, Xuding Zhu

Motivation

Aim: Common generalization of Nash-Williams' Formula and decomposition results for planar graphs.

Motivation

Aim: Common generalization of Nash-Williams' Formula and decomposition results for planar graphs.

Thm. (Nash-Williams [1965]) G decomposes into k forests $\Leftrightarrow$ $|E(H)| \le k(|V(H)|-1)$ for every subgraph H.

Motivation

Aim: Common generalization of Nash-Williams' Formula and decomposition results for planar graphs.

Thm. (Nash-Williams [1965]) G decomposes into k forests $\Leftrightarrow$ $|E(H)| \leq k(|V(H)|-1)$ for every subgraph H.

Def. fractional arboricity $\mathrm{Arb}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$.

(Payan [1986])

Motivation

Aim: Common generalization of Nash-Williams' Formula and decomposition results for planar graphs.

Thm. (Nash-Williams [1965]) G decomposes into k forests $\Leftrightarrow$ $|E(H)| \leq k(|V(H)|-1)$ for every subgraph H.

Def. fractional arboricity $\mathrm{Arb}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$.

(Payan [1986]) N-W: arboricity $\Upsilon(G) = \lceil \mathrm{Arb}(G) \rceil$.

Motivation

Aim: Common generalization of Nash-Williams' Formula and decomposition results for planar graphs.

Thm. (Nash-Williams [1965]) G decomposes into k forests $\Leftrightarrow$ $|E(H)| \leq k(|V(H)|-1)$ for every subgraph H.

Def. fractional arboricity $\text{Arb}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$.

(Payan [1986]) N-W: arboricity $\Upsilon(G) = \lceil \text{Arb}(G) \rceil$.

Idea: Three forests are needed when $\text{Arb}(G) = 2 + \epsilon$; can we restrict the third forest?

Motivation

Aim: Common generalization of Nash-Williams' Formula and decomposition results for planar graphs.

Thm. (Nash-Williams [1965]) G decomposes into k forests $\Leftrightarrow$ $|E(H)| \le k(|V(H)|-1)$ for every subgraph H.

Def. fractional arboricity $\mathrm{Arb}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$.

(Payan [1986]) N-W: arboricity $\Upsilon(G) = \lceil \mathrm{Arb}(G) \rceil$.

Idea: Three forests are needed when $\mathrm{Arb}(G) = 2 + \epsilon$; can we restrict the third forest?

Def. G is d-bounded if $\Delta(G) \le d$.

Motivation

Aim: Common generalization of Nash-Williams' Formula and decomposition results for planar graphs.

Thm. (Nash-Williams [1965]) G decomposes into k forests $\Leftrightarrow$ $|E(H)| \leq k(|V(H)|-1)$ for every subgraph H.

Def. fractional arboricity $\mathrm{Arb}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$.

(Payan [1986]) N-W: arboricity $\Upsilon(G) = \lceil \mathrm{Arb}(G) \rceil$.

Idea: Three forests are needed when $\mathrm{Arb}(G) = 2 + \epsilon$; can we restrict the third forest?

Def. G is d-bounded if $\Delta(G) \leq d$.

Nine Dragon Tree (NDT) Conjecture:
(Montassier, Ossona de Mendez, Raspaud, Zhu [2010])
$\mathrm{Arb}(G) \leq k + \frac{d}{k+d+1}$ $\Rightarrow$ G decomposes into $k+1$ forests, with the last being d-bounded.

Decomposition of Planar Graphs

Sparse planar graphs (girth at least g) decompose into a forest and a d-bounded subgraph.

Decomposition of Planar Graphs

Sparse planar graphs (girth at least g) decompose into a forest and a d-bounded subgraph.

	g	d
Borodin-Kostochka-Sheikh-Yu [2008]	9	1
Wang-Zhang [2010+]	8	1
He-Hou-Lih-Shao-Wang-Zhu [2002]	7	2
Kleitman [2006, unpub.]	6	2
He-Hou-Lih-Shao-Wang-Zhu [2002]	5	4

Decomposition of Planar Graphs

Sparse planar graphs (girth at least g) decompose into a forest and a d-bounded subgraph.

	g	d
Borodin-Kostochka-Sheikh-Yu [2008]	9	1
Wang-Zhang [2010+]	8	1
He-Hou-Lih-Shao-Wang-Zhu [2002]	7	2
Kleitman [2006, unpub.]	6	2
He-Hou-Lih-Shao-Wang-Zhu [2002]	5	4

Thm. Gonçalves [2009] Planar $G \vdash$ three forests, one 4-bounded (conj. Balogh-Kochol-Pluhar-Yu [2005]).

Decomposition of Planar Graphs

Sparse planar graphs (girth at least g) decompose into a forest and a d-bounded subgraph.

	g	d
Borodin-Kostochka-Sheikh-Yu [2008]	9	1
Wang-Zhang [2010+]	8	1
He-Hou-Lih-Shao-Wang-Zhu [2002]	7	2
Kleitman [2006, unpub.]	6	2
He-Hou-Lih-Shao-Wang-Zhu [2002]	5	4

Thm. Gonçalves [2009] Planar $G \vdash$ three forests, one 4-bounded (conj. Balogh-Kochol-Pluhar-Yu [2005]).

Motivation: Applies to "game coloring number", a bound on game chromatic number χ_g (always $\chi_g(G) \le \mathrm{col}_g(G)$).

- Zhu [1999]: If G decomposes into G_1 and G_2, then $\mathrm{col}_g(G) \le \mathrm{col}_g(G_1) + \Delta(G_2)$.
- Faigle-Kern-Kierstead-Trotter [1993]: If G is a forest, then $\mathrm{col}_g(G) \le 4$.

Maximum Average Degree

- Planar graphs are sparse; a planar n-vertex graph has $\leq 3n - 6$ edges and hence average degree < 6 (same for every subgraph).

Maximum Average Degree

- Planar graphs are sparse; a planar n-vertex graph has $\leq 3n - 6$ edges and hence average degree < 6 (same for every subgraph).

Sparseness measured by Maximum average degree.

Def. $\mathrm{Mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$.

Maximum Average Degree

- Planar graphs are sparse; a planar n-vertex graph has $\leq 3n-6$ edges and hence average degree < 6 (same for every subgraph).

Sparseness measured by Maximum average degree.

Def. $\mathrm{Mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$.

- Euler's Formula $\Rightarrow$ $\mathrm{Mad}(G) < \frac{2g}{g-2}$ when G is planar with girth g.

Maximum Average Degree

- Planar graphs are sparse; a planar n-vertex graph has $\leq 3n - 6$ edges and hence average degree < 6 (same for every subgraph).

Sparseness measured by Maximum average degree.

Def. $\mathrm{Mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$.

- Euler's Formula $\Rightarrow$ $\mathrm{Mad}(G) < \frac{2g}{g-2}$ when G is planar with girth g.

Some results on planar graphs (with girth at least g) use only the bound on $\mathrm{Mad}(G)$ and hold more generally.

Maximum Average Degree

- Planar graphs are sparse; a planar n-vertex graph has $\leq 3n - 6$ edges and hence average degree < 6 (same for every subgraph).

Sparseness measured by Maximum average degree.

Def. $\mathrm{Mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$.

- Euler's Formula $\Rightarrow$ $\mathrm{Mad}(G) < \frac{2g}{g-2}$ when G is planar with girth g.

Some results on planar graphs (with girth at least g) use only the bound on $\mathrm{Mad}(G)$ and hold more generally.

We seek sparseness conditions (bounds on $\mathrm{Mad}(G)$ or $\mathrm{Arb}(G)$) to guarantee decomposition results.

Prior result

Thm. (Montassier–Pêcher–Raspaud–West–Zhu [2010])
If $\mathrm{Mad}(G) < 2 + \frac{2d}{d+4-6/(d+2)}$, then G decomposes into a forest and a d-bounded graph. When $d \leq 3$, both graphs can be required to be forests.

Prior result

Thm. (Montassier–Pêcher–Raspaud–West–Zhu [2010])
If $\mathrm{Mad}(G) < 2 + \frac{2d}{d+4-6/(d+2)}$, then G decomposes into a forest and a d-bounded graph. When $d \le 3$, both graphs can be required to be forests.

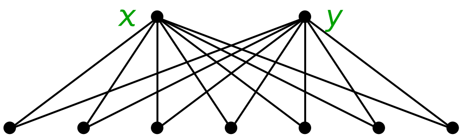
How large a $\mathrm{Mad}(G)$ bound permits this conclusion?

Prior result

Thm. (Montassier–Pêcher–Raspaud–West–Zhu [2010])
If $\mathrm{Mad}(G) < 2 + \frac{2d}{d+4-6/(d+2)}$, then G decomposes into a forest and a d-bounded graph. When $d \le 3$, both graphs can be required to be forests.

How large a $\mathrm{Mad}(G)$ bound permits this conclusion?

Ex. $K_{2,2d+2}$ has no such decomposition into F and D.

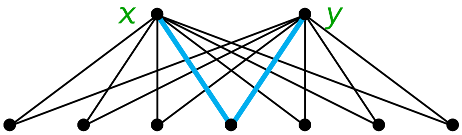


Prior result

Thm. (Montassier–Pêcher–Raspaud–West–Zhu [2010])
If $\mathrm{Mad}(G) < 2 + \frac{2d}{d+4-6/(d+2)}$, then G decomposes into a forest and a d-bounded graph. When $d \le 3$, both graphs can be required to be forests.

How large a $\mathrm{Mad}(G)$ bound permits this conclusion?

Ex. $K_{2,2d+2}$ has no such decomposition into F and D.



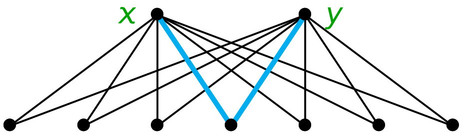
At most one 2-vertex has degree 2 in F.

Prior result

Thm. (Montassier–Pêcher–Raspaud–West–Zhu [2010])
If $\mathrm{Mad}(G) < 2 + \frac{2d}{d+4-6/(d+2)}$, then G decomposes into a forest and a d-bounded graph. When $d \leq 3$, both graphs can be required to be forests.

How large a $\mathrm{Mad}(G)$ bound permits this conclusion?

Ex. $K_{2,2d+2}$ has no such decomposition into F and D.



At most one 2-vertex has degree 2 in F.

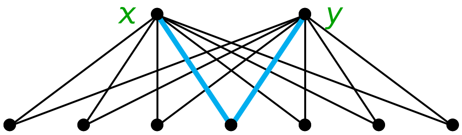
$\therefore\ d_F(x) + d_F(y) \leq 2d + 3$, so $d_D(x) + d_D(y) \geq 2d + 1$.

Prior result

Thm. (Montassier–Pêcher–Raspaud–West–Zhu [2010])
If $\mathrm{Mad}(G) < 2 + \frac{2d}{d+4-6/(d+2)}$, then G decomposes into a forest and a d-bounded graph. When $d \le 3$, both graphs can be required to be forests.

How large a $\mathrm{Mad}(G)$ bound permits this conclusion?

Ex. $K_{2,2d+2}$ has no such decomposition into F and D.



At most one 2-vertex has degree 2 in F.

$\therefore$ $d_F(x) + d_F(y) \le 2d + 3$, so $d_D(x) + d_D(y) \ge 2d + 1$.

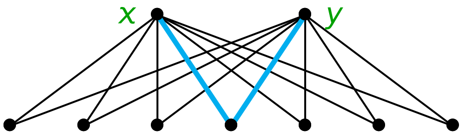
- $\mathrm{Mad}(K_{2,2d+2}) = \frac{8d+8}{2d+4} = 2 + \frac{2d}{d+2}$.

Prior result

Thm. (Montassier–Pêcher–Raspaud–West–Zhu [2010])
If $\mathrm{Mad}(G) < 2 + \frac{2d}{d+4-6/(d+2)}$, then G decomposes into a forest and a d-bounded graph. When $d \le 3$, both graphs can be required to be forests.

How large a $\mathrm{Mad}(G)$ bound permits this conclusion?

Ex. $K_{2,2d+2}$ has no such decomposition into F and D.



At most one 2-vertex has degree 2 in F.

$\therefore\ d_F(x) + d_F(y) \le 2d + 3$, so $d_D(x) + d_D(y) \ge 2d + 1$.

• $\mathrm{Mad}(K_{2,2d+2}) = \frac{8d+8}{2d+4} = 2 + \frac{2d}{d+2}$. (Same whenever we subdivide each edge of a $(2d+2)$-regular multigraph.)

Our results

Our result implies that $\mathrm{Mad}(G) < 2 + \frac{2d}{d+2}$ suffices for G to decompose into a forest and a d-bounded graph. This implies the results in the table for planar graphs.

Our results

Our result implies that $\mathrm{Mad}(G) < 2 + \frac{2d}{d+2}$ suffices for G to decompose into a forest and a d-bounded graph.
This implies the results in the table for planar graphs.

girth g	$\Rightarrow$ $\mathrm{Mad}(G) < \frac{2g}{g-2}$	d	$2 + \frac{2d}{d+2}$
8	$< 8/3$	1	$8/3$
6	< 3	2	3
5	$< 10/3$	4	$10/3$

Our results

Our result implies that $\mathrm{Mad}(G) < 2 + \frac{2d}{d+2}$ suffices for G to decompose into a forest and a d-bounded graph.

For $A \subseteq V(G)$, let $\|A\| = |E(G[A])|$.

Our results

Our result implies that $\text{Mad}(G) < 2 + \frac{2d}{d+2}$ suffices for G to decompose into a forest and a d-bounded graph.

For $A \subseteq V(G)$, let $\|A\| = |E(G[A])|$.

Condition	Equivalent condition (when imposed $\forall A \subseteq V(G)$)
$\text{Arb}(G) \leq k + \frac{d}{k+d+1}$	$(k+1)(k+d)\,|A| - (k+d+1)\,\|A\| - (k+1)(k+d) \geq 0$
$\text{Mad}(G) < 2k + \frac{2d}{k+d+1}$	$(k+1)(k+d)\,|A| - (k+d+1)\,\|A\| - 1 \geq 0$

Our results

Our result implies that $\mathrm{Mad}(G) < 2 + \frac{2d}{d+2}$ suffices for G to decompose into a forest and a d-bounded graph.

For $A \subseteq V(G)$, let $\|A\| = |E(G[A])|$.

Condition	Equivalent condition (when imposed $\forall A \subseteq V(G)$)
$\mathrm{Arb}(G) \leq k + \frac{d}{k+d+1}$	$(k+1)(k+d)\,|A| - (k+d+1)\,\|A\| - (k+1)(k+d) \geq 0$
(k,d)-sparse	$(k+1)(k+d)\,|A| - (k+d+1)\,\|A\| - k^2 \geq 0$
$\mathrm{Mad}(G) < 2k + \frac{2d}{k+d+1}$	$(k+1)(k+d)\,|A| - (k+d+1)\,\|A\| - 1 \geq 0$

Our results

Our result implies that $\mathrm{Mad}(G) < 2 + \frac{2d}{d+2}$ suffices for G to decompose into a forest and a d-bounded graph.

For $A \subseteq V(G)$, let $\|A\| = |E(G[A])|$.

Condition	Equivalent condition (when imposed $\forall A \subseteq V(G)$)
$\mathrm{Arb}(G) \leq k + \frac{d}{k+d+1}$	$(k+1)(k+d)\,|A| - (k+d+1)\,\|A\| - (k+1)(k+d) \geq 0$
(k,d)-sparse	$(k+1)(k+d)\,|A| - (k+d+1)\,\|A\| - k^2 \geq 0$
$\mathrm{Mad}(G) < 2k + \frac{2d}{k+d+1}$	$(k+1)(k+d)\,|A| - (k+d+1)\,\|A\| - 1 \geq 0$

$\mathrm{Arb}(G) \leq k + \frac{d}{k+d+1} \Rightarrow$ (k,d)-sparse $\Rightarrow \mathrm{Mad}(G) < 2k + \frac{2d}{k+d+1}$

Our results

Our result implies that $\mathrm{Mad}(G) < 2 + \frac{2d}{d+2}$ suffices for G to decompose into a forest and a d-bounded graph.

For $A \subseteq V(G)$, let $\|A\| = |E(G[A])|$.

Condition	Equivalent condition (when imposed $\forall A \subseteq V(G)$)
$\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$	$(k+1)(k+d)\,\lvert A\rvert - (k+d+1)\,\|A\| - (k+1)(k+d) \ge 0$
(k,d)-sparse	$(k+1)(k+d)\,\lvert A\rvert - (k+d+1)\,\|A\| - k^2 \ge 0$
$\mathrm{Mad}(G) < 2k + \frac{2d}{k+d+1}$	$(k+1)(k+d)\,\lvert A\rvert - (k+d+1)\,\|A\| - 1 \ge 0$

$\mathrm{Arb}(G) \le k + \frac{d}{k+d+1} \;\Rightarrow\; (k,d)\text{-sparse} \;\Rightarrow\; \mathrm{Mad}(G) < 2k + \frac{2d}{k+d+1}$

Thm. If G is (k,d)-sparse, and $d > k$, then G decomposes into k forests and a d-bounded graph.

Our results

Our result implies that $\mathrm{Mad}(G) < 2 + \frac{2d}{d+2}$ suffices for G to decompose into a forest and a d-bounded graph.

For $A \subseteq V(G)$, let $\|A\| = |E(G[A])|$.

Condition	Equivalent condition (when imposed $\forall A \subseteq V(G)$)
$\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$	$(k+1)(k+d)\,|A| - (k+d+1)\,\|A\| - (k+1)(k+d) \ge 0$
(k,d)-sparse	$(k+1)(k+d)\,|A| - (k+d+1)\,\|A\| - k^2 \ge 0$
$\mathrm{Mad}(G) < 2k + \frac{2d}{k+d+1}$	$(k+1)(k+d)\,|A| - (k+d+1)\,\|A\| - 1 \ge 0$

$\mathrm{Arb}(G) \le k + \frac{d}{k+d+1} \;\Rightarrow\; (k,d)$-sparse $\;\Rightarrow\; \mathrm{Mad}(G) < 2k + \frac{2d}{k+d+1}$

Thm. If G is (k,d)-sparse, and $d > k$, then G decomposes into k forests and a d-bounded graph.

Thm. For $d = k+1$, and for $d \le 6$ when $k = 1$, the NDT Conjecture holds. That is, $\mathrm{Arb}(G) \le k + \frac{d}{k+d+1} \;\Rightarrow\; G$ decomposes into k forests and a d-bounded forest.

(k, d)-decomposition

Def. Say that G is (k, d)-decomposable if G decomposes into k forests and a d-bounded graph.

(k, d)-decomposition

Def. Say that G is (k, d)-decomposable if G decomposes into k forests and a d-bounded graph.

Thm. If G is (k, d)-sparse, and $d > k$, then G is (k, d)-decomposable.

Cor. If $\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$, and $d > k$, then G is (k, d)-decomposable.

(k,d)-decomposition

Def. Say that G is (k,d)-decomposable if G decomposes into k forests and a d-bounded graph.

Thm. If G is (k,d)-sparse, and $d > k$, then G is (k,d)-decomposable.

Cor. If $\mathrm{Arb}(G) \leq k + \frac{d}{k+d+1}$, and $d > k$, then G is (k,d)-decomposable.

Thm. For $d \leq k$, if $\mathrm{Arb}(G) \leq k + \frac{d}{2k+2}$, then G is (k,d)-decomposable.

(k,d)-decomposition

Def. Say that G is (k,d)-decomposable if G decomposes into k forests and a d-bounded graph.

Thm. If G is (k,d)-sparse, and $d > k$, then G is (k,d)-decomposable.

Cor. If $\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$, and $d > k$, then G is (k,d)-decomposable.

Thm. For $d \le k$, if $\mathrm{Arb}(G) \le k + \frac{d}{2k+2}$, then G is (k,d)-decomposable.

Thm. (Lap Chi Lau and Tamás Király; weaker) The tighter bound on $\mathrm{Arb}(G)$ that in the theorems above suffices for $(k,d-1)$-decomposition is shown to suffice for (k,d)-decomposition.

More General Statement

A seemingly more general statement facilitates proof by induction on $|E(G)| + |V(G)|$. Fix d and k.

More General Statement

A seemingly more general statement facilitates proof by induction on $|E(G)| + |V(G)|$. Fix d and k.

Def. A capacity function on a graph G is a function $f\colon V(G) \to \{0, \ldots, d\}$. A graph D with vertex set $V(G)$ is f-bounded if $d_D(v) \le f(v)$ for $v \in V(G)$.

More General Statement

A seemingly more general statement facilitates proof by induction on $|E(G)| + |V(G)|$. Fix d and k.

Def. A capacity function on a graph G is a function $f\colon V(G) \to \{0, \dots, d\}$. A graph D with vertex set $V(G)$ is f-bounded if $d_D(v) \le f(v)$ for $v \in V(G)$. For $A \subseteq V(G)$, let

$$\beta_f(A) = (k+1)\sum_{v \in A}(k + f(v)) - (k+d+1)\,\|A\| - k^2.$$

More General Statement

A seemingly more general statement facilitates proof by induction on $|E(G)| + |V(G)|$. Fix d and k.

Def. A capacity function on a graph G is a function $f\colon V(G) \to \{0, \dots, d\}$. A graph D with vertex set $V(G)$ is f-bounded if $d_D(v) \le f(v)$ for $v \in V(G)$. For $A \subseteq V(G)$, let

$$\beta_f(A) = (k+1)\sum_{v \in A}(k + f(v)) - (k+d+1)\,\|A\| - k^2.$$

Def. A capacity function f on G is feasible if $\beta_f(A) \ge 0$ for all nonempty $A \subseteq V(G)$.

More General Statement

A seemingly more general statement facilitates proof by induction on $|E(G)| + |V(G)|$. Fix d and k.

Def. A capacity function on a graph G is a function $f\colon V(G) \to \{0, \dots, d\}$. A graph D with vertex set $V(G)$ is f-bounded if $d_D(v) \le f(v)$ for $v \in V(G)$. For $A \subseteq V(G)$, let

$$\beta_f(A) = (k+1)\sum_{v\in A}(k + f(v)) - (k+d+1)\,\|A\| - k^2.$$

Def. A capacity function f on G is feasible if $\beta_f(A) \ge 0$ for all nonempty $A \subseteq V(G)$.

Thm. A graph with a feasible capacity function f decomposes into k forests and an f-bounded graph.

More General Statement

A seemingly more general statement facilitates proof by induction on $|E(G)| + |V(G)|$. Fix d and k.

Def. A capacity function on a graph G is a function $f\colon V(G) \to \{0, \dots, d\}$. A graph D with vertex set $V(G)$ is f-bounded if $d_D(v) \le f(v)$ for $v \in V(G)$. For $A \subseteq V(G)$, let

$$\beta_f(A) = (k+1)\sum_{v\in A}(k + f(v)) - (k+d+1)\,\|A\| - k^2.$$

Def. A capacity function f on G is feasible if $\beta_f(A) \ge 0$ for all nonempty $A \subseteq V(G)$.

Thm. A graph with a feasible capacity function f decomposes into k forests and an f-bounded graph.

We will use the induction hypoth. for contracted graphs, induced subgraphs, and edge-deleted subgraphs.

Lemma 1 - Contractions

Def. For $B \subseteq V(G)$, let G_B be the graph obtained by contracting B into a new vertex z.

Lemma 1 - Contractions

Def. For $B \subseteq V(G)$, let G_B be the graph obtained by contracting B into a new vertex z.

Lem. 1 If f is feasible on G, and $B \subset V(G)$ with $|B| \geq 2$ and $\beta_f(B) \leq k$, then f' is feasible on G_B, where $f'(z) = 0$ and f' agrees with f on $V(G) - B$.

Lemma 1 - Contractions

Def. For $B \subseteq V(G)$, let G_B be the graph obtained by contracting B into a new vertex z.

Lem. 1 If f is feasible on G, and $B \subset V(G)$ with $|B| \geq 2$ and $\beta_f(B) \leq k$, then f' is feasible on G_B, where $f'(z) = 0$ and f' agrees with f on $V(G) - B$.

Pf. Consider $A \subseteq V(G_B)$. If $z \notin A$, then $\beta_{f'}(A) = \beta_f(A) \geq 0$.

Lemma 1 - Contractions

Def. For $B \subseteq V(G)$, let G_B be the graph obtained by contracting B into a new vertex z.

Lem. 1 If f is feasible on G, and $B \subset V(G)$ with $|B| \geq 2$ and $\beta_f(B) \leq k$, then f' is feasible on G_B, where $f'(z) = 0$ and f' agrees with f on $V(G) - B$.

Pf. Consider $A \subseteq V(G_B)$. If $z \notin A$, then $\beta_{f'}(A) = \beta_f(A) \geq 0$.

If $z \in A$, then let $A' = (A - z) \cup B$.

Lemma 1 - Contractions

Def. For $B \subseteq V(G)$, let G_B be the graph obtained by contracting B into a new vertex z.

Lem. 1 If f is feasible on G, and $B \subset V(G)$ with $|B| \geq 2$ and $\beta_f(B) \leq k$, then f' is feasible on G_B, where $f'(z) = 0$ and f' agrees with f on $V(G) - B$.

Pf. Consider $A \subseteq V(G_B)$. If $z \notin A$, then $\beta_{f'}(A) = \beta_f(A) \geq 0$.

If $z \in A$, then let $A' = (A - z) \cup B$.

Claim: $\beta_f(A') = \beta_{f'}(A) - (k+1)k + \beta_f(B) + k^2.$

Lemma 1 - Contractions

Def. For $B \subseteq V(G)$, let G_B be the graph obtained by contracting B into a new vertex z.

Lem. 1 If f is feasible on G, and $B \subset V(G)$ with $|B| \geq 2$ and $\beta_f(B) \leq k$, then f' is feasible on G_B, where $f'(z) = 0$ and f' agrees with f on $V(G) - B$.

Pf. Consider $A \subseteq V(G_B)$. If $z \notin A$, then $\beta_{f'}(A) = \beta_f(A) \geq 0$.

If $z \in A$, then let $A' = (A - z) \cup B$.

Claim: $\beta_f(A') = \beta_{f'}(A) - (k+1)k + \beta_f(B) + k^2$.

Each edge in $G[A']$ is in $G_B[A]$ or $G[B]$ and contributes $(k+1+d)$ to both. Since $f'(z) = 0$, capacities are equal. Right gains $(k+1)k$ from z and extra constant $-k^2$.

Lemma 1 - Contractions

Def. For $B \subseteq V(G)$, let G_B be the graph obtained by contracting B into a new vertex z.

Lem. 1 If f is feasible on G, and $B \subset V(G)$ with $|B| \geq 2$ and $\beta_f(B) \leq k$, then f' is feasible on G_B, where $f'(z) = 0$ and f' agrees with f on $V(G) - B$.

Pf. Consider $A \subseteq V(G_B)$. If $z \notin A$, then $\beta_{f'}(A) = \beta_f(A) \geq 0$.

If $z \in A$, then let $A' = (A - z) \cup B$.

Claim: $\beta_f(A') = \beta_{f'}(A) - (k+1)k + \beta_f(B) + k^2$.

Each edge in $G[A']$ is in $G_B[A]$ or $G[B]$ and contributes $(k + 1 + d)$ to both. Since $f'(z) = 0$, capacities are equal. Right gains $(k+1)k$ from z and extra constant $-k^2$.

Now $\beta_f(B) \leq k$ implies $\beta_{f'}(A) \geq \beta_f(A') \geq 0$. ■

Lemma 2 - Combining Restriction & Contraction

Def. Say that G is (k, f)-decomposable if G decomposes into k forests and an f-bounded graph.

Lemma 2 - Combining Restriction & Contraction

Def. Say that G is (k, f)-decomposable if G decomposes into k forests and an f-bounded graph.

Lem. 2 Fix a capacity fcn f on G and $B \subseteq V(G)$. If $G[B]$ is $(k, f|_B)$-decomposable and G_B is (k, f')-decomposable, with f' as before, then G is (k, f)-decomposable.

Lemma 2 - Combining Restriction & Contraction

Def. Say that G is (k, f)-decomposable if G decomposes into k forests and an f-bounded graph.

Lem. 2 Fix a capacity fcn f on G and $B \subseteq V(G)$. If $G[B]$ is $(k, f|_B)$-decomposable and G_B is (k, f')-decomposable, with f' as before, then G is (k, f)-decomposable.

Pf. Pick a $(k, f|_B)$-decomp. (F, D) of $G[B]$ and a (k, f')-decomp. (F', D') of G_B. Each edge of G is in $G[B]$ or G_B, with those of $[B, \overline{B}]$ incident to z in G_B.

Lemma 2 - Combining Restriction & Contraction

Def. Say that G is (k, f)-decomposable if G decomposes into k forests and an f-bounded graph.

Lem. 2 Fix a capacity fcn f on G and $B \subseteq V(G)$. If $G[B]$ is $(k, f|_B)$-decomposable and G_B is (k, f')-decomposable, with f' as before, then G is (k, f)-decomposable.

Pf. Pick a $(k, f|_B)$-decomp. (F, D) of $G[B]$ and a (k, f')-decomp. (F', D') of G_B. Each edge of G is in $G[B]$ or G_B, with those of $[B, \overline{B}]$ incident to z in G_B.

Claim: $(F \cup F', D \cup D')$ is a (k, f)-decomp. of G.

Lemma 2 - Combining Restriction & Contraction

Def. Say that G is (k, f)-decomposable if G decomposes into k forests and an f-bounded graph.

Lem. 2 Fix a capacity fcn f on G and $B \subseteq V(G)$. If $G[B]$ is $(k, f|_B)$-decomposable and G_B is (k, f')-decomposable, with f' as before, then G is (k, f)-decomposable.

Pf. Pick a $(k, f|_B)$-decomp. (F, D) of $G[B]$ and a (k, f')-decomp. (F', D') of G_B. Each edge of G is in $G[B]$ or G_B, with those of $[B, \overline{B}]$ incident to z in G_B.

Claim: $(F \cup F', D \cup D')$ is a (k, f)-decomp. of G.

Since $f'(z) = 0 \;\; \Rightarrow \;\; d_{D'}(z) = 0$, all of $[B, \overline{B}]$ is in F'. Hence $D \cup D'$ satisfies the capacities f.

Lemma 2 - Combining Restriction & Contraction

Def. Say that G is (k, f)-decomposable if G decomposes into k forests and an f-bounded graph.

Lem. 2 Fix a capacity fcn f on G and $B \subseteq V(G)$. If $G[B]$ is $(k, f|_B)$-decomposable and G_B is (k, f')-decomposable, with f' as before, then G is (k, f)-decomposable.

Pf. Pick a $(k, f|_B)$-decomp. (F, D) of $G[B]$ and a (k, f')-decomp. (F', D') of G_B. Each edge of G is in $G[B]$ or G_B, with those of $[B, \overline{B}]$ incident to z in G_B.

Claim: $(F \cup F', D \cup D')$ is a (k, f)-decomp. of G.

Since $f'(z) = 0 \;\; \Rightarrow \;\; d_{D'}(z) = 0$, all of $[B, \overline{B}]$ is in F'. Hence $D \cup D'$ satisfies the capacities f.

For forests $F_i \in F$ and $F_i' \in F'$ the union is a forest in G; otherwise, contracting the part in F_i of a resulting cycle yields a cycle through z in F_i'. ■

Start of Main Proof

Thm. If $d > k$ and G is a graph with a feasible capacity function f, then G is (k, f)-decomposable.

Start of Main Proof

Thm. If $d > k$ and G is a graph with a feasible capacity function f, then G is (k, f)-decomposable.

Pf. Induction on $|V(G)| + |E(G)|$, trivial when $|E(G)| \leq k$. Suppose that G is larger but not (k, f)-decomposable.

Start of Main Proof

Thm. If $d > k$ and G is a graph with a feasible capacity function f, then G is (k, f)-decomposable.

Pf. Induction on $|V(G)| + |E(G)|$, trivial when $|E(G)| \leq k$. Suppose that G is larger but not (k, f)-decomposable.

Claim 1: $\beta_f(B) \geq k + 1$ when $B \subset V(G)$ with $|B| \geq 2$.

Start of Main Proof

Thm. If $d > k$ and G is a graph with a feasible capacity function f, then G is (k, f)-decomposable.

Pf. Induction on $|V(G)| + |E(G)|$, trivial when $|E(G)| \leq k$. Suppose that G is larger but not (k, f)-decomposable.

Claim 1: $\beta_f(B) \geq k + 1$ when $B \subset V(G)$ with $|B| \geq 2$.

If $\beta_f(B) \leq k$, then f' on G_B defined as before is feasible, by Lemma 1. Since $B \subset V(G)$, also $f|_B$ feasible on $G[B]$.

Start of Main Proof

Thm. If $d > k$ and G is a graph with a feasible capacity function f, then G is (k, f)-decomposable.

Pf. Induction on $|V(G)| + |E(G)|$, trivial when $|E(G)| \leq k$. Suppose that G is larger but not (k, f)-decomposable.

Claim 1: $\beta_f(B) \geq k + 1$ when $B \subset V(G)$ with $|B| \geq 2$.

If $\beta_f(B) \leq k$, then f' on G_B defined as before is feasible, by Lemma 1. Since $B \subset V(G)$, also $f|_B$ feasible on $G[B]$.

These graphs are smaller than G, so G_B is $(k, f|_B)$-decomposable and G_B is (k, f')-decomposable. By Lemma 2, G is (k, f)-decomposable.

Start of Main Proof

Thm. If $d > k$ and G is a graph with a feasible capacity function f, then G is (k, f)-decomposable.

Pf. Induction on $|V(G)| + |E(G)|$, trivial when $|E(G)| \le k$. Suppose that G is larger but not (k, f)-decomposable.

Claim 1: $\beta_f(B) \ge k + 1$ when $B \subset V(G)$ with $|B| \ge 2$.

If $\beta_f(B) \le k$, then f' on G_B defined as before is feasible, by Lemma 1. Since $B \subset V(G)$, also $f|_B$ feasible on $G[B]$.

These graphs are smaller than G, so G_B is $(k, f|_B)$-decomposable and G_B is (k, f')-decomposable. By Lemma 2, G is (k, f)-decomposable.

Having proved Claim 1, let $S = \{v \in V(G) \colon f(v) > 0\}$. We next show that S is an independent set.

S is Independent - Using Edge-Deletion

Claim: S is independent, where $S = \{v \in V(G) : f(v) > 0\}$.

S is Independent - Using Edge-Deletion

Claim: S is independent, where $S = \{v \in V(G) \colon f(v) > 0\}$.

If $u, v \in S$ are adjacent, then let $\hat{f}$ be the capacity function on $G - uv$ that agrees with f except for $\hat{f}(u) = f(u) - 1$ and $\hat{f}(v) = f(v) - 1$.

S is Independent - Using Edge-Deletion

Claim: S is independent, where $S = \{v \in V(G) : f(v) > 0\}$.

If $u, v \in S$ are adjacent, then let $\hat{f}$ be the capacity function on $G - uv$ that agrees with f except for $\hat{f}(u) = f(u) - 1$ and $\hat{f}(v) = f(v) - 1$.

If $\hat{f}$ is feasible, then since $G - uv$ is smaller than G, it has a $(k, \hat{f})$-decomposition; add uv to the $\hat{f}$-bounded subgraph to obtain a (k, f)-decomposition of G.

S is Independent - Using Edge-Deletion

Claim: S is independent, where $S = \{v \in V(G) \colon f(v) > 0\}$.

If $u, v \in S$ are adjacent, then let $\hat{f}$ be the capacity function on $G - uv$ that agrees with f except for $\hat{f}(u) = f(u) - 1$ and $\hat{f}(v) = f(v) - 1$.

If $\hat{f}$ is feasible, then since $G - uv$ is smaller than G, it has a $(k, \hat{f})$-decomposition; add uv to the $\hat{f}$-bounded subgraph to obtain a (k, f)-decomposition of G.

To show $\hat{f}$ feasible, consider $A \subseteq V(G - uv) = V(G)$.
If $u, v \notin A$, then $\beta_{\hat{f}}(A) = \beta_f(A) \geq 0$.

S is Independent - Using Edge-Deletion

Claim: S is independent, where $S = \{v \in V(G) \colon f(v) > 0\}$.

If $u, v \in S$ are adjacent, then let $\hat{f}$ be the capacity function on $G - uv$ that agrees with f except for $\hat{f}(u) = f(u) - 1$ and $\hat{f}(v) = f(v) - 1$.

If $\hat{f}$ is feasible, then since $G - uv$ is smaller than G, it has a $(k, \hat{f})$-decomposition; add uv to the $\hat{f}$-bounded subgraph to obtain a (k, f)-decomposition of G.

To show $\hat{f}$ feasible, consider $A \subseteq V(G - uv) = V(G)$.
If $u, v \notin A$, then $\beta_{\hat{f}}(A) = \beta_f(A) \geq 0$.

If $u, v \in A$, then defn of $\hat{f}$, loss of one edge, and $d > k$ yield $\beta_{\hat{f}}(A) = \beta_f(A) - 2(k+1) + (k+d+1) \geq \beta_f(A) \geq 0$.

S is Independent - Using Edge-Deletion

Claim: S is independent, where $S = \{v \in V(G) : f(v) > 0\}$.

If $u, v \in S$ are adjacent, then let $\hat{f}$ be the capacity function on $G - uv$ that agrees with f except for $\hat{f}(u) = f(u) - 1$ and $\hat{f}(v) = f(v) - 1$.

If $\hat{f}$ is feasible, then since $G - uv$ is smaller than G, it has a $(k, \hat{f})$-decomposition; add uv to the $\hat{f}$-bounded subgraph to obtain a (k, f)-decomposition of G.

To show $\hat{f}$ feasible, consider $A \subseteq V(G - uv) = V(G)$.
If $u, v \notin A$, then $\beta_{\hat{f}}(A) = \beta_f(A) \geq 0$.

If $u, v \in A$, then defn of $\hat{f}$, loss of one edge, and $d > k$ yield $\beta_{\hat{f}}(A) = \beta_f(A) - 2(k+1) + (k + d + 1) \geq \beta_f(A) \geq 0$.

If one of $\{u, v\}$ is in A, then $A \subset V(G)$.
If $|A| \geq 2$, then $\beta_{\hat{f}}(A) = \beta_f(A) - (k+1) \geq 0$.
If $|A| = 1$, then $\beta_{\hat{f}}(A) \geq (k+1)k - 0 - k^2 \geq k$ (no loops).

Completion of Proof

Hence S is independent, where $S = \{v \in V(G) \colon f(v) > 0\}$.

Completion of Proof

Hence S is independent, where $S = \{v \in V(G) \colon f(v) > 0\}$.

We show that G now decomposes into k forests (a (k, f)-decomposition with the second graph empty).

Completion of Proof

Hence S is independent, where $S = \{v \in V(G) : f(v) > 0\}$.

We show that G now decomposes into k forests (a (k, f)-decomposition with the second graph empty).

If $\Upsilon(G) > k$, then $\exists$ minimal A with $\|A\| \geq k(|A| - 1) + 1$.

Completion of Proof

Hence S is independent, where $S = \{v \in V(G) \colon f(v) > 0\}$.

We show that G now decomposes into k forests (a (k, f)-decomposition with the second graph empty).

If $\Upsilon(G) > k$, then $\exists$ minimal A with $\|A\| \geq k(|A| - 1) + 1$.

Minimality $\Rightarrow$ every $v \in A$ has at least $k+1$ nbrs in A.
Let $A' = S \cap A$. Since S is independent, $\|A\| \geq (k+1)\,|A'|$.

Completion of Proof

Hence S is independent, where $S = \{v \in V(G) : f(v) > 0\}$.

We show that G now decomposes into k forests (a (k, f)-decomposition with the second graph empty).

If $\Upsilon(G) > k$, then $\exists$ minimal A with $\|A\| \geq k(|A| - 1) + 1$.

Minimality $\Rightarrow$ every $v \in A$ has at least $k+1$ nbrs in A.
Let $A' = S \cap A$. Since S is independent, $\|A\| \geq (k+1)\,|A'|$.

$\therefore\ (k+1+d)\,\|A\| \geq (k+1)k(|A|-1) + (k+1) + d(k+1)\,|A'|$.

Completion of Proof

Hence S is independent, where $S = \{v \in V(G) \colon f(v) > 0\}$.

We show that G now decomposes into k forests (a (k, f)-decomposition with the second graph empty).

If $\Upsilon(G) > k$, then $\exists$ minimal A with $\|A\| \geq k(|A| - 1) + 1$.

Minimality $\Rightarrow$ every $v \in A$ has at least $k+1$ nbrs in A.
Let $A' = S \cap A$. Since S is independent, $\|A\| \geq (k+1)\,|A'|$.

$\therefore\ (k+1+d)\,\|A\| \geq (k+1)k(|A|-1) + (k+1) + d(k+1)\,|A'|.$

Now we compute

$$\begin{aligned}
\beta_f(A) &= (k+1)k\,|A| + (k+1)f(A') - (k+1+d)\,\|A\| - k^2 \\
&\leq (k+1)[k-1-d|A'|] + (k+1)d|A'| - k^2 \\
&= (k+1)[k-1] - k^2 \ = \ -1.
\end{aligned}$$

Completion of Proof

Hence S is independent, where $S = \{v \in V(G) \colon f(v) > 0\}$.

We show that G now decomposes into k forests (a (k, f)-decomposition with the second graph empty).

If $\Upsilon(G) > k$, then $\exists$ minimal A with $\|A\| \geq k(|A| - 1) + 1$.

Minimality $\Rightarrow$ every $v \in A$ has at least $k+1$ nbrs in A.
Let $A' = S \cap A$. Since S is independent, $\|A\| \geq (k+1)\,|A'|$.

$\therefore\ (k+1+d)\,\|A\| \geq (k+1)k(|A|-1) + (k+1) + d(k+1)\,|A'|$.

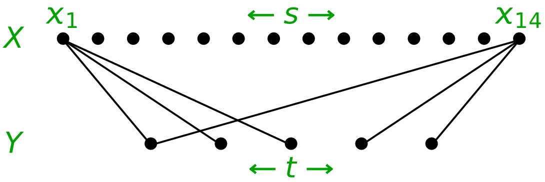
Now we compute

$$\beta_f(A) = (k+1)k\,|A| + (k+1)f(A') - (k+1+d)\,\|A\| - k^2$$
$$\leq (k+1)[k-1-d|A'|] + (k+1)d|A'| - k^2$$
$$= (k+1)[k-1] - k^2 \ = \ -1.$$

Contradiction! Thus G is (k, f)-decomposable. ■

Sharpness Example

Ex. Construct an X, Y-bigraph G with $|X| = s$ and $|Y| = t$ where $s = t(k+d) - k + 1$. Put $x_i \leftrightarrow \{y_i, \ldots, y_{i+k}\}$ (indices modulo t), so $|E(G)| = (k+1)(k+d)t - k^2 + 1$.



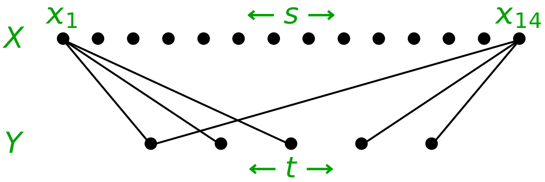
$k = 2, d = 1$

$t = 5, s = 14$

$42 > 18 \cdot 2 + 5$

Sharpness Example

Ex. Construct an X, Y-bigraph G with $|X| = s$ and $|Y| = t$ where $s = t(k + d) - k + 1$. Put $x_i \leftrightarrow \{y_i, \ldots, y_{i+k}\}$ (indices modulo t), so $|E(G)| = (k + 1)(k + d)t - k^2 + 1$.



$k = 2, d = 1$

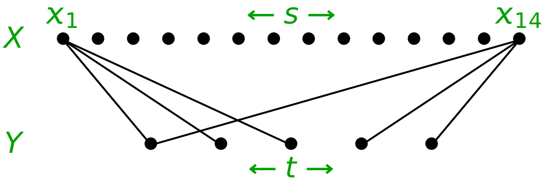
$t = 5, s = 14$

$42 > 18 \cdot 2 + 5$

No (k, d)-decomposition: Deleting at most dt edges leaves at least $k(k + d)t + kt - k^2 + 1$ edges, but k forests cover at most $k[t(k + d + 1) - k]$ edges.

Sharpness Example

Ex. Construct an X, Y-bigraph G with $|X| = s$ and $|Y| = t$ where $s = t(k+d) - k + 1$. Put $x_i \leftrightarrow \{y_i, \dots, y_{i+k}\}$ (indices modulo t), so $|E(G)| = (k+1)(k+d)t - k^2 + 1$.



$k = 2, d = 1$

$t = 5, s = 14$

$42 > 18 \cdot 2 + 5$

No (k, d)-decomposition: Deleting at most dt edges leaves at least $k(k+d)t + kt - k^2 + 1$ edges, but k forests cover at most $k[t(k+d+1) - k]$ edges.

Almost feasible: A set A minimizing $\beta_f(A)$ contains all nbrs of $X \cap A$ and also has $Y \cap A = \{y_0, \dots, y_{t'-1}\}$. Finally, $\beta_f(A) \geq 0$ for $A \subset V(G)$, but $\beta_f(V(G)) = -1$. ■

NDT Conjecture for $d = k + 1$

Require $(k, f)^*$-decomposition: the leftover f-bounded graph is also a forest. Feasibility of f is not sufficient: an edge of multiplicity $k + 2$, with $f(u) = f(v) = d$, has $\beta_f(A) \geq 0$ for all A but no decomp. into $k + 1$ forests.

NDT Conjecture for $d = k + 1$

Require $(k,f)^*$-decomposition: the leftover f-bounded graph is also a forest. Feasibility of f is not sufficient: an edge of multiplicity $k+2$, with $f(u) = f(v) = d$, has $\beta_f(A) \geq 0$ for all A but no decomp. into $k+1$ forests.

Def. For capacity function f, let $S = \{v \in V(G) \colon f(v) = d\}$ and $\alpha_f(A) = k(|A| - 1) - \|A\| + |A \cap S|$. Say f is strongly feasible when $\beta_f(A), \alpha_f(A) \geq 0$ and $\alpha_f(A) \geq 1$ for $A \subseteq S$.

NDT Conjecture for $d = k + 1$

Require $(k, f)^*$-decomposition: the leftover f-bounded graph is also a forest. Feasibility of f is not sufficient: an edge of multiplicity $k + 2$, with $f(u) = f(v) = d$, has $\beta_f(A) \geq 0$ for all A but no decomp. into $k + 1$ forests.

Def. For capacity function f, let $S = \{v \in V(G) \colon f(v) = d\}$ and $\alpha_f(A) = k(|A| - 1) - \|A\| + |A \cap S|$. Say f is strongly feasible when $\beta_f(A), \alpha_f(A) \geq 0$ and $\alpha_f(A) \geq 1$ for $A \subseteq S$.

Def. A graph is strongly $(k, f)^*$-decomposable if it has a $(k, f)^*$-decomp. in which each component of the f-bounded forest has at most one vertex v with $f(v) < d$.

NDT Conjecture for $d = k + 1$

Require $(k, f)^*$-decomposition: the leftover f-bounded graph is also a forest. Feasibility of f is not sufficient: an edge of multiplicity $k+2$, with $f(u) = f(v) = d$, has $\beta_f(A) \geq 0$ for all A but no decomp. into $k+1$ forests.

Def. For capacity function f, let $S = \{v \in V(G) \colon f(v) = d\}$ and $\alpha_f(A) = k(|A| - 1) - \|A\| + |A \cap S|$. Say f is strongly feasible when $\beta_f(A), \alpha_f(A) \geq 0$ and $\alpha_f(A) \geq 1$ for $A \subseteq S$.

Def. A graph is strongly $(k, f)^*$-decomposable if it has a $(k, f)^*$-decomp. in which each component of the f-bounded forest has at most one vertex v with $f(v) < d$.

Thm. If $d = k + 1$ and f is strongly feasible, then G is strongly $(k, f)^*$-decomposable.

NDT Conjecture for $d = k + 1$

Require $(k,f)^*$-decomposition: the leftover f-bounded graph is also a forest. Feasibility of f is not sufficient: an edge of multiplicity $k+2$, with $f(u) = f(v) = d$, has $\beta_f(A) \ge 0$ for all A but no decomp. into $k+1$ forests.

Def. For capacity function f, let $S = \{v \in V(G) \colon f(v) = d\}$ and $\alpha_f(A) = k(|A| - 1) - \|A\| + |A \cap S|$. Say f is strongly feasible when $\beta_f(A), \alpha_f(A) \ge 0$ and $\alpha_f(A) \ge 1$ for $A \subseteq S$.

Def. A graph is strongly $(k,f)^*$-decomposable if it has a $(k,f)^*$-decomp. in which each component of the f-bounded forest has at most one vertex v with $f(v) < d$.

Thm. If $d = k + 1$ and f is strongly feasible, then G is strongly $(k,f)^*$-decomposable.

Cor. The NDT Conjecture holds for $d = k + 1$.

Pf. $\mathrm{Arb}(G) \le k + \frac{d}{k+d+1} \Rightarrow f$ is feasible.
$S = V(G) \Rightarrow \alpha_f(A) \ge 1$ becomes $\|A\| \le (k+1)(|A| - 1)$. ∎

Ghosts

The capacity function f facilitates the induction, but the uniform case $f(v) = d$ for all v is equally general.

Ghosts

The capacity function f facilitates the induction, but the uniform case $f(v) = d$ for all v is equally general.

Def. A ghost is a vertex of degree $k + 1$ having only one neighbor (via all $k + 1$ edges).

Ghosts

The capacity function f facilitates the induction, but the uniform case $f(v) = d$ for all v is equally general.

Def. A ghost is a vertex of degree $k + 1$ having only one neighbor (via all $k + 1$ edges).

Prop. If feasibility implies (k, d)-decomposition in the uniform case, then feasibility of f suffices in general.

Ghosts

The capacity function f facilitates the induction, but the uniform case $f(v) = d$ for all v is equally general.

Def. A ghost is a vertex of degree $k + 1$ having only one neighbor (via all $k + 1$ edges).

Prop. If feasibility implies (k, d)-decomposition in the uniform case, then feasibility of f suffices in general.

Pf. Given a feasible f on G, form G' by adding $d - f(v)$ ghost nbrs to each v, and let $f'(w) = d$ for $w \in V(G')$.

Ghosts

The capacity function f facilitates the induction, but the uniform case $f(v) = d$ for all v is equally general.

Def. A ghost is a vertex of degree $k + 1$ having only one neighbor (via all $k + 1$ edges).

Prop. If feasibility implies (k, d)-decomposition in the uniform case, then feasibility of f suffices in general.

Pf. Given a feasible f on G, form G' by adding $d - f(v)$ ghost nbrs to each v, and let $f'(w) = d$ for $w \in V(G')$.
Increasing capacity from $f(v)$ to d while adding $f(v) - d$ ghost nbrs has no net effect on β for sets containing v.

$\beta_{f'}(A') = \beta_f(G) + (k+1)\sum_{v\in A}(d-f(v)) - (k+1)\sum_{v\in A}(d-f(v))$.

Ghosts

The capacity function f facilitates the induction, but the uniform case $f(v) = d$ for all v is equally general.

Def. A ghost is a vertex of degree $k + 1$ having only one neighbor (via all $k + 1$ edges).

Prop. If feasibility implies (k, d)-decomposition in the uniform case, then feasibility of f suffices in general.

Pf. Given a feasible f on G, form G' by adding $d - f(v)$ ghost nbrs to each v, and let $f'(w) = d$ for $w \in V(G')$.
Increasing capacity from $f(v)$ to d while adding $f(v) - d$ ghost nbrs has no net effect on β for sets containing v.

$$\beta_{f'}(A') = \beta_f(G) + (k+1)\sum_{v \in A}(d - f(v)) - (k+1)\sum_{v \in A}(d - f(v)).$$

Thus the uniform function f' is feasible, and G' has a (k, d)-decomposition. Deleting the added ghost vertices yields a (k, f)-decomposition of G. ∎

Mad vs. Arb

Prior results in this area have used discharging; we use it to prove the NDT Conjecture for $k = 1$ and $d \leq 6$.

Mad vs. Arb

Prior results in this area have used discharging; we use it to prove the NDT Conjecture for $k = 1$ and $d \leq 6$.

Def. Let $m_{k,d} = 2k + \frac{2d}{k+d+1}$. For $A \subseteq V(G)$, define sparseness $\beta_G(A) = (k+1)(k+d)\,|A| - (k+d+1)\,\|A\|$.

Mad vs. Arb

Prior results in this area have used discharging; we use it to prove the NDT Conjecture for $k=1$ and $d \leq 6$.

Def. Let $m_{k,d} = 2k + \frac{2d}{k+d+1}$. For $A \subseteq V(G)$, define sparseness $\beta_G(A) = (k+1)(k+d)\,|A| - (k+d+1)\,\|A\|$.

$$\text{Arb}(G) \leq m_{k,d}/2 \quad \Leftrightarrow \quad \beta_G(A) \geq (k+1)(k+d)\ \forall\, A \subseteq V(G)$$

$$\text{Mad}(G) < m_{k,d} \quad \Leftrightarrow \quad \beta_G(A) \geq 1 \quad \forall\, A \subseteq V(G)$$

Mad vs. Arb

Prior results in this area have used discharging; we use it to prove the NDT Conjecture for $k = 1$ and $d \leq 6$.

Def. Let $m_{k,d} = 2k + \frac{2d}{k+d+1}$. For $A \subseteq V(G)$, define sparseness $\beta_G(A) = (k+1)(k+d)\,|A| - (k+d+1)\,\|A\|$.

$$\mathrm{Arb}(G) \leq m_{k,d}/2 \quad \Leftrightarrow \quad \beta_G(A) \geq (k+1)(k+d)\ \forall\, A \subseteq V(G)$$

$$\mathrm{Mad}(G) < m_{k,d} \quad \Leftrightarrow \quad \beta_G(A) \geq 1 \qquad \forall\, A \subseteq V(G)$$

Discharging works well with bounds on $\mathrm{Mad}(G)$, and $\mathrm{Mad}(G) < m_{k,d} \Leftrightarrow (k,d)$-sparse when $k = 1$.
Aim: Replace $\mathrm{Arb} \leq m_{k,d}/2$ (NDT hypothesis) with (k,d)-sparse plus something else.

Mad vs. Arb

Prior results in this area have used discharging; we use it to prove the NDT Conjecture for $k = 1$ and $d \leq 6$.

Def. Let $m_{k,d} = 2k + \frac{2d}{k+d+1}$. For $A \subseteq V(G)$, define sparseness $\beta_G(A) = (k+1)(k+d)\,|A| - (k+d+1)\,\|A\|$.

$$\text{Arb}(G) \leq m_{k,d}/2 \quad \Leftrightarrow \quad \beta_G(A) \geq (k+1)(k+d)\ \forall\, A \subseteq V(G)$$

$$\text{Mad}(G) < m_{k,d} \quad \Leftrightarrow \quad \beta_G(A) \geq 1 \qquad \forall\, A \subseteq V(G)$$

Discharging works well with bounds on $\text{Mad}(G)$, and $\text{Mad}(G) < m_{k,d} \Leftrightarrow (k, d)$-sparse when $k = 1$.
Aim: Replace $\text{Arb} \leq m_{k,d}/2$ (NDT hypothesis) with (k, d)-sparse plus something else.

Violating $\Upsilon(G) \leq k + 1$ requires $\|A\| \geq (k+1)(|A| - 1) + 1$ for some A. If G is (k, d)-sparse ($\beta_G(A) \geq k^2$), then

Mad vs. Arb

Prior results in this area have used discharging; we use it to prove the NDT Conjecture for $k = 1$ and $d \leq 6$.

Def. Let $m_{k,d} = 2k + \frac{2d}{k+d+1}$. For $A \subseteq V(G)$, define sparseness $\beta_G(A) = (k+1)(k+d)\,|A| - (k+d+1)\,\|A\|$.

$$\text{Arb}(G) \leq m_{k,d}/2 \quad \Leftrightarrow \quad \beta_G(A) \geq (k+1)(k+d) \; \forall\, A \subseteq V(G)$$

$$\text{Mad}(G) < m_{k,d} \quad \Leftrightarrow \quad \beta_G(A) \geq 1 \quad \forall\, A \subseteq V(G)$$

Discharging works well with bounds on $\text{Mad}(G)$, and $\text{Mad}(G) < m_{k,d} \;\Leftrightarrow\; (k,d)$-sparse when $k = 1$.
Aim: Replace $\text{Arb} \leq m_{k,d}/2$ (NDT hypothesis) with (k,d)-sparse plus something else.

Violating $\Upsilon(G) \leq k+1$ requires $\|A\| \geq (k+1)(|A|-1)+1$ for some A. If G is (k,d)-sparse ($\beta_G(A) \geq k^2$), then

$$(k+1)(k+d)\,|A| - (k+d+1)[(k+1)(|A|-1)+1] \geq k^2,$$

which simplifies to $|A| \leq \frac{k}{k+1}(d+1)$.

A Revised Conjecture

Def. Graph G is feasible if $\beta_G(A) \geq k^2$ for $\varnothing \neq A \subseteq V(G)$.
A set $A \subseteq V(G)$ is overfull if $\|A\| > (k+1)(|A|-1)$.

A Revised Conjecture

Def. Graph G is feasible if $\beta_G(A) \geq k^2$ for $\varnothing \neq A \subseteq V(G)$.
A set $A \subseteq V(G)$ is overfull if $\|A\| > (k+1)(|A|-1)$.

$\mathrm{Arb}(G) \leq m_{k,d}/2$ implies feasibility and forbids overfull sets. Feasibility prohibits overfull sets with more than $\frac{(d+1)k}{k+1}$ vertices. Hence the NDT Conjecture follows from:

A Revised Conjecture

Def. Graph G is feasible if $\beta_G(A) \geq k^2$ for $\varnothing \neq A \subseteq V(G)$. A set $A \subseteq V(G)$ is overfull if $\|A\| > (k+1)(|A|-1)$.

$\text{Arb}(G) \leq m_{k,d}/2$ implies feasibility and forbids overfull sets. Feasibility prohibits overfull sets with more than $\frac{(d+1)k}{k+1}$ vertices. Hence the NDT Conjecture follows from:

Conj. If G is feasible and has no overfull set with at most $\frac{(d+1)k}{k+1}$ vertices, then G is $(k,d)^*$-decomposable.

A Revised Conjecture

Def. Graph G is feasible if $\beta_G(A) \geq k^2$ for $\varnothing \neq A \subseteq V(G)$. A set $A \subseteq V(G)$ is overfull if $\|A\| > (k+1)(|A|-1)$.

$\text{Arb}(G) \leq m_{k,d}/2$ implies feasibility and forbids overfull sets. Feasibility prohibits overfull sets with more than $\frac{(d+1)k}{k+1}$ vertices. Hence the NDT Conjecture follows from:

Conj. If G is feasible and has no overfull set with at most $\frac{(d+1)k}{k+1}$ vertices, then G is $(k,d)^*$-decomposable.

We prove this when $k = 1$ and $d \leq 6$. The advantage when $k = 1$ is that $k^2 = 1$, so feasibility reduces to $\beta_G(A) > 0$ for all A, which is equivalent to $\text{Mad}(G) < m_{1,d}$ (discharging is suitable).

A Revised Conjecture

Def. Graph G is feasible if $\beta_G(A) \geq k^2$ for $\varnothing \neq A \subseteq V(G)$.
A set $A \subseteq V(G)$ is overfull if $\|A\| > (k+1)(|A|-1)$.

$\mathrm{Arb}(G) \leq m_{k,d}/2$ implies feasibility and forbids overfull sets. Feasibility prohibits overfull sets with more than $\frac{(d+1)k}{k+1}$ vertices. Hence the NDT Conjecture follows from:

Conj. If G is feasible and has no overfull set with at most $\frac{(d+1)k}{k+1}$ vertices, then G is $(k,d)^*$-decomposable.

We prove this when $k=1$ and $d \leq 6$. The advantage when $k=1$ is that $k^2=1$, so feasibility reduces to $\beta_G(A) > 0$ for all A, which is equivalent to $\mathrm{Mad}(G) < m_{1,d}$ (discharging is suitable).

Ghosts help control $(k,d)^*$-decompositions, since the edges at a ghost must lie in $k+1$ distinct forests.

Minimal Counterexamples

Def. For fixed k, d, a minimal counterexample to the conjecture is one having the fewest ghosts among those with the fewest non-ghosts. A set $A \subseteq V(G)$ is nontrivial if it has at least two but not all non-ghosts.

Minimal Counterexamples

Def. For fixed k, d, a minimal counterexample to the conjecture is one having the fewest ghosts among those with the fewest non-ghosts. A set $A \subseteq V(G)$ is nontrivial if it has at least two but not all non-ghosts.

Lem. Properties of a minimal counterexample G:

1) G is $(k+1)$-edge-connected.
2) If $d_G(v) \le 2k+1$, then v has no ghost neighbor.
3) If $\beta_G(A) \le k(k+1)$, then G_A is feasible.
4) If G_A is $(k, d)^*$-decomposable, then so is G.
5) If A is nontrivial, then $\beta_G(A) > k(k+1)$.
6) If $\beta_G(A) \le k(k+1)$ and A has just one non-ghost v, then $d_G(v) \ge (k+1)(d+1)$.
7) If $d_G(v) < (k+1)(d+1)$, then v has no non-ghost neighbor with degree $k+1$.
8) If v has $q (\ge 1)$ ghost nbrs, then $d_G(v) > kq + k + d$.
9) Edges with multiplicity $k+1$ have a ghost endpoint.

Discharging Argument

Aim: Show that if G has the properties derived for a minimal counterexample, then $\mathrm{Mad}(G) \geq m_{k,d}$. We need a bit more but first suggest the discharging argument.

Discharging Argument

Aim: Show that if G has the properties derived for a minimal counterexample, then $\mathrm{Mad}(G) \geq m_{k,d}$. We need a bit more but first suggest the discharging argument.

To prove $\mathrm{Mad}(G) \geq m_{k,d}$, initially give each vertex charge equal to its degree. Then redistribute charge so the final charge $\mu(v)$ at each vertex v is at least $m_{k,d}$.

Discharging Argument

Aim: Show that if G has the properties derived for a minimal counterexample, then $\mathrm{Mad}(G) \geq m_{k,d}$. We need a bit more but first suggest the discharging argument.

To prove $\mathrm{Mad}(G) \geq m_{k,d}$, initially give each vertex charge equal to its degree. Then redistribute charge so the final charge $\mu(v)$ at each vertex v is at least $m_{k,d}$.

Since G is $(k+1)$-edge-connected, $\delta(G) \geq k+1$.

Discharging Argument

Aim: Show that if G has the properties derived for a minimal counterexample, then $\text{Mad}(G) \geq m_{k,d}$. We need a bit more but first suggest the discharging argument.

To prove $\text{Mad}(G) \geq m_{k,d}$, initially give each vertex charge equal to its degree. Then redistribute charge so the final charge $\mu(v)$ at each vertex v is at least $m_{k,d}$.

Since G is $(k+1)$-edge-connected, $\delta(G) \geq k+1$.

Rule 1: A $(k+1)$-vertex takes charge $\frac{m_{k,d}}{k+1} - 1$ along each incident edge. This amount is $\frac{k+d-1}{k+d+1}$.

Discharging Argument

Aim: Show that if G has the properties derived for a minimal counterexample, then $\mathrm{Mad}(G) \geq m_{k,d}$. We need a bit more but first suggest the discharging argument.

To prove $\mathrm{Mad}(G) \geq m_{k,d}$, initially give each vertex charge equal to its degree. Then redistribute charge so the final charge $\mu(v)$ at each vertex v is at least $m_{k,d}$.

Since G is $(k+1)$-edge-connected, $\delta(G) \geq k+1$.

Rule 1: A $(k+1)$-vertex takes charge $\frac{m_{k,d}}{k+1} - 1$ along each incident edge. This amount is $\frac{k+d-1}{k+d+1}$.

Thus a ghost takes $m_{k,d} - (k+1)$ in total from its nbr. By force, $\mu(v) = m_{k,d}$ when $d_G(v) = k+1$, since $(k+1)$-vertices are not adjacent.

Other Vertex Degrees

If all neighbors of v have degree $k+1$, then each edge takes $\frac{k+d-1}{k+d+1}$, so $\mu(v) = d_G(v)\frac{2}{k+d+1}$.
Hence $\mu(v) \geq m_{k,d}$ if and only if $d_G(v) \geq (k+1)(k+d)$.

Other Vertex Degrees

If all neighbors of v have degree $k+1$, then each edge takes $\frac{k+d-1}{k+d+1}$, so $\mu(v) = d_G(v)\frac{2}{k+d+1}$.
Hence $\mu(v) \geq m_{k,d}$ if and only if $d_G(v) \geq (k+1)(k+d)$.

How to handle degrees between $k+1$ and $(k+1)(k+d)$?
Vertices with degree at most $2k+1$ need some charge.
Vertices with degree less than $(k+1)(k+d)$ cannot give away too much.

Other Vertex Degrees

If all neighbors of v have degree $k+1$, then each edge takes $\frac{k+d-1}{k+d+1}$, so $\mu(v) = d_G(v)\frac{2}{k+d+1}$.
Hence $\mu(v) \geq m_{k,d}$ if and only if $d_G(v) \geq (k+1)(k+d)$.

How to handle degrees between $k+1$ and $(k+1)(k+d)$?
Vertices with degree at most $2k+1$ need some charge.
Vertices with degree less than $(k+1)(k+d)$ cannot give away too much.

A vertex v with degree less than $(k+1)(k+d)$ and more than $k+1$ has no non-ghost $(k+1)$-neighbor, but it has some neighbors with higher degrees and hence won't give as much as $(k+1)\frac{k+d+1}{k+d-1}$. More information is needed about the degrees of neighboring vertices.

The Case $k = 1$

When $(k, d) = (1, 1)$, only 2-vertices need charge. Their nbrs have high enough degree that Rule 1 suffices.
This proves the NDT Conjecture for $(k, d) = (1, 1)$ (Montassier et al.).

The Case $k = 1$

When $(k, d) = (1, 1)$, only 2-vertices need charge. Their nbrs have high enough degree that Rule 1 suffices. This proves the NDT Conjecture for $(k, d) = (1, 1)$ (Montassier et al.).

When $k = 1$ and $d > 1$, only 2-vertices and 3-vertices need charge. This leads to a sufficient condition.

The Case $k = 1$

When $(k, d) = (1, 1)$, only 2-vertices need charge. Their nbrs have high enough degree that Rule 1 suffices. This proves the NDT Conjecture for $(k, d) = (1, 1)$ (Montassier et al.).

When $k = 1$ and $d > 1$, only 2-vertices and 3-vertices need charge. This leads to a sufficient condition.

Thm. For $d > k = 1$, if each 3-vertex in a minimal counterexample G has a neighbor with degree at least $d + 2$, then $\mathrm{Mad}(G) \geq m_{1,d} = 2 + \frac{2d}{d+2}$.

The Case $k = 1$

When $(k, d) = (1, 1)$, only 2-vertices need charge. Their nbrs have high enough degree that Rule 1 suffices. This proves the NDT Conjecture for $(k, d) = (1, 1)$ (Montassier et al.).

When $k = 1$ and $d > 1$, only 2-vertices and 3-vertices need charge. This leads to a sufficient condition.

Thm. For $d > k = 1$, if each 3-vertex in a minimal counterexample G has a neighbor with degree at least $d + 2$, then $\mathrm{Mad}(G) \geq m_{1,d} = 2 + \frac{2d}{d+2}$.

Pf. In addition to Rule 1 to give each 2-vertex $\frac{d}{d+2}$ along each edge, we add a rule to satisfy 3-vertices.

The Case $k = 1$

When $(k, d) = (1, 1)$, only 2-vertices need charge. Their nbrs have high enough degree that Rule 1 suffices. This proves the NDT Conjecture for $(k, d) = (1, 1)$ (Montassier et al.).

When $k = 1$ and $d > 1$, only 2-vertices and 3-vertices need charge. This leads to a sufficient condition.

Thm. For $d > k = 1$, if each 3-vertex in a minimal counterexample G has a neighbor with degree at least $d + 2$, then $\mathrm{Mad}(G) \geq m_{1,d} = 2 + \frac{2d}{d+2}$.

Pf. In addition to Rule 1 to give each 2-vertex $\frac{d}{d+2}$ along each edge, we add a rule to satisfy 3-vertices.

Rule 2: If $d_G(v) = 3$, and v has neighbor u with $d_G(u) \geq d + 2$, then v receives $\frac{d-2}{d+2}$ from u.

Everyone is Happy

Rules 1 and 2 yield $\mu(v) \geq m_{1,d}$ when $d_G(v) \leq 3$ (from $3 + \frac{d-2}{d+2} = 2 + \frac{2d}{d+2}$). Since $\frac{d-2}{d+2} < \frac{d}{d+2}$, the general argument for vertices with degree at least $2d + 2$ (which equals $(k+1)(k+d)$) also remains valid.

Everyone is Happy

Rules 1 and 2 yield $\mu(v) \geq m_{1,d}$ when $d_G(v) \leq 3$ (from $3 + \frac{d-2}{d+2} = 2 + \frac{2d}{d+2}$). Since $\frac{d-2}{d+2} < \frac{d}{d+2}$, the general argument for vertices with degree at least $2d+2$ (which equals $(k+1)(k+d)$) also remains valid.

If $4 \leq d_G(v) \leq d+1$, then v has no non-ghost 2-nbr, and Rule 2 takes no charge from v. If v has q ghost 2-neighbors with $q \geq 1$, then $d_G(v) \geq q+d+2$. Since $q \leq d/2$, we compute $\mu(v) \geq 2 + \frac{2d}{d+2} + \frac{d}{2} > m_{1,d}$.

Everyone is Happy

Rules 1 and 2 yield $\mu(v) \geq m_{1,d}$ when $d_G(v) \leq 3$ (from $3 + \frac{d-2}{d+2} = 2 + \frac{2d}{d+2}$). Since $\frac{d-2}{d+2} < \frac{d}{d+2}$, the general argument for vertices with degree at least $2d+2$ (which equals $(k+1)(k+d)$) also remains valid.

If $4 \leq d_G(v) \leq d+1$, then v has no non-ghost 2-nbr, and Rule 2 takes no charge from v. If v has q ghost 2-neighbors with $q \geq 1$, then $d_G(v) \geq q+d+2$. Since $q \leq d/2$, we compute $\mu(v) \geq 2 + \frac{2d}{d+2} + \frac{d}{2} > m_{1,d}$.

If $d+2 \leq d_G(v) \leq 2d+1$, again no non-ghost 2-nbr; v may give charge to q ghost neighbors and to $d_G(v) - 2q$ neighbors of degree 3.

Everyone is Happy

Rules 1 and 2 yield $\mu(v) \geq m_{1,d}$ when $d_G(v) \leq 3$ (from $3 + \frac{d-2}{d+2} = 2 + \frac{2d}{d+2}$). Since $\frac{d-2}{d+2} < \frac{d}{d+2}$, the general argument for vertices with degree at least $2d+2$ (which equals $(k+1)(k+d)$) also remains valid.

If $4 \leq d_G(v) \leq d+1$, then v has no non-ghost 2-nbr, and Rule 2 takes no charge from v. If v has q ghost 2-neighbors with $q \geq 1$, then $d_G(v) \geq q+d+2$. Since $q \leq d/2$, we compute $\mu(v) \geq 2 + \frac{2d}{d+2} + \frac{d}{2} > m_{1,d}$.

If $d+2 \leq d_G(v) \leq 2d+1$, again no non-ghost 2-nbr; v may give charge to q ghost neighbors and to $d_G(v) - 2q$ neighbors of degree 3.

$$\begin{aligned}\mu(v) &\geq d_G(v) - \frac{d}{d+2}2q - [d_G(v) - 2q]\frac{d-2}{d+2} \\ &= \frac{4(d_G(v)-q)}{d+2} \geq \frac{4(d+2)}{d+2} = 4 > m_{1,d}.\end{aligned}$$

Hence the final charge at each vertex is at least $m_{1,d}$. ■

What is Left to Do?

We have reduced the case $k = 1$ of the NDT Conjecture to proving that each 3-vertex in a minimal counterexample has a neighbor with degree at least $d + 2$. We have done this for $d \leq 6$.

What is Left to Do?

We have reduced the case $k = 1$ of the NDT Conjecture to proving that each 3-vertex in a minimal counterexample has a neighbor with degree at least $d + 2$. We have done this for $d \leq 6$.

Lem. β_G is submodular on the subsets of $V(G)$.

What is Left to Do?

We have reduced the case $k = 1$ of the NDT Conjecture to proving that each 3-vertex in a minimal counterexample has a neighbor with degree at least $d + 2$. We have done this for $d \leq 6$.

Lem. β_G is submodular on the subsets of $V(G)$.

Lem. If $d_G(v) = 3$ and v has no nbr with degree at least $d + 2$ (when $d \leq 6$), then v has nbrs u and u' such that $\beta_G(A) \geq d + 3$ whenever $\{u, u'\} \subseteq A \subseteq V(G) - \{v\}$.

What is Left to Do?

We have reduced the case $k = 1$ of the NDT Conjecture to proving that each 3-vertex in a minimal counterexample has a neighbor with degree at least $d + 2$. We have done this for $d \leq 6$.

Lem. β_G is submodular on the subsets of $V(G)$.

Lem. If $d_G(v) = 3$ and v has no nbr with degree at least $d + 2$ (when $d \leq 6$), then v has nbrs u and u' such that $\beta_G(A) \geq d + 3$ whenever $\{u, u'\} \subseteq A \subseteq V(G) - \{v\}$.

This leads to a contradiction, so the desired nbr exists.

What is Left to Do?

We have reduced the case $k = 1$ of the NDT Conjecture to proving that each 3-vertex in a minimal counterexample has a neighbor with degree at least $d + 2$. We have done this for $d \le 6$.

Lem. β_G is submodular on the subsets of $V(G)$.

Lem. If $d_G(v) = 3$ and v has no nbr with degree at least $d + 2$ (when $d \le 6$), then v has nbrs u and u' such that $\beta_G(A) \ge d + 3$ whenever $\{u, u'\} \subseteq A \subseteq V(G) - \{v\}$.

This leads to a contradiction, so the desired nbr exists.

Conj. Strong NDT Conjecture: If $\text{Arb}(G) \le k + \frac{d}{k+d+1}$, then G decomposes into $k + 1$ forests, one of which has at most d edges in every component.

What is Left to Do?

We have reduced the case $k = 1$ of the NDT Conjecture to proving that each 3-vertex in a minimal counterexample has a neighbor with degree at least $d + 2$. We have done this for $d \leq 6$.

Lem. β_G is submodular on the subsets of $V(G)$.

Lem. If $d_G(v) = 3$ and v has no nbr with degree at least $d + 2$ (when $d \leq 6$), then v has nbrs u and u' such that $\beta_G(A) \geq d + 3$ whenever $\{u, u'\} \subseteq A \subseteq V(G) - \{v\}$.

This leads to a contradiction, so the desired nbr exists.

Conj. Strong NDT Conjecture: If $\mathrm{Arb}(G) \leq k + \frac{d}{k+d+1}$, then G decomposes into $k + 1$ forests, one of which has at most d edges in every component.

Thm. The Strong NDT Conj. is true when $(k, d) = (1, 2)$.