

Maximum Face-Constrained Coloring of Plane Graphs*

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Abstract

Let $f(G)$ be the maximum number of colors in a vertex coloring of a simple plane graph G such that no face has distinct colors on all its vertices. If G has n vertices and chromatic number k , then $f(G) \geq \lceil n/k \rceil + 1$. For $k \in \{2, 3\}$, this bound is sharp for all n (except $n \leq 3$ when $k = 2$). For $k = 4$, the bound is within 1 for all n .

Keywords: planar graph, coloring, polychromatic; AMS codes: 05C15, 05C10

1 Introduction

In this paper, we consider colorings of the vertices of a simple plane graph subject to constraints arising from the faces instead of the usual constraints arising from the edges. Two natural optimization problems are to minimize the number of colors so that no face is monochromatic or to maximize the number of colors so that no face is polychromatic, where a set is *polychromatic* under a coloring if its elements have distinct colors.

The first of these problems is a classical hypergraph coloring problem. The edges of the *face hypergraph* of a plane graph are the vertex sets of the faces. These form a special subclass of the class of “planar hypergraphs” introduced by Zykov [16]. Further properties of face hypergraphs are studied in [9]. Minimizing the number of colors used on the vertices so that no face is monochromatic is the ordinary chromatic number problem for the face hypergraph.

The Four Color Theorem ([2, 12]) implies that the face hypergraph of a simple plane graph have chromatic number 2, as follows. Begin with a simple plane graph G . Add edges to obtain a supergraph G' that is a triangulation. Find a proper 4-coloring c' of G' . Combine the first color class with the second, and combine the third with the fourth. This yields a 2-coloring c of $V(G)$. Since each face of G contains a face of G' , the vertex set of a face in G contains the vertices of a triangle in G' . Hence at least three colors under c' appear on

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each face of G , which means that on each face there is a vertex of each color under c . On the other hand, two colors are required whenever the graph has at least two vertices. It is worth noting that there are other proofs of this bound that do not require the Four Color Theorem, such as [10].

Thus the first optimization problem is easy, and we henceforth study the maximum number of colors in colorings without polychromatic faces. This could be called the “anti-chromatic number” of the face hypergraph by analogy with terminology for edge-coloring problems. Given a fixed graph G , coloring $E(K_n)$ with fewest colors while avoiding a monochromatic copy of G is the *Ramsey problem* for G , and using the most colors while avoiding a polychromatic copy of G is the *anti-Ramsey problem*. Anti-Ramsey problems are studied in [1, 3, 4, 6, 7, 8, 13].

For a plane graph G , we use $f(G)$ to denote the maximum number of colors in a coloring of $V(G)$ such that no face is polychromatic. We consider only simple plane graphs; allowing double edges could bring $f(G)$ as low as 1.

We specify that G is a plane graph because $f(G)$ may depend on the embedding. For the graph consisting of k triangles with one common vertex, let G be the outerplane embedding, and let G' be the embedding in which the regions enclosed by the triangles form a chain under inclusion, as illustrated in Fig. 1. Let n denote the number of vertices, which is $2k + 1$.

In this example, $f(G) = k + 1 = \lceil n/2 \rceil$, since the three vertices of a triangle cannot have distinct colors. On the other hand, $f(G') = \lceil 3k/2 \rceil = \lceil 3(n - 1)/4 \rceil$. This follows inductively, since the vertices in the outer two triangles use at most three colors not used on the rest, and it is possible to use three such colors there. It is natural to ask whether this difference of $\lfloor (n - 2)/4 \rfloor$ between $f(G)$ and $f(G')$ is the maximum difference for distinct embeddings of a planar n -vertex graph.

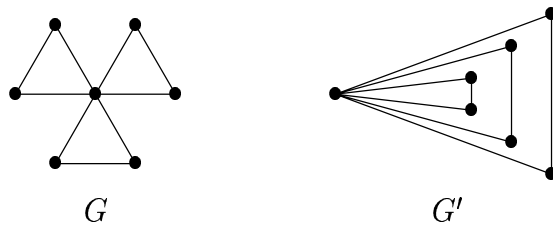


Fig. 1. Two embeddings of a planar graph.

We refer to a coloring with no polychromatic faces as a *valid* coloring. For arbitrary simple plane graphs, a general lower bound on $f(G)$ follows by considering a maximum stable set in G , the size of which is denoted by $\alpha(G)$. Maria Axenovich independently observed this lower bound.

Proposition 1 *If G is a n -vertex simple plane graph with $\alpha(G) < n - 1$, then*

$$f(G) \geq \alpha(G) + 1 \geq \left\lceil \frac{n}{\chi(G)} \right\rceil + 1.$$

Proof: We obtain a valid coloring of G by giving the $\alpha(G)$ vertices in a maximum stable set distinct colors and giving all other vertices the same color, distinct from those in the stable set. Since G is simple and $\alpha(G) < n - 1$, every face has at least two vertices outside the stable set and hence has at least two vertices of the same color.

Finally, $\alpha(G)$ is at least the size of the largest color class in a proper $\chi(G)$ -coloring of G . ■

Since $\chi(G) \leq 4$ for planar graphs, we obtain $f(G) \geq \lceil n/4 \rceil + 1$. When the chromatic number is 3 or 2, the lower bound improves to $\lceil n/3 \rceil + 1$ or $\lceil n/2 \rceil + 1$, respectively. Our objective in this paper is to show that Proposition 1 is sharp or nearly sharp. For $k \in \{2, 3\}$ and arbitrary n , we present a graph G with n vertices and chromatic number k such that $f(G) = \lceil n/k \rceil + 1$ (except $k = 2$ and $n \leq 3$). For $k = 4$, the graphs we present have $f(G) = \lceil n/k \rceil + 2$ for arbitrary n . Hence the bound is sharp for $k \in \{2, 3\}$ and is within 1 for $k = 4$. We also show that the bound $f(G) \geq \alpha(G) + 1$ is sharp or nearly sharp for all choices of n and $\alpha(G)$.

2 Sharpness for $\chi(G) = 3$ and $\chi(G) = 4$

Henceforth, we consider k -chromatic simple plane graphs with n vertices. When n is a multiple of k , equality in the bound of Proposition 1 requires $\alpha(G) = n/k$. Mike Albertson provided us a family of 4-chromatic planar graphs satisfying this condition. We show that these graphs almost achieve the lower bound for $k = 4$. The analogous 3-chromatic graphs do achieve the lower bound for $k = 3$.

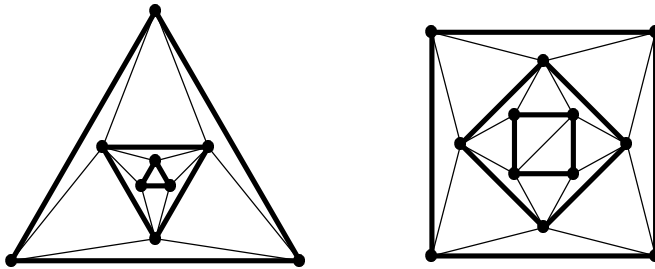


Fig. 2. $G_{3,3}$ and $G_{3,4}$.

For $k \in \{3, 4\}$, the graph $G_{m,k}$ is constructed as follows (Fig. 2 shows $G_{3,3}$ and $G_{3,4}$). Begin with m concentric pairwise disjoint k -cycles C_1, \dots, C_m (indexed from the center, shown in bold); these are the *rungs* of $G_{m,k}$. For each $i \in \{1, \dots, m-1\}$, add a cycle of length $2k$ on $V(C_i) \cup V(C_{i+1})$ triangulating the region between C_i and C_{i+1} . To complete $G_{m,4}$, add one chord of C_1 .

Since $\alpha(G_{m,3}) = m$ and $\alpha(G_{m,4}) = 2 \lceil m/2 \rceil$, Proposition 1 implies that $f(G_{m,3}) \geq m+1$ and $f(G_{m,4}) \geq 2 \lceil m/2 \rceil$. Using only the chromatic number bound, we still have $f(G_{m,k}) \geq m+1$. When $k=3$ equality holds, but when $k=4$ there is always a valid coloring with one additional color. Adding an outside chord of C_m to produce a triangulation does not prevent this, since the coloring we produce can be twisted to give the endpoints of the added chord the same color. Thus for simple plane graphs with $4m$ vertices our theorem only determines the minimum of f within 1.

Theorem 2 *If $m \geq 1$, then $f(G_{m,3}) = m+1$ and $f(G_{m,4}) = m+2$, except that $f(G_{2,4}) = 3$.*

Proof: Since $G_{m,k}$ is k -colorable, $f(G_{m,k}) \geq m+1$. By case analysis, this is sharp for $G_{2,4}$ (the details appear in [11]).

For $k=4$ and $m \neq 2$, we improve the lower bound to $f(G_{m,4}) \geq m+2$ as follows. For $2 \leq i \leq m-1$, use color i on all of $V(C_i)$. On C_1 , use color 2 on the adjacent opposite vertices, and use colors 1 and 0 on the remaining two vertices. On C_m , use color $m-1$ on two opposite vertices and colors m and $m+1$ on the remaining two vertices.

We have used $m+2$ colors. Between two monochromatic rungs, every triangle has two vertices on one rung and hence of the same color. Since the vertices with colors 0 and 1 are nonadjacent, every triangle involving them has two vertices of color 2; similarly, every triangle involving color m or $m+1$ has two vertices of color $m-1$.

For the upper bound, we prove by induction on m that $f(G_{m,3}) \leq m+1$ and $f(G_{m,4}) \leq m+2$, even when the constraint from the outside face is ignored (this facilitates the induction step). This is sharp (except for $G_{2,4}$) and remains so even when the constraint from the outside face is included.

The bound holds by inspection when $m=1$. For the induction step, consider two consecutive rungs in a valid coloring. Two adjacent vertices on one rung form a triangle with a vertex of the other. This implies that two adjacent vertices on one rung cannot have distinct colors that don't appear on the neighboring rung. In particular, a rung has at most $\lfloor k/2 \rfloor$ colors not appearing on the neighboring rung.

If C_m has at most one color on its vertices that does not appear on C_1, \dots, C_{m-1} , then the induction step is immediate. Our observation that there are at most $\lfloor k/2 \rfloor$ "new" colors (not on an earlier rung) thus completes the proof for $k=3$. We are left with the case that $k=4$ and C_m has two new colors on it (on opposite vertices).

Each vertex on C_m having a new color forms a triangle with two vertices on C_{m-1} ; these two vertices must have the same (old) color. Also, each vertex on C_{m-1} forms a triangle with two vertices of C_m ; these triangles force all of C_{m-1} plus the two vertices of C_m without new colors to have the same color.

If the color on C_{m-1} appears somewhere earlier, then we delete $V(C_m) \cup V(C_{m-1})$ and apply the induction hypothesis; the last two rungs contribute only two new colors.

If the color on C_{m-1} does not appear earlier, then consider C_{m-2} (if it exists). Since all vertices of C_{m-1} have a color not appearing on C_{m-2} , the triangles between them force all of C_{m-2} to have the same color.

Continuing in this manner, we have monochromatic rungs until we reach a rung whose color appears earlier or until we exhaust all rungs. If we exhaust all rungs, then we have one color for each rung except two for C_m , and the claim holds. Otherwise, we apply the induction hypothesis to the coloring left by deleting these rungs, including the one that introduces no new color. The number of colors we have deleted equals the number of rungs deleted, so the induction hypothesis yields the desired bound. ■

Remarks. Theorem 2 yields near-tightness of the bounds in Proposition 1. Since the computation of $f(G_{m,k})$ does not require the constraint from the outside face, we can add vertices and edges in the outside face and study the change in f due to the new vertices.

Consider the bound $\min f(G) \geq \lceil n/k \rceil + 1$ over n -vertex k -chromatic plane graphs. Section 3 studies the case $k = 2$. For $k \in \{3, 4\}$, let $m = \lfloor n/k \rfloor$ and $b = n - mk$. By Theorem 2, we need only consider $1 \leq b \leq k - 1$. Modify $G_{m,k}$ to form a plane graph G' by identifying an external vertex x of $G_{m,k}$ with a vertex of K_{b+1} . To compute $f(G')$ we need only consider constraints on the new vertices. For $b \in \{1, 2, 3\}$, the new bounded faces ensure that the new vertices together receive at most one color not used on x . Hence Theorem 2 yields $f(G') \leq m + 2$ when $k = 3$ and $f(G') \leq m + 3$ when $k = 4$. Since $b \geq 1$, the lower bound $\lceil n/k \rceil + 1$ yields $f(G') \geq m + 2$, and hence the claimed sharpness holds.

Now consider the bound $\min f(G) \geq a + 1$ over n -vertex plane graphs with independence number a . Since $\alpha(G_{m,4}) = 2 \lceil m/2 \rceil$ and $\alpha(G_{m,3}) = m$, Theorem 2 implies that the bound is tight when n is a multiple of 3 or an odd multiple of 4. Adding s pendant vertices to an external vertex of $G_{m,k}$ increases both the independence number and the value of f by s . Call the resulting graph $H_{m,k,s}$. If $a \geq n/3$ and $n = 3a - 2s$ for some integer s , then $H_{a-s,3,s}$ achieves the bound. If $n/4 \leq a < n/3$ and $n = 4a - 3s$ for some integer s , then let $H_{a-s,4,s}$ is within one of the bound. Similar constructions show approximate tightness for other pairs (n, a) . ■

3 Sharpness for Bipartite Graphs

We construct a sequence of bipartite graphs. For $n \geq 4$, the graph G_n will satisfy $f(G_n) = \lceil n/2 \rceil + 1$. (For $n < 4$, there is no simple bipartite n -vertex graph G with $\alpha(G) < n - 1$, and the lower bound is larger.) Each graph in this sequence is a maximal bipartite plane graph with at least one vertex of degree 2. A bipartite plane graph is maximal if and only if every face has length 4.

Initialize the sequence with $G_4 = C_4$ and $G_5 = K_{2,3}$. For $n \geq 6$, we form G_n from G_{n-2} as indicated in Fig. 3. Choose a face $uvwz$ of G_{n-2} having a vertex v of degree 2. Insert into this face an edge xy , making y adjacent to v and x adjacent to u and w . Each new face created is a 4-cycle, and the new vertex y has degree 2.

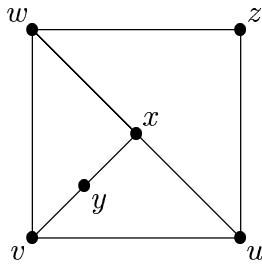


Fig. 3. Adding vertices x and y .

Theorem 3 For $n \geq 4$, $f(G_n) = \lceil n/2 \rceil + 1$.

Proof: The lower bound follows from Proposition 1. For the upper bound, we use induction on n . The basis step uses the upper bound $n - 1$, which equals $\lceil n/2 \rceil + 1$ for C_4 and $K_{2,3}$.

For $n > 5$, consider a valid coloring c of G_n . Let the two vertices added to form G_n from G_{n-2} be x and y , inserted into face $uvwz$ of G_{n-2} as in Fig. 3.

Call a vertex *lonely* if its color appears on no other vertex. We first modify c so that x and y are not both lonely. If they are, then u, v, w must all have the same color. Since v has degree 2 in G_{n-2} , switching the colors on y and v leaves all constraints satisfied in G and y no longer lonely.

When x and y are not both lonely, they contribute at most one “new” color. Hence the claim follows from the induction hypothesis if we can modify the coloring, without changing the number of colors, so that deleting x and y leaves a valid coloring of G_{n-2} . For this it suffices to arrange that two vertices in $\{u, v, w, z\}$ have the same color.

Suppose that these four vertices have distinct colors. Now $c(x)$ must be on one of u, w, z and cannot equal $c(v)$. Also x is not lonely. We find the desired coloring in one of two ways.

If $c(v) \neq c(y)$, then satisfying both faces involving y requires $c(y) \in \{c(u), c(w), c(x)\}$ (the choice depends on which of $\{c(u), c(w), c(z)\}$ equals $c(x)$). Now we recolor all vertices

having color $c(v)$ to have color $c(y)$, and we put color $c(v)$ on y . This yields a new valid coloring in which v has the same color as a vertex in $\{u, w, z\}$, and still x is not lonely.

On the other hand, if $c(v) = c(y)$, then y is not lonely. Since $c(x)$ is on only one of u, w, z , we may assume that $c(x) \neq c(w)$. Now we recolor all vertices having color $c(w)$ to have color $c(x)$, and we put color $c(w)$ on x . This yields a new valid coloring in which w has the same color as u or z , and still y is not lonely. ■

4 An Open Question

Although these results settle the question in terms of chromatic number (except for the gap of 1 when G is 4-chromatic), there remain interesting families of planar graphs in which we do not know the maximum of f .

Grötzsch's Theorem (see [5, 15]) states that triangle-free planar graphs are 3-colorable, so for such a graph G we have $f(G) \geq \lceil n/3 \rceil + 1$. Steinberg and Tovey [14] proved that $\alpha(G)$ is strictly greater than $n/3$ when G is triangle-free and planar, which very slightly strengthens the lower bound. On the other hand, we have no constructions that prevent us from using considerably more colors.

Conjecture 4 *If G is a triangle-free simple n -vertex planar graph with $n \geq 4$, then $f(G) \geq \lceil n/2 \rceil + 1$.*

This bound is attained by the bipartite graphs of Section 3. When the girth is at least 6, the bound can be proved. Intuitively, large faces make it easy to prevent polychromatic faces without constraining many vertices.

Proposition 5 *If G is a planar n -vertex graph with girth at least 6 and $n \geq 4$, then $f(G) \geq \lceil n/2 \rceil + 1$.*

Proof: Given a planar graph G with girth at least 6, we add two triangular chords in each face to form a planar graph G' so that in each face of G , there are two disjoint faces of G' , sharing not even a vertex.

As remarked in the introduction, we can 2-color $V(G')$ so that every face of G' has vertices of both colors. By the construction of G' , this coloring puts each color at least twice on each face of G . Changing all the vertices in the larger color class to distinct colors produces a coloring of $V(G)$ using at least $\lceil n/2 \rceil + 1$ colors and having no polychromatic face. ■

Acknowledgements

We thank André Kündgen and Vitaly Voloshin for their observations and suggestions.

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