Cut-edges and Regular Subgraphs in Odd-degree Regular Graphs

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slides available on DBW preprint page

Joint work with

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What about $2k$-factors for $k > 1$? Harder to guarantee, since every $2k$-factor contains a 2-factor.

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All the results also apply to multigraphs.

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$$q(S, T) \leq \ell(|S| - |T|) + d_{G-S}(T)$$

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Parity Lemma: No $2k$-factor $\Rightarrow$ $\exists$ disjoint S, T with

$$q(S, T) \geq 2k(|S| - |T|) + d_{G-S}(T) + 2.$$

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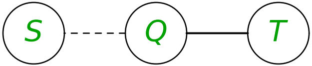

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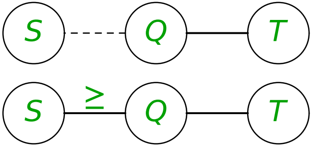

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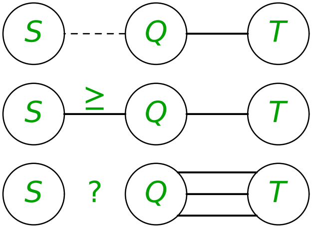

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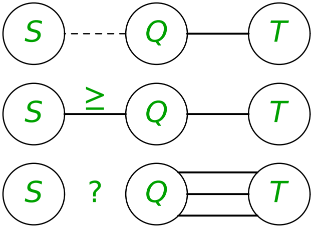

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$$3q(S,T) = 3(q_1+q_2+q_3) \le 2c + 2\|R,S\| + d_{G-S}(T)$$

Lower Bound on Number of Cut-Edges, c

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Hence $c \geq (2r+1-3k)(|T| - |S|) + 3$.

Finally, every Q counted by $q(S,T)$ adds at least 1 to $d_{G-S}(T)$, since $\|T,V(Q)\|$ is odd.

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When does equality hold?

Characterization

For $k \leq (2r+1)/3$, a $(2r+1)$-regular multigraph with $c = 2r+4-3k$ and no $2k$-factor must satisfy equality in all the inequalities producing $c \geq 2r+1-3(k-1)$.

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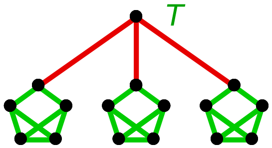
Thm. For $k \le (2r+1)/3$, a $(2r+1)$-regular G with $c = 2r+4-3k$ has no $2k$-factor iff $V(G)$ splits to R, S, T so
(a) S and T are independent sets with $|T| > |S|$,
(b) all cut-edges join T to distinct components of $G[R]$,
(c) all edges at S lead to T (maybe via “blisters”),
(d) exactly $k(|T| - |S|) - 1$ components of $G[R]$
are joined to T by exactly three edges each,
(e) other comps. of R are $(2r+1)$-regular, w/o cut-edge,
(f) if $k < (2r+1)/3$, then $|T| - |S| = 1$.

Fewest Cut-Edges with No $2k$-factor



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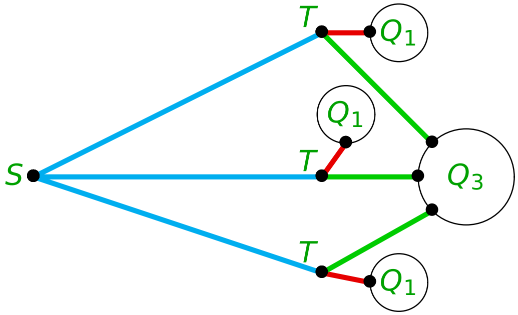

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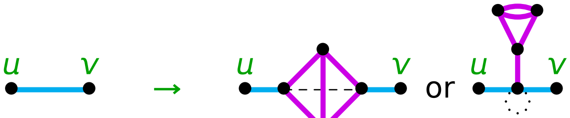

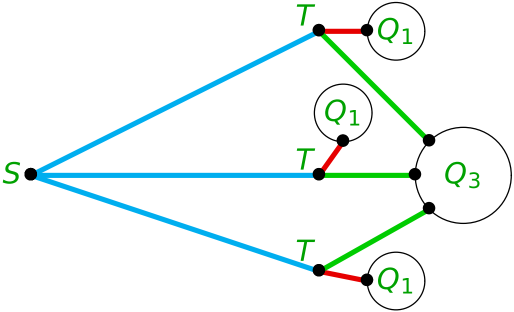

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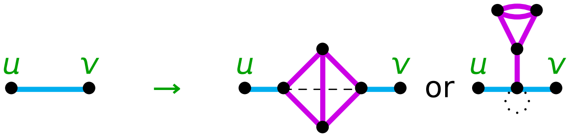

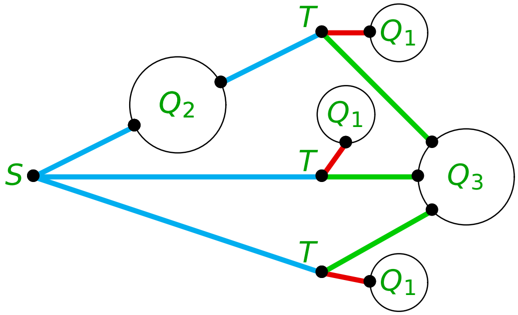

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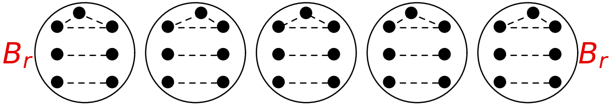

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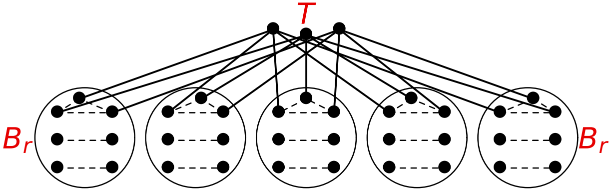

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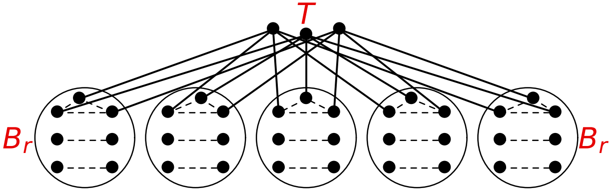

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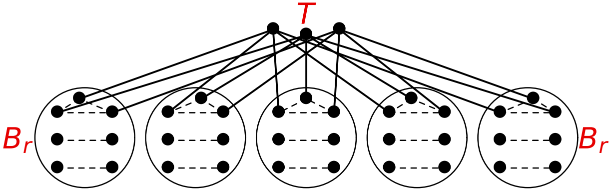

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G has no $2k$-factor if $q(S,T) > d_{G-S}(T) + 2k(|S| - |T|)$, which is equivalent to $6k > 4r+2$, or $k > (2r+1)/3$. ∎

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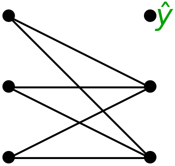

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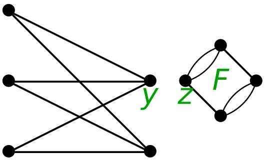

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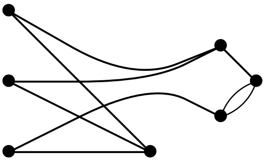

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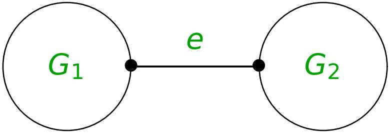
Pf. Use induction on c.

The induction step $c > 0$ is easy and reduces the problem to the base case $c = 0$ for subcubic multigraphs w/o cut-edges.

Induction Step ($c > 0$)

G has c cut-edges and deficit d.



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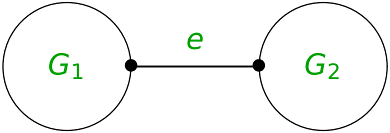

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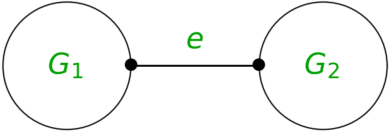

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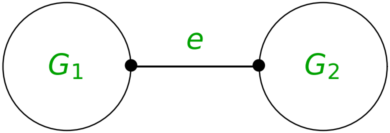

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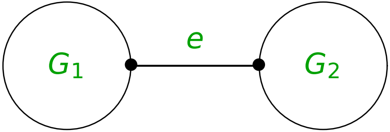

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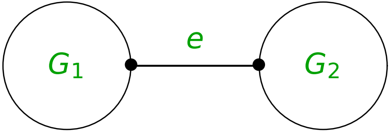

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Furthermore, equality in the bound for G requires equality for both G_1 and G_2.

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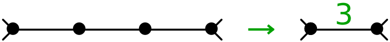

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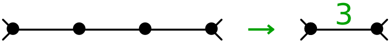

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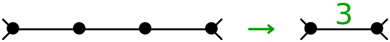

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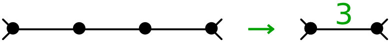

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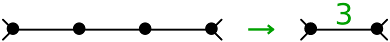

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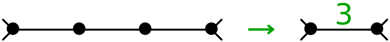

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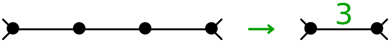

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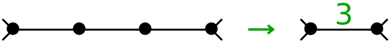

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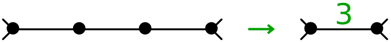

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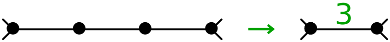

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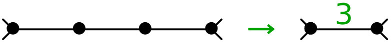

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If $d = 3$ and no 2-factor (and $c = 0$), show that $G \in \mathcal{F}$. ■

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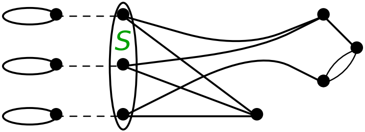

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