

Cut-edges and Regular Subgraphs in Odd-degree Regular Graphs

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slides available on DBW preprint page

Joint work with

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What about $2k$ -factors for $k > 1$? Harder to guarantee, since every $2k$ -factor contains a 2-factor.

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All the results also apply to multigraphs.

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Parity Lemma: No $2k$ -factor $\Rightarrow \exists$ disjoint S, T with $q(S, T) \geq 2k(|S| - |T|) + d_{G-S}(T) + 2$.

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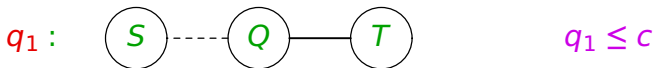
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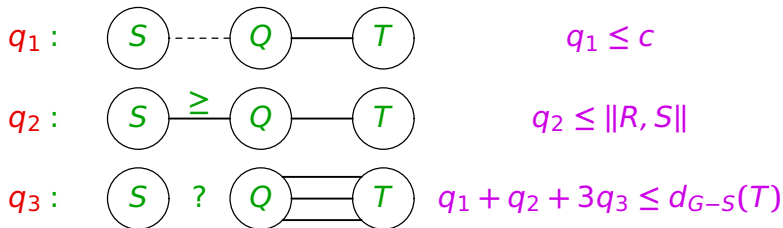
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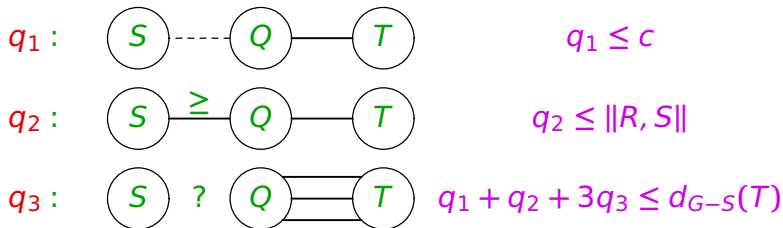
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$$3q(S, T) = 3(q_1 + q_2 + q_3) \leq 2c + 2\|R, S\| + d_{G-S}(T)$$

Lower Bound on Number of Cut-Edges, c

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Hence $c \geq (2r + 1 - 3k)(|T| - |S|) + 3$.

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Is it sharp? If so, when does equality hold?

Characterization

For $k \leq (2r + 1)/3$, a $(2r + 1)$ -regular multigraph with $c = 2r + 4 - 3k$ and no $2k$ -factor must satisfy equality in all the inequalities producing $c \geq 2r + 1 - 3(k - 1)$.

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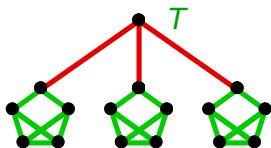
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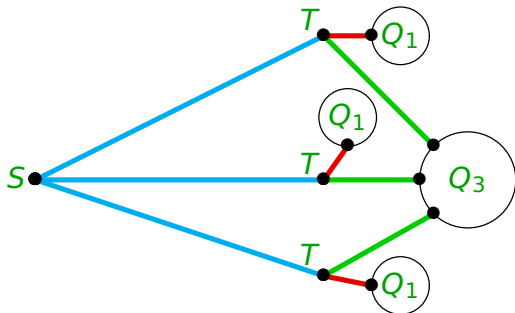
Thm. For $k \leq (2r + 1)/3$, a $(2r + 1)$ -regular G with $c = 2r + 4 - 3k$ has no $2k$ -factor iff $V(G)$ splits to R, S, T so

- (a) S and T are independent sets with $|T| > |S|$,
- (b) all cut-edges join T to distinct components of $G[R]$,
- (c) all edges at S lead to T (maybe via “blisters”),
- (d) exactly $k(|T| - |S|) - 1$ components of $G[R]$ are joined to T by exactly three edges each,
- (e) other comps. of R are $(2r + 1)$ -regular, w/o cut-edge,
- (f) if $k < (2r + 1)/3$, then $|T| - |S| = 1$.

Fewest Cut-Edges with No $2k$ -factor

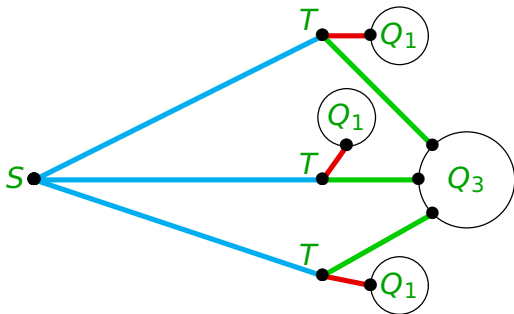
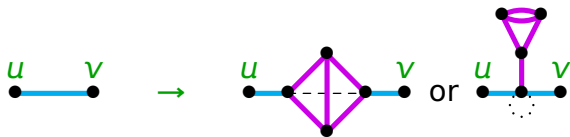


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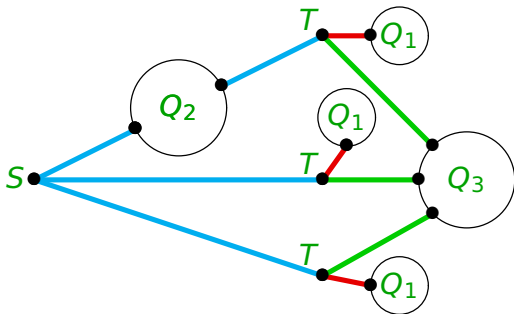
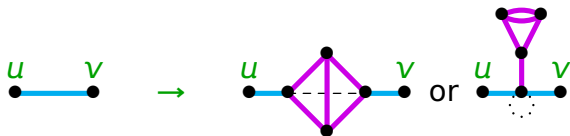
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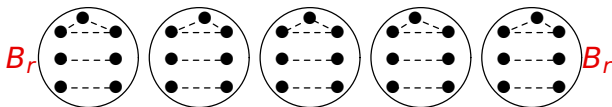
Necessity of $k \leq (2r + 1)/3$

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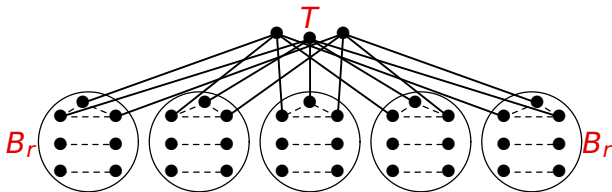
Pf. Let $B_r = \overline{C_3 + rP_2}$ ($2r + 3$ vertices, three with degree $2r$ and the rest with degree $2r + 1$). $2r + 1$ copies.



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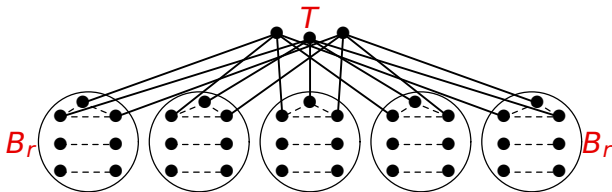


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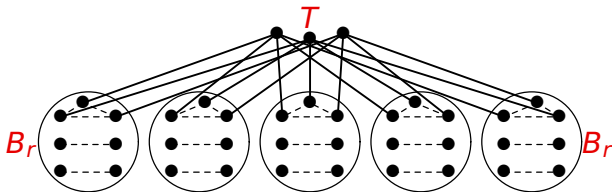
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G has no $2k$ -factor if $q(S, T) > d_{G-S}(T) + 2k(|S| - |T|)$, which is equivalent to $6k > 4r + 2$, or $k > (2r + 1)/3$. ■

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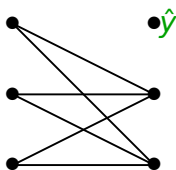
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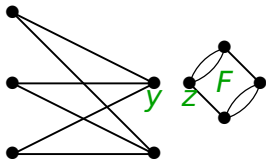
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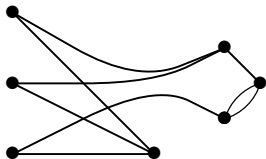
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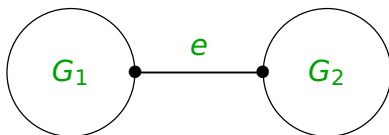
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Pf. Use induction on c .

The induction step $c > 0$ is easy and reduces the problem to the base case $c = 0$ for subcubic multigraphs w/o cut-edges.

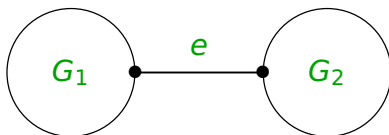
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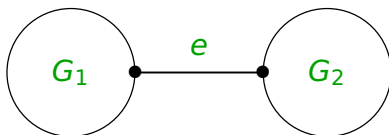
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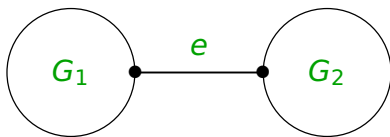


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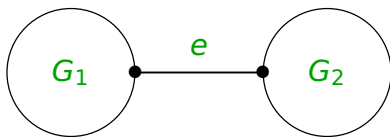
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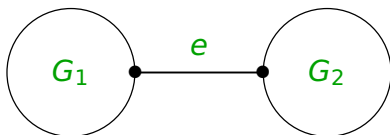
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Furthermore, equality in the bound for G requires equality for both G_1 and G_2 .

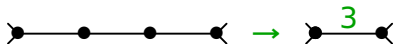
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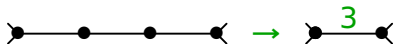
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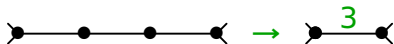


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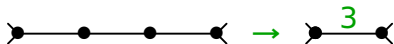


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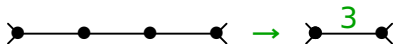
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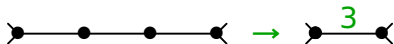
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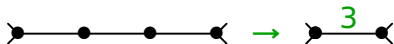
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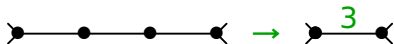
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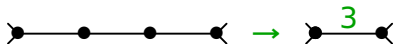
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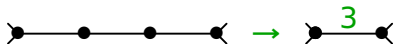
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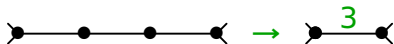
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If $d = 3$ and no 2-factor (and $c = 0$), show that $G \in \mathcal{F}$. ■

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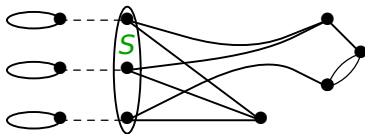
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