Cut-edges and regular factors in regular graphs of odd degree

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Abstract

We study $2k$-factors in $(2r+1)$-regular graphs. Hanson, Loten, and Toft proved that every $(2r+1)$-regular graph with at most $2r$ cut-edges has a 2-factor. We generalize their result by proving for $k \le (2r+1)/3$ that every $(2r+1)$-regular graph with at most $2r - 3(k-1)$ cut-edges has a $2k$-factor. Both the restriction on k and the restriction on the number of cut-edges are sharp. We characterize the graphs that have exactly $2r - 3(k-1) + 1$ cut-edges but no $2k$-factor. For $k > (2r+1)/3$, there are graphs without cut-edges that have no $2k$-factor, as studied by Bollobás, Saito, and Wormald.

1 Introduction

An *ℓ-factor* in a graph is an ℓ-regular spanning subgraph. In this paper we study the relationship between cut-edges and $2k$-factors in regular graphs of odd degree. In fact, all our results are for multigraphs, allowing loops and multiedges, so the model we mean by "graph" allows loops and multiedges.

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The relationship between edge-connectivity and 1-factors in regular graphs is well known. Petersen [11] proved that every 3-regular graph with no cut-edge decomposes into a 1-factor and a 2-factor, noting that the conclusion also holds when all cut-edges lie along a path. Schönberger [13] proved that in a 3-regular graph with no cut-edge, every edge lies in some 1-factor. Berge [5] obtained the same conclusion for r-regular $(r-1)$-edge-connected graphs of even order. Finally, a result of Plesník [12] implies most of these statements: If G is an r-regular $(r-1)$-edge-connected multigraph with even order, and G' is obtained from G by discarding at most $r-1$ edges, then G' has a 1-factor. The edge-connectivity condition is sharp: Katerinis [9] determined the minimum number of vertices in an r-regular graph of even order that is $(r-2)$-edge-connected but has no 1-factor. Belck [4] and Bollobás, Saito, and Wormald [6] (independently) determined all (r,t,k) such that every r-regular t-edge-connected graph has a k-factor; Niessen and Randerath [10] further refined this in terms of also the number of vertices.

Petersen was in fact more interested in 2-factors. The result about 3-regular graphs whose cut-edges lie on a path implies that every 3-regular graph with at most two cut-edges has a 2-factor. Also, there are 3-regular graphs with three cut-edges having no 2-factor (communicated to Petersen by Sylvester in 1889). As a tool in a result about interval edge-coloring, Hanson, Loten, and Toft [8] generalized Petersen's result to regular graphs with larger odd degree.

Theorem 1.1 ([8])**.** *For $r \in \mathbb{N}$, every $(2r+1)$-regular graph with at most $2r$ cut-edges has a 2-factor.*

Petersen [11] also proved that every regular graph of even degree has a 2-factor. Thus when $k \le r$ every $2r$-regular graph has a $2k$-factor. As a consequence, regular factors of degree $2k$ become harder to guarantee as k increases. That is, a decomposition of a $(2r+1)$-regular graph into a 2-factor and $(2r-1)$-factor is easiest to find, while decomposition into a $2r$-factor and 1-factor is hardest to find (and implies the others).

In this paper, we generalize Theorem 1.1 to find the corresponding best possible guarantee for $2k$-factors. Limiting the number of cut-edges suffices when k is not too large.

Theorem 1.2. *For $r,k \in \mathbb{N}$ with $k \le (2r+1)/3$, every $(2r+1)$-regular graph with at most $2r-3(k-1)$ cut-edges has a $2k$-factor. Furthermore, both inequalities are sharp.*

Earlier, Xiao and Liu [16] proved a relationship between cut-edges and $2k$-factors, showing that a $(2kr+s)$-regular graph with at most $k(2r-3)+s$ cut-edges has a $2k$-factor avoiding any given edge. Their number of cut-edges in terms of degree and k is similar to ours, since $(2kr+s)-1-3(k-1) = k(2r-3)+s+2$, but their range of validity of k in terms of the degree of the full graph is more restricted than ours.

Our result is sharp in two ways. First, when $k \le (2r+1)/3$ we construct $(2r+1)$-regular graphs with no $2k$-factor that have just one more cut-edge than the bound in our theorem. For the case $r = 1$, Sylvester found examples of such graphs. We continue further and complete the Petersen–Sylvester investigation by describing all the extremal graphs when $k \le (2r+1)/3$; that is, all the $(2r+1)$-regular graphs that have exactly $2r+1-3(k-1)$ cut-edges and have no $2k$-factor.

Theorem 1.3. *For $r, k \in \mathbb{N}$ with $k \le (2r+1)/3$, a $(2r+1)$-regular graph with exactly $2r+1-3(k-1)$ cut-edges fails to have a $2k$-factor if and only if it satisfies the constructive structural description stated in Theorem 3.2.*

The second aspect of sharpness concerns the inequality $k \le (2r+1)/3$. When $k > (2r+1)/3$, the condition in Theorem 1.2 cannot be satisfied, and in fact there are $(2r+1)$-regular graphs that have no $2k$-factor even though they have no cut-edges. A $2k$-factor can instead be guaranteed by edge-connectivity requirements. The result of Berge [5] implies that $(2r+1)$-regular $2r$-edge-connected graphs have 1-factors and hence factors of all even degrees, by the 2-factor theorem of Petersen [11]. Therefore, when $k > (2r+1)/3$ the natural question becomes what edge-connectivity suffices to guarantee a $2k$-factor.

As mentioned earlier, this problem was solved by Bollobás, Saito, and Wormald [6], who determined all triples (r, t, k) such that every r-regular t-edge-connected multigraph has a k-factor (the triples are the same for simple graphs). As noted by Häggkvist [7] and by Niessen and Randerath [10], earlier Belck [4] obtained the result (in 1950). Earlier still, Baebler [3] proved the weaker result that $2k$-edge-connected $(2r+1)$-regular graphs have $2k$-factors.

The special case of the result of [6] that applies here (even-regular factors of odd-regular multigraphs) is that all $(2r+1)$-regular $2t$-edge-connected or $(2t+1)$-edge-connected multigraphs have $2k$-factors if and only if $k \le \frac{t}{2t+1}(2r+1)$. The general construction given in [6], which covers additional cases, is quite complicated. Here we provide a very simple construction that shows necessity of their condition for even-regular factors of odd-regular graphs. That is, for $1 \le t < r$ and $k > \frac{t}{2t+1}(2r+1)$ we present an easily described $(2t+1)$-connected simple $(2r+1)$-regular graph having no $2k$-factor.

Our arguments use only the necessary and sufficient condition for the existence of ℓ-factors that was initially proved by Belck [4] and is a special case of the f-Factor Theorem of Tutte [14, 15]. When T is a set of vertices in a graph G, let $d_G(T) = \sum_{v \in T} d_G(v)$, where $d_G(v)$ is the degree of v in G. With $|T|$ for the size of a vertex set T, we also write $\|T\|$ for the number of edges induced by T and $\|A, B\|$ for the number of edges having endpoints in both A and B (when $A \cap B = \varnothing$). The characterization is the following.

Theorem 1.4 ([4]). *A multigraph G has a ℓ-factor if and only if*

$$q(S,T) - d_{G-S}(T) \leq \ell(|S| - |T|) \tag{1}$$

for all disjoint subsets $S, T \subset V(G)$, where $q(S,T)$ is the number of components Q of $G-S-T$ such that $\|V(Q),T\| + \ell\,|V(Q)|$ is odd.

Since we consider only the situation where $\ell = 2k$, the criterion for a component Q of $G - S - T$ to be counted by $q(S,T)$ simplifies to $\|V(Q),T\|$ being odd.

We note that the study of f-factors is more general, where for a function $f\colon V(G) \to \mathbb{N}\cup\{0\}$ an *f-factor* of G is a spanning subgraph in which the degree of any vertex v is required to be $f(v)$. As noted earlier, Tutte [14, 15] generalized Belck's Theorem to a necessary and sufficient condition for the existence of an f-factor. There are further deep and complicated results about the existence of f-factors that can yield various cases of our result. Results about factors and their generalizations are thoroughly explored in the survey and book by Akiyama and Kano [1, 2]. Nevertheless, our purpose here is to complete the Petersen–Sylvester investigation using only simple counting arguments and the classical results about ℓ-factors developed in the 1950s.

2 Cut-edges and $2k$-factors

In this section we generalize Theorem 1.1 to $2k$-factors. The notion of a Parity Lemma for violations of (1) is well-known; for example the following is included in Theorem 1 of Tutte [14].

Lemma 2.1 (Parity Lemma). *If a multigraph G has no $2k$-factor, then it has disjoint vertex subsets S and T such that*

$$q(S,T) \geq d_{G-S}(T) + 2k(|S| - |T|) + 2, \tag{2}$$

where $q(S,T)$ is the number of components Q of $G - S - T$ such that $\|V(Q),T\|$ is odd.

Proof. For disjoint $S, T \subseteq V(G)$, let $R = V(G) - S - T$. Note that $q(S,T)$ has the same parity as $\|R,T\|$. Also, $\|R,T\|$ has the same parity as $d_{G-S}(T)$, since $d_{G-S}(T)$ counts edges from R to T once and edges in T twice. Hence the two sides of (1) have the same parity. □

Theorem 2.2. *For $r, k \in \mathbb{N}$ with $k \leq (2r+1)/3$, every $(2r+1)$-regular multigraph with at most $2r - 3(k-1)$ cut-edges has a $2k$-factor.*

Proof. Let G be a $(2r+1)$-regular multigraph having no $2k$-factor, and let c be the number of cut-edges in G. We prove $c > 2r - 3(k-1)$. By Theorem 1.4 and Lemma 2.1, lack of a $2k$-factor requires disjoint sets $S, T \subseteq V(G)$ such that (2) holds, where $q(S,T)$ counts the components Q of $G - S - T$ such that $\|V(Q), T\|$ is odd; call such a component T-*odd*.

Each T-odd component contributes at least 1 to $d_{G-S}(T)$. Hence (2) cannot hold with $|S| \geq |T|$, and we may assume $|T| > |S|$.

Let q_1 be the number of T-odd components having one edge to T and no edges to S; the edge to T is a cut-edge, so $q_1 \leq c$. Let q_2 be the number of T-odd components having one edge to T and at least one edge to S; note that $q_2 \leq \|R, S\|$, where $R = V(G) - S - T$. The remaining q_3 T-odd components have at least three edges to T. Thus the edges joining R and T yield $q_1 + q_2 + 3q_3 \leq d_{G-S}(T)$. Note also that $q(S,T) = q_1 + q_2 + q_3$. Summing the edge-counting inequality with two copies of the inequalities for q_1 and q_2 yields

$$3q(S,T) = 3(q_1 + q_2 + q_3) \leq 2c + 2\,\|R, S\| + d_{G-S}(T).$$

Combining this inequality with (2) yields

$$2c + 2\,\|R, S\| + d_{G-S}(T) \geq 3d_{G-S}(T) + 6k(|S| - |T|) + 6,$$

which simplifies to

$$\|R, S\| \geq 3 - c + d_{G-S}(T) + 3k(|S| - |T|). \tag{3}$$

On the other hand, since G is $(2r+1)$-regular,

$$d_{G-S}(T) = (2r+1)|T| - \|T, S\| \geq (2r+1)|T| - [(2r+1)|S| - \|R, S\|]\,.$$

Using this inequality, (3), and $|T| - |S| \geq 1$, the given hypothesis $2r + 1 - 3k \geq 0$ yields

$$\|R, S\| \geq 3 - c + (2r+1-3k)(|T| - |S|) + \|R, S\| \geq 3 - c + (2r+1-3k) + \|R, S\|\,.$$

This simplifies to $c \geq 2r + 1 - 3(k-1)$, as claimed. ∎

3 Fewest cut-edges with no $2k$-factor

To describe the extremal graphs, we begin with a definition. Keep in mind that here “graph” allows loops and multiedges.

Definition 3.1. In a $(2r+1)$-regular graph G, the result of *blistering* an edge $e \in E(G)$ by a $(2r+1)$-regular graph H having no cut-edge is a graph G' obtained from the disjoint union $G + H$ by deleting e and an edge $e' \in E(H)$ (where e' may be a loop if $r > 1$), followed by adding two disjoint edges to make each endpoint of e adjacent to one endpoint of e'. The resulting graph G' is $(2r+1)$-regular.

Figure 1 shows a graph G' obtained by blistering an edge joining S and T in a 3-regular graph G with three cut-edges and no 2-factor. The components of $G' - S - T$ labeled Q_i are components counted by q_i, for $i \in \{1, 2, 3\}$. In this case the graph H consists of the component labeled Q_2 plus an edge e' joining the vertices that are shown having neighbors in S and T. Replacing Q_2 and those two edges with an edge e that was blistered gives the original graph G.

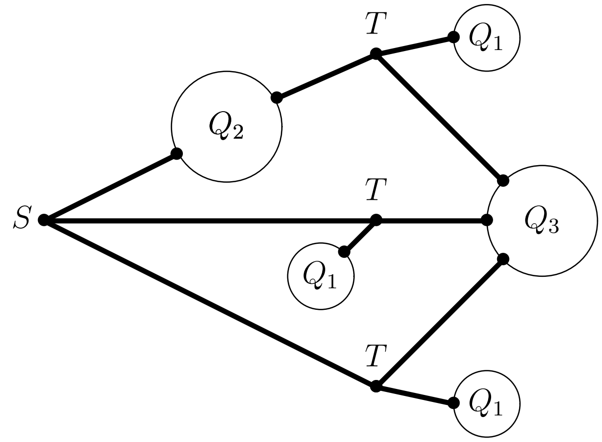

Figure 1: A 3-regular graph having three cut-edges and no 2-factor.

In the statement below, (c) notes the possibility of blistered edges from S to T.

Theorem 3.2. *For $k \le (2r+1)/3$, a $(2r+1)$-regular graph with $2r+4-3k$ cut-edges has no $2k$-factor if and only if the vertex set $V(G)$ has a partition into sets R, S, T such that*
(a) S and T are independent sets with $|T| > |S|$,
(b) all cut-edges join T to distinct components of $G[R]$,
(c) all edges incident to S are incident to T or to a component of $G[R]$ having one edge to T,
(d) exactly $k(|T| - |S|) - 1$ components of $G[R]$ are joined to T by exactly three edges each,
(e) other components of $G[R]$ are $(2r+1)$-regular components of G with no cut-edges, and
(f) if $k < (2r+1)/3$, then $|T| - |S| = 1$.

Proof. Sufficiency: Let G be a graph G with $2r+4-3k$ cut-edges, and suppose that such a partition $\{R, S, T\}$ of $V(G)$ exists. Let q_2 be the number of components of $G[R]$ that blister edges from S to T (as in condition (c)). Each cut-edge joins T to a T-odd component, by (b). The $k(|T| - |S|) - 1$ components of $G[R]$ joined to T by three edges (according to (d)) are also T-odd, as are the q_2 components of $G[R]$ arising as blisters. Hence $q(S,T) \ge 2r+4-3k+k(|T|-|S|)-1+q_2$. The number of edges joining S and T is $(2r+1)\,|S| - q_2$, by (c). Using also (a), we have $d_{G-S}(T) = (2r+1)(|T|-|S|)+q_2$. We compute

$$\begin{aligned} q(S,T) - d_{G-S}(T) &\geq (2r+1-3k) + 2 + (k-2r-1)(|T|-|S|) \\ &= -(2r+1-3k)(|T|-|S|-1) + 2k(|S|-|T|) + 2 = 2k(|S|-|T|) + 2, \end{aligned}$$

where the last equality uses (f) and the restriction $k \leq (2r+1)/3$. Hence the given partition R, S, T satisfies (2), and G has no $2k$-factor.

Necessity: Suppose that G has $2r+1-3(k-1)$ cut-edges and no $2k$-factor; we obtain the described partition of $V(G)$. The proof of Theorem 2.2 considers a $(2r+1)$-regular graph with no $2k$-factor and produces $c \geq 2r+4-3k$, where c is the number of cut-edges. To avoid having more cut-edges, we must have equality in all the inequalities used to produce this lower bound.

Recall that $q(S,T)$ counts the components Q of $G[R]$ with $\|V(Q),T\|$ odd. Also $q(S,T) = q_1 + q_2 + q_3$, where q_1, q_2, q_3 count the components having one edge to T and none to S, one edge to T and at least one to S, and at least three edges to T, respectively. Equality in the computation of Theorem 2.2 requires all of the following.

$$q_1 = c \tag{4}$$

$$q_2 = \|R,S\| \tag{5}$$

$$q_1 + q_2 + 3q_3 = d_{G-S}(T) \tag{6}$$

$$(2r+1)|S| = \|T,S\| + \|R,S\| \tag{7}$$

$$|T| - |S| \geq 1, \text{with equality when } k < (2r+1)/3 \tag{8}$$

By (6), contributions to $d_G(T)$ not in $\|T,S\|$ are counted in $\|T,R\|$, so T is independent. By (7), all edges incident to S are also incident to T or to R, so S is independent, proving (a). The first observation in proving Theorem 2.2 was $|T| > |S|$, and equality in the last step requires $|T| - |S| = 1$ when $2r+1 > 3k$, as stated in (8) and desired in (f). By (4), the cut-edges join T to distinct components of $G[R]$, proving (b).

By (5) and (7), $q_2 = 0$ implies $(2r+1)|S| = \|T,S\|$; in this case all edges incident to S are incident also to T. Since $(2r+1)|S| = \|T,S\| + q_2$, each component of $G[R]$ counted by q_2 generates only one edge from R to S. Thus each such component blisters an edge joining S and T in a smaller such graph. This explains all the edges counted by $\|S,R\|$. Hence all edges incident to S are as described in (c).

Having proved (a) and (c), we have accounted for $(2r+1)|S|$ edges incident to T. There are also exactly $2r+1-3(k-1)$ cut-edges, all joining T to components of $G[R]$. This leaves

$$(2r+1)|T| - (2r+1) + 3(k-1) - (2r+1)|S| \tag{9}$$

edges incident to T that are not cut-edges and join T to components of $G[R]$ not counted by q_1 or q_2. Setting $|T| - |S| = 1$ when $k < (2r+1)/3$ or $2r+1 = 3k$ when $k = (2r+1)/3$, (9) simplifies to $3[k(|T| - |S|) - 1]$. By (6), all remaining edges incident to T connect vertices of T to T-odd components of $G[R]$ counted by q_3, using exactly three edges for each such component. Hence there are exactly $k(|T| - |S|) - 1$ such components of $G[R]$, proving (d). This completes the description of the T-odd components.

Since we have described all edges incident to S and T, any remaining components of $G[R]$ are actually $(2r+1)$-regular components of G without cut-edges, proving (e). They do not affect the number of T-odd components or the existence of a $2k$-factor. □

Theorem 3.2 can be viewed as a constructive procedure for generating all extremal examples from certain base graphs. Given r and k with $k \le (2r+1)/3$, we start with a bipartite graph having parts T and $R \cup S$, where $|T| - |S| \ge 1$, with equality if $k < (2r+1)/3$. Also, vertices in $T \cup S$ have degree $2r+1$, and R has $2r+4-3k$ vertices of degree 1 and $k(|T|-|S|)-1$ vertices of degree 3. We expand the vertices of R to obtain a $(2r+1)$-regular multigraph G. This is a base graph. We can then blister edges from S to T and/or add $(2r+1)$-regular 2-edge-connected components.

The case $|T| = 1$ and $|S| = 0$ gives the graphs found by Sylvester. When $k > (2r+1)/3$, the final inequality in the proof of Theorem 2.2 is not valid. In this range no restriction on cut-edges can guarantee a $2k$-factor; we present a simple general construction. As mentioned earlier, this is a sharpness example for the result of Bollobás, Saito, and Wormald [6] that every $(2r+1)$-regular $2t$-edge-connected or $(2t+1)$-edge-connected multigraph has a $2k$-factor if and only if $k \le \frac{t}{2t+1}(2r+1)$. It is simpler than their more general construction.

Theorem 3.3. *For $1 \le t < r$ and $k > \frac{t}{2t+1}(2r+1)$, there is a $(2t+1)$-connected $(2r+1)$-regular graph having no $2k$-factor.*

Proof. Let $H_{r,t}$ be the complement of $C_{2t+1} + (r-t+1)K_2$. That is, $H_{r,t}$ is obtained from the complete graph K_{2r+3} by deleting the edges of a $(2t+1)$-cycle and $r-t+1$ other pairwise disjoint edges not incident to the cycle. In $H_{r,t}$ the vertices of the deleted cycle have degree $2r$, and the remaining vertices have degree $2r+1$. Let G be the graph formed from the disjoint union of $2r+1$ copies of $H_{r,t}$ by adding a set T of $2t+1$ vertices and $2r+1$ matchings joining T to the vertices of the deleted cycle in each copy of $H_{r,t}$ (see Figure 2).

Deleting $2t$ vertices cannot separate any copy of $H_{r,t}$ from T, and any two vertices of T are connected by $2r+1$ disjoint paths through the copies of $H_{r,t}$, so G is $(2t+1)$-connected.

Suppose that G has a $2k$-factor F. Every edge cut in an even factor is crossed by an even number of edges, since the factor decomposes into cycles. Hence F has at most $2t$ edges joining T to each copy of $H_{r,t}$. On the other hand, since T is independent, F must have $2k|T|$ edges leaving T. Thus $2k(2t+1) \leq 2t(2r+1)$. This contradicts the hypothesis $k > \frac{t}{2t+1}(2r+1)$. $\square$

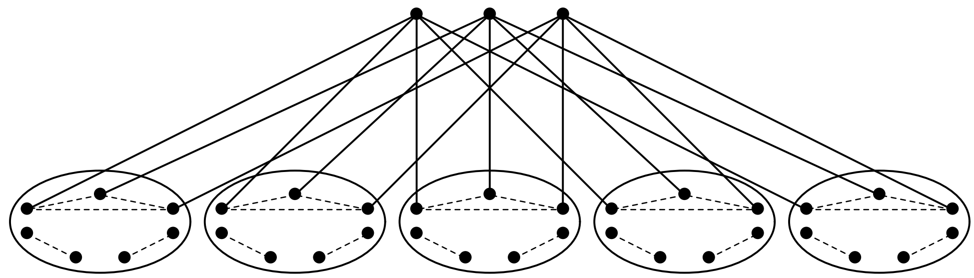

Figure 2: $(2r+1)$-regular, $(2t+1)$-connected, no $2k$-factor ($(r,t,k) = (2,1,2)$ shown).

References

[1] J. Akiyama and M. Kano, Factors and factorizations of graphs—a survey. *J. Graph Theory* 9 (1985), 1–42.

[2] J. Akiyama and M. Kano, *Factors and factorizations of graphs: Proof techniques in factor theory. Lecture Notes in Mathematics* Vol 2031. Springer, Heidelberg, 2011.

[3] F. Baebler, Über die Zerlegung regulärer Streckenkromplexe ungerader Ordnung (German). *Comment. Math. Helv.* 10 (1937–38), 275–287.

[4] H.-B. Belck, Reguläre Faktoren von Graphen (German). *J. Reine Angew. Math.* 188 (1950), 228–252.

[5] C. Berge, *Graphs and Hypergraphs* (North-Holland, 1973). (translation and revision of Graphes et Hypergraphes, Dunod, 1970), p. 162.

[6] B. Bollobás, A. Saito, and N.C. Wormald, Regular factors of regular graphs. *J. Graph Theory* 9 (1985), 97–103.

[7] R. Häggkvist, Factors galore: Extending theorems of Petersen, Baebler, Belck and Gallai, lecture at "Combinatorics in Cambridge", August 4, 2003.

[8] D. Hanson, C.O.M. Loten, and B. Toft, On interval colourings of bi-regular bipartite graphs. *Ars Combin.* 50 (1998), 23–32.

[9] P. Katerinis, Maximum matchings in a regular graph of specified connectivity and bounded order. *J. Graph Theory* 11 (1987), 53–58.

[10] T. Niessen and B. Randerath, Regular factors of simple regular graphs and factor-spectra. *Discrete Math.* 185 (1998), 89–103.

[11] J. Petersen, Die Theorie der regulären graphs. *Acta Math.* 15 (1891), 193–220.

[12] J. Plesník, Connectivity of regular graphs and the existence of 1-factors. *Mat. Casopis Sloven. Akad. Vied* 22 (1972), 310–318.

[13] T. Schönberger, Ein Beweis des Petersenschen Graphensatzes. *Acta Scientia Mathematica Szeged* 7 (1934), 51–57.

[14] W.T. Tutte, The factors of graphs. *Canad. J. Math.* 4 (1952), 314–328.

[15] W.T. Tutte, A short proof of the factor theorem for finite graphs. *Canad. J. Math.* 6 (1954), 347–352.

[16] L. Xiao and Y. Liu, Even regular factor of regular graphs and number of cut edges. *Southeast Asian Bull. Math.* 31 (2007), 1019–1026.