On the Erdős–Simonovits–Sós Conjecture about the anti-Ramsey number of a cycle

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Abstract

Given a positive integer n and a family $\mathcal{F}$ of graphs, let $f(n,\mathcal{F})$ denote the maximum number of colors in an edge-coloring of K_n such that no subgraph of K_n belonging to $\mathcal{F}$ has distinct colors on its edges. Erdős, Simonovits, and Sós [6] conjectured for fixed k with $k \geq 3$ that $f(n, C_k) \in \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n + O(1)$. This has been proved for $k \leq 7$. For general k, in this paper we improve the previous bound of $(k-2)n - \binom{k-1}{2}$ to $f(n, C_k) \leq \left(\frac{k+1}{2} - \frac{2}{k-1}\right) n - (k-2)$. For even k, we further improve it to $\frac{k}{2}n - (k-2)$. We also prove that $f(n, \{C_k, C_{k+1}, C_{k+2}\}) \leq \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n - 1$, which is sharp.

1 Introduction

A subgraph in a coloring of the edges of the complete graph K_n is *polychromatic* if the colors on its edges are distinct; it is a *polychromatic copy of H* if also it is isomorphic to H. Let n be a positive integer, and let $\mathcal{F}$ be a family of graphs. We study the *anti-Ramsey number* $f(n,\mathcal{F})$; this is the maximum number of colors in a coloring of $E(K_n)$ that has no polychromatic copy of any graph in $\mathcal{F}$. (The classical "Ramsey problem" can be interpreted as finding the minimum number of colors in a coloring of $E(K_n)$ that avoids monochromatic copies of graphs in $\mathcal{F}$.) We write $f(n,H)$ for $f(n,\{H\})$.

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Erdős, Simonovits, and Sós [6] introduced anti-Ramsey numbers. By relating them to Turán numbers, they showed that $f(n, \mathcal{F})/\binom{n}{2} \to 1 - \frac{1}{r-1}$ as $n \to \infty$, where $r = \min\{\chi(H - e): e \in E(H) \text{ and } H \in \mathcal{F}\}$. This determines $f(n, H)$ asymptotically when $r \geq 3$.

When $r = 2$, the limit yields only $f(n, H) \in o(n^2)$. This leaves open the asymptotics of anti-Ramsey numbers for bipartite graphs, for graphs that become bipartite upon deletion of an edge, and for families of such graphs. Exact formulas or asymptotics are known for $f(n, H)$ when H is a path ([12]), a star ([8]), some types of trees ([10]), the family of all trees of fixed size ([10]), or $K_{2,t}$ ([2, 7]).

Erdős, Simonovits, and Sós [6] initiated the study of $f(n, C_k)$. For fixed k with $k \geq 3$, they conjectured that $f(n, C_k) \in \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n + O(1)$ and proved this for $k = 3$. Alon [1] proved it for $k = 4$, showing that $f(n, C_4) = \lfloor 4n/3 \rfloor - 1$. It is proved for $k \leq 7$ in [9]. In Section 6 we explain the relationship between that result and our general bounds.

For general k, Alon [1] proved that $f(n, C_k) \leq (k-2)n - \binom{k-1}{2}$. In this paper, we improve this bound to $f(n, C_k) \leq \left(\frac{k+1}{2} - \frac{2}{k-1}\right) n - (k-2)$, and for even k we improve it further to $f(n, C_k) \leq \frac{k}{2}n - (k-2)$. We also prove that the bound conjectured for $f(n, C_k)$ does hold when we further restrict the colorings on $E(K_n)$ by also forbidding slightly longer cycles. In particular, we prove that $f(n, \{C_k, C_{k+1}, C_{k+2}\}) \leq \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n - 1$.

2 Preliminaries

Given a graph G, we use $n(G)$ for $|V(G)|$, $e(G)$ for $|E(G)|$, and $G[S]$ for the subgraph induced by vertex set S. A *u, v-path* is a path with endpoints u and v. We use the following notions.

Definition 1 Given a graph G and a coloring c of $E(G)$, a *representing graph* for c is a spanning subgraph L of G having exactly one edge of each color under c (L may have isolated vertices). For a family $\mathcal{F}$, an *$\mathcal{F}$-good* coloring is a coloring of the edges of a complete graph with no polychromatic copy of any graph in $\mathcal{F}$. We write *H-good* for $\{H\}$-good.

We begin with the Erdős-Simonovits-Sós construction. When $r = k - 1$, the number of colors equals $\left(\frac{k-2}{2} + \frac{1}{k-1}\right) n - 1$, and the construction never uses more than $\left(\frac{k-2}{2} + \frac{1}{k-1}\right) n - 1$ colors. In [9], it is proved that $f(n, C_k) \leq \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n - 1$ when $k \leq 7$.

Theorem 2 *[6] If $n = (k-1)q + r$, where $1 \leq r \leq k-1$, then*

$$f(n, C_k) \geq \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n - \left(\frac{k-1-r}{2} + \frac{1}{k-1}\right) r.$$

Proof. Partition the vertices into sets $V_1, \ldots, V_q$ of size $k-1$ and one set V_{q+1} of size r. The edges with endpoints in the same set receive $q\binom{k-1}{2} + \binom{r}{2}$ distinct colors. On the remaining edges are q additional colors $c_1, \ldots, c_q$, with color $c_{\min\{i,j\}}$ on the edges with endpoints in V_i and V_j when $i \neq j$. The total $q\binom{k-1}{2}+\binom{r}{2}+q$ equals the given formula when $n=(k-1)q+r$.

Each V_i is too small to contain C_k, and every cycle that visits more than one of the subsets has two edges of color c_i, where V_i is the smallest-indexed set that it visits. Hence the coloring is C_k-good. □

In this construction, deleting a polychromatic copy of K_{k-1} yields the analogous construction for $n-k+1$ vertices. The difference in the number of colors is $\binom{k-1}{2}+1$, which equals $\left(\frac{k-2}{2}+\frac{1}{k-1}\right)(k-1)$. The idea of deleting (roughly) $k-1$ vertices and using induction motivates our proof. We use this approach in proving the conjecture for $k \leq 4$, which also serves as a basis for induction on k in our general result.

Lemma 3 *[1] Every coloring of $E(K_n)$ having no polychromatic k-cycle or l-cycle also has no polychromatic $(k+l-2)$-cycle. In particular, every C_k-good coloring also is $C_{2+s(k-2)}$-good, for every positive integer s.*

Proof. Let C be a $(k+l-2)$-cycle in such a coloring. Let x and y be vertices at distance $k-1$ along C. The edge xy completes cycles of lengths k and l with the two x,y-paths along C. Since neither of these is polychromatic, also C is not polychromatic.

The second claim is now immediate by induction on s. □

The proof of Lemma 3 is essentially the same as in Alon [1]. Theorem 4 was proved for C_3 in [6] and for C_4 in [1]; the proof here is simpler.

Theorem 4 *([6, 1]) $f(n, C_3)=n-1$ and $f(n, C_4)=\lfloor 4n/3 \rfloor - 1$.*

Proof. For $k \leq 4$, the construction of Theorem 2 uses exactly $\left\lfloor \left(\frac{k-2}{2}+\frac{1}{k-1}\right) n \right\rfloor - 1$ colors.

For $k=3$, Lemma 3 implies that a C_3-good coloring has no polychromatic cycles of any length. Hence a representing graph has at most $n-1$ edges.

For $k=4$, we use induction on n. If $n \leq 3$, then $\binom{n}{2} \leq \lfloor 4n/3 \rfloor - 1$. For $n \geq 4$, let c be a C_4-good coloring of $E(K_n)$. If c uses more than $n-1$ colors, then c has a polychromatic 3-cycle C. Let H be a representing graph containing C. By Lemma 3, H has only odd cycles.

Let F be a component of $H-V(C)$. Let $V(C)=\{u,v,w\}$. If $N_H(u)$ and $N_H(v)$ both intersect $V(F)$, then H has three pairwise internally-disjoint u,v-paths, which yields an even cycle. If $x,y \in N_H(u) \cap V(F)$, then the nontrivial x,y-path P in F must have odd length. Now v,w,u,x and v,u,y,P form edge-disjoint v,x-paths of odd length in H. Adding vx completes a polychromatic even cycle in c with one of them.

Thus H has at most one edge from C to each component F of $H-V(C)$. By the induction hypothesis, $e(F) \leq \lfloor 4n(F)/3 \rfloor - 1$. Summing over components of $H-V(C)$ and counting edges to $V(C)$ yields $e(H) \leq 3+\sum_F \lfloor 4n(F)/3 \rfloor \leq 3+\lfloor 4(n-3)/3 \rfloor = \lfloor 4n/3 \rfloor - 1$. □

In applying induction on the number of vertices, we will want to limit the number of edges between a cycle and the rest of the vertices in a representing graph when some cycle lengths are forbidden. Our basic lemma for this setting is of independent interest.

For vertex-disjoint subgraphs J, J' in a graph G, let $E_G[J, J']$ denote the set of edges having one endpoint in $V(J)$ and the other in $V(J')$, and let $e_G(J, J')$ denote its size. We write J as v when the subgraph is a single vertex v. We drop the subscript when only one graph is being discussed.

Lemma 5 *Let C be a cycle of length p in a graph G, and let P be a path in $G - V(C)$. If G has no cycle whose length is congruent to 2 modulo p, then $e(P, C) \leq p$.*

Proof. Let $x_1, \ldots, x_m$ be the vertices of P, in order, and let $N_i = N(x_i) \cap V(C)$. With respect to a consistent orientation of C, let N_i^s denote the shift of N_i by s positions. If $j - i \equiv r \bmod p$ with $0 \leq r \leq p-1$ and there is a vertex in $N_i^{m-i} \cap N_j^{m-j}$, then x_i and x_j have neighbors on C that are separated by distance r along C. Replacing this portion of C with the edges from its endpoints to x_i and x_j and the x_i, x_j-path along P yields a cycle in G with length congruent to 2 modulo p.

Therefore, the sets $N_1^{m-1}, N_2^{m-2}, \ldots, N_m^0$ are pairwise disjoint. Since they all lie in $V(C)$, their sizes sum to at most p. Since $|N_i^s| = |N_i|$, the sum of their sizes is $e(P, C)$. □

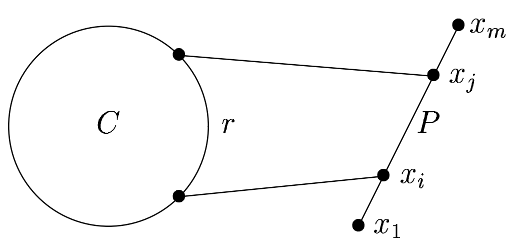

Figure 1. Disjointness of shifted neighborhoods.

The hypothesis on G in Lemma 5 can be weakened when a bound is placed on the length of P. That is, the result $e(P, C) \leq p$ holds whenever G has no $(p+2)$-cycle if P has length at most p. We will need only the form proved above.

3 A Greedy Structure

Since $\left(\frac{k-2}{2} + \frac{1}{k-1}\right)$ increases with k, in our induction step we can restrict attention to colorings of $E(K_n)$ with polychromatic cycles shorter than C_k. We will use length $k-2$ when $\mathcal{F} = \{C_k\}$ and length $k-1$ when $\mathcal{F} = \{C_k, C_{k+1}, C_{k+2}\}$. We focus first on $\mathcal{F} = \{C_k\}$.

Definition 6 Let C be a polychromatic $(k-2)$-cycle in a C_k-good coloring c of $E(K_n)$, and let H be a representing graph containing C. Let a be the number of chords of C in H, and let b be the number of edges of H with exactly one endpoint in $V(C)$. The list (C, H, F) is *C_k-greedy in c* if C and H are chosen to lexicographically maximize the ordered pair (a, b) and F is a component of $H - V(C)$ with maximum order.

From the defining conditions, a C_k-greedy (C, H, F) has the following properties:
(1) every color on an edge of K_n induced by $V(C)$ is on some edge of H induced by $V(C)$,
(2) every color on an edge of K_n incident to $V(C)$ is on some edge of H incident to $V(C)$,
(3) no color appearing in F appears on any edge of K_n incident to $V(C)$.
When we use these properties, we say "*by greediness*".

Property (1) of greediness is used in Theorem 15 and in the proofs of the optimal bounds for $k \leq 7$ in [9]. Property (2) is used heavily in the subsequent lemmas here. Most of the lemmas do not use property (3); we use this when reducing the proof of the bound for general k (Theorem 10) to the case where $H - V(C)$ has no edges. The variations in the lemmas for even k are used for the improvement of the general bound for even k (again Theorem 10).

Lemma 7 *Let (C, H, F) be C_k-greedy in c. Let P be an x, y-path in F, and let u be a vertex of C. If the length of P is a multiple of $k-2$, then $c(ux) = c(uy)$. If k is even, $ux \in E(H)$, and the length of P is an odd multiple of $(k-2)/2$, then $c(ux) = c(u'y)$, where u' is the vertex opposite u on C.*

Proof. The first statement does not assume that ux or uy lies in $E(H)$. Nevertheless, greediness implies that $c(ux)$ and $c(uy)$ do not appear on the edges of P. If $c(ux) \neq c(uy)$, then adding ux and uy to P produces a polychromatic cycle whose length is congruent to 2 modulo k. Lemma 3 forbids this, so $c(ux) = c(uy)$.

For the second statement, greediness again forbids $c(u'y)$ from the edges of P. Also, $c(u'y)$ appears on at most one of the u, u'-paths in C. If $c(ux) \neq c(u'y)$, then adding one of these paths plus ux and $u'y$ to P yields a polychromatic cycle forbidden by Lemma 3. □

Lemma 8 *Let (C, H, F) be C_k-greedy in c. Let $q = k-2$ when k is odd and $q = (k-2)/2$ when k is even. If F has a cycle C' of length at least q, then there exists H' such that (C, H', F) is C_k-greedy in c and all edges of $E_{H'}[C, F]$ are incident to $V(C')$.*

Proof. Let ux be an edge of H with $u \in V(C)$ and $x \in V(F) - V(C')$. Let P be a shortest path in F from x to $V(C')$. Since C' has length at least q, we can extend P along C' to obtain a path P' whose length is a multiple of q. Let y be the endpoint of P' on C'. By repeated application of Lemma 7, there is a vertex $v \in V(C)$ such that $c(ux) = c(vy)$. Replace ux

with vy. Replacing all of $E_H[C, F - V(C')]$ yields the desired graph H'. □

The *circumference* of a graph is the length of its longest cycle (or is ∞ if the graph is acyclic). Our final tool is based on Woodall's proof [13] (see [3, p137–8] for an exposition) of the Erdős–Gallai bound [5] on the number of edges in a n-vertex graph with circumference at most l. When W is a set of vertices in a graph G, let $e_G(W)$ denote the number of edges of G incident to W.

Lemma 9 *Let (C, H, F) be C_k-greedy in c, and let $p = k - 2$. If F has circumference at most l, then $e_H(W) \le \lambda\,|W|$ for some $W \subseteq V(F)$, where $\lambda = \max\{1 + \frac{p}{2}, \frac{l}{2} + \frac{p}{l/2+1}\}$.*

Proof. From the set of longest paths in F, choose P to lexicographically maximize the pair (a, d), where P is a u, v-path, $a = e_H(v, C)$, and $d = d_F(u)$. Index the vertices of P as $x_1, \ldots, x_m$ in order, with $u = x_1$ and $v = x_m$.

Let $W = \{x_i \colon x_{i+1} \in N_F(x_1)\}$; note that $|W| = d$ and $x_1 \in W$ and $v \notin W$. We claim that $e_H(W) \le \lambda\,|W|$. Note that $e_H(W) = e_H(W, C) + e_F(W)$.

For $x_i \in W$, let Q_i be the path formed by the v, x_{i+1}-path in P, the edge $x_{i+1}x_1$, and the x_1, x_i-path in P. If x_i has a neighbor in F outside $V(P)$, then Q_i extends, contradicting the choice of P. Therefore, $N_F(W) \subseteq V(P)$. Since Q_i has the same length as P, the choice of x_1 yields $d_F(x_i) \le d$. Hence $e_F(W) \le d^2$.

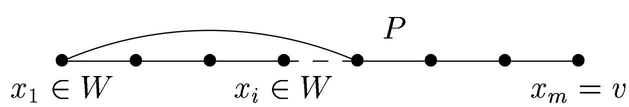

Figure 2. Longest paths from v in F.

If x_i has a neighbor x_j with $j > l$ in F, then $x_i x_j$ and $x_{i+1}x_1$ form a cycle of length j with portions of P, contradicting the circumference hypothesis. Therefore, both $W \subseteq L$ and $N(W) \subseteq L$, where $L = \{x_1, \ldots, x_l\}$. The sum $\sum_{x \in W} d_F(x)$ counts each edge of $F[W]$ twice and each edge of $E_F[W, L - W]$ once. Since $d_F(x) \le d$ for $x \in W$, we have $2e_F(W) - e_F(W, L - W) \le d^2$, which yields $e_F(W) \le \frac{1}{2}(d^2 + d(l - d)) = dl/2$.

Now consider $e_H(W, C)$. By Lemmas 3 and 5, $e(P, C) \le p$. Since v has the most edges to $V(C)$ among endpoints of longest paths in F, we have $e_H(W, C) \le p\frac{d}{d+1}$.

We have shown that $e_H(W) \le dg(d)$, where $g(d) = \min\{d, l/2\} + p/(d + 1)$. We bound $g(d)$. In the range $d \ge l/2$, the maximum occurs when $d = l/2$. In the range $d \le l/2$, the maximum occurs when $d = 1$ or $d = l/2$. Hence $g(d) \le \max\{1 + \frac{p}{2}, \frac{l}{2} + \frac{p}{l/2+1}\}$. □

4 The Bound for General k

Our general bound for odd k is valid for all k, but for even k we prove a stronger bound.

Theorem 10 *If $n \ge 2$, then $f(n, C_k) \le \beta_k n - (k-2)$, where $\beta_k = \left(\frac{k+1}{2} - \frac{2}{k-1}\right)$ for odd k and $\beta_k = k/2$ for even k.*

Proof. We use induction on $n+k$. By Theorem 4, we may assume that $k \ge 5$. When $2 \le n \le k-1$, we have $f(n, C_k) = \binom{n}{2}$. Here it suffices to show that $\beta_k n - (k-2) - \binom{n}{2} \ge 0$. The left side is $2\beta_k - k + 1$ when $n = 2$ and is $1 + (2\beta_k - k)(k-1)/2$ when $n = k-1$. Both values are nonnegative when $k \ge 3$ whether k is odd or even, and hence this quadratic inequality holds for $2 \le n \le k-1$.

Hence we may assume that $n \ge k \ge 5$. Let c be a C_k-good coloring of $E(K_n)$. Since the desired bound exceeds $f(n, C_{k-2})$, we may assume that c has a polychromatic $(k-2)$-cycle. Hence we may select a C_k-greedy (C, H, F). We define *long cycle* to be a cycle of length at least q as defined in Lemma 8 ($q = k-2$ when k is odd, and $q = (k-2)/2$ when k is even).

Case 1: F has a long cycle C'. By Lemma 8, we may assume that all of $E_H[C, F]$ is incident to $V(C')$. Since $V(C')$ lies on a path in F, Lemma 5 yields $e_H(C, F) \le k-2$.

Consider the colorings obtained by restricting c to $V(F)$ and to $V(H) - V(F)$. Since $n(F) \ge 2$ and $n(H - V(F)) \ge 2$, the induction hypothesis applies. Since F and $H - V(F)$ are contained in representing graphs for these colorings, we have the desired bound:

$$e(H) = e(F) + e(H - V(F)) + e_H(C, F) \le \beta_k n - (k-2).$$

Case 2: F has a cycle but no long cycle. We apply Lemma 9 with $l = q-1$ and $p = k-2$. Now $1 + p/2 = k/2$. The quantity $\frac{l}{2} + \frac{p}{l/2+1}$ reduces to β_k when k is odd and to something at most β_k when k is even and at least 8. This case cannot occur when $k = 6$, because then $q = 2$ and every cycle is long. Hence $\max\{1 + \frac{p}{2}, \frac{l}{2} + \frac{p}{l/2+1}\} = \beta_k$ for $k \ge 5$, and we obtain $W \subseteq V(F)$ such that $e_H(W) \le \beta_k |W|$. We discard W and apply the induction hypothesis to the restriction of c to $V(H) - W$ to obtain the desired bound on $e(H)$.

Case 3: F is a tree with at least two vertices. Let v be a vertex of F having the most neighbors in H on C. Let P be a maximal path in F starting at v; let u be its other endpoint. By Lemma 5, $e(P, C) \le k-2$. The choice of v yields $e_H(u, C) \le (k-2)/2$. Also u has exactly one neighbor in F, so $d_H(u) \le k/2$. We let $W = \{u\}$ and delete W as in Case 2.

Case 4: $H - V(C)$ has no edges. The greedy choice of F makes this the only remaining case. Since $H - V(C)$ has no edges, $d_H(u) = e_H(u, C)$ when $u \notin V(C)$. Since $n \ge k$ and C is a $(k-2)$-cycle, we can choose distinct vertices u and v outside C. If $d_H(u) > k/2$, then we can choose distinct vertices $x, y \in N_H(u)$, with successors x', y' on C in a consistent orientation of C. If $x', y' \in N_H(v)$, then replacing xx' and yy' on C with $\{xu, uy, x'v, vy'\}$

yields a forbidden k-cycle in H. Hence at most one vertex of C can be both a neighbor of v and a successor of a neighbor of u. This yields $d_H(u) + d_H(v) \le k-1$. Hence $d_H(u) \le k/2$ or $d_H(v) \le k/2$, and we finish as in Case 3. $\square$

5 Forbidding cycles of lengths k through $k+2$

Let $\mathcal{C}_k = \{C_k, C_{k+1}, C_{k+2}\}$. We prove that $f(n, \mathcal{C}_k) \le \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n - 1$. This is optimal infinitely often, since the construction of Theorem 2 shows that equality holds whenever $k-1$ divides n. The proof uses many of the ideas in our general bound on $f(n, C_k)$, but we need analogues of the lemmas in that proof. We say that (C, H, F) is *$\mathcal{C}_k$-greedy in c* if it is C_{k+1}-greedy in c; in particular, C will be a polychromatic $(k-1)$-cycle. We start with a stronger version of Lemma 5.

Lemma 11 *Let C be a cycle of length p in a graph G having no cycle with length greater than 3 and congruent to any of $\{1, 2, 3\}$ modulo p, and let P be a path in $G - V(C)$.*
(1) If u is an endpoint of P, then $e(P, C) + e(u, C) \le p$.
(2) If no two consecutive vertices on P have a common neighbor on C, then $e(P, C) \le \lfloor p/2 \rfloor$.
(3) If S is a subset of $V(P)$ with no two consecutive vertices on P, then $e(S, C) \le \lfloor p/2 \rfloor$.

Proof. Let $x_1, \ldots, x_m$ be the vertices of P, in order, and let $N_i = N(x_i) \cap V(C)$. With respect to a consistent orientation of C, let N_i^s denote the shift of N_i by s positions.

(1) As in Lemma 5, if there is a vertex in $N_i^{m-i} \cap N_j^{m-j}$, then G has a cycle with length congruent to 2 modulo p. Therefore, the sets $N_1^{m-1}, N_2^{m-2}, \ldots, N_m^0$ are pairwise disjoint. Letting $u = x_m$, also N_m^{-1} is disjoint from all of these, since a common vertex in N_m^{-1} and N_j^{m-j} yields a cycle with length congruent to 1 modulo $p-1$. Since they all lie in $V(C)$, the sizes of these disjoint sets sum to at most p. Since $|N_i^s| = |N_i|$, the sum equals $e(P, C) + e(u, C)$.

(2) Since G has no $(p+1)$-cycle, for each i the sets N_i^r and N_i^{r+1} are disjoint. If $N_i^{m-i} \cup N_i^{m-i+1}$ and $N_j^{m-j} \cup N_j^{m-j+1}$ have a common vertex x, then $x \in N_i^{m-i+1} \cap N_j^{m-j}$ or $x \in N_i^{m-i} \cap N_j^{m-j+1}$. With $i < j$, the first case yields a cycle with length congruent to 1 modulo p. The second case yields a non-triangle cycle with length congruent to 3 modulo p or has $i = j-1$ and yields consecutive vertices on P with a common neighbor on C. All these cases are forbidden by the hypotheses, so the sets $\{N_i^{m-i} \cup N_i^{m-i+1} \colon 1 \le i \le m\}$ are pairwise disjoint subsets of $V(C)$. Each edge from P to C is counted exactly twice in these sets, so $e(P, C) \le \lfloor p/2 \rfloor$.

(3) If S contains no two consecutive vertices on P, then in the argument for (2) the case of consecutive vertices with a common neighbor cannot arise. Hence the sets $\{N_i^{m-i} \cup$

$N_i^{m-i+1}\colon x_i \in S\}$ are pairwise disjoint, and $e(C,S) \le \lfloor p/2 \rfloor$. $\square$

As in Lemma 5, in Lemma 11 the hypothesis on G can be weakened to having no cycle with length in $\{p, p+1, p+2\}$ when the length of P is at most $p-1$, but we only need the statement proved above.

We will apply Lemma 11 to a graph with an edge-coloring having no polychromatic cycle with length in $\{p+1, p+2, p+3\}$. By Lemma 3, such a coloring has no polychromatic non-triangle cycle with length congruent to one of $\{1,2,3\}$ modulo p, so we may apply Lemma 11 to a representing graph.

We also need variants of Lemmas 7 and 8. Henceforth, let $q = \lfloor (k-1)/2 \rfloor$.

Lemma 12 *Let (C,H,F) be $\mathcal{C}_k$-greedy in c. Let P be an x,y-path in F, and let u be a vertex of C. If the length of P is a multiple of $k-1$, then $c(ux) = c(uy)$. If $ux \in E(H)$, and the length of P is congruent to q modulo $k-1$, then $c(ux) = c(u'y)$, where u' is a vertex at distance q from u on C.*

Proof. The first claim holds by Lemma 7 because (C,H,F) is C_{k+1}-greedy in c.

For the second claim, the first allows us to assume that P has length q. Now C_{k+1}-greediness forbids $c(u'y)$ from the edges of P, and $c(u'y)$ can appear on only one of the two u,u'-paths in C. If $c(ux) \ne c(u'y)$, then adding one of these paths plus ux and $u'y$ to P produces a polychromatic cycle of length k or $k+1$, both of which are forbidden. $\square$

Lemma 13 *Let (C,H,F) be $\mathcal{C}_k$-greedy in c. If F has a cycle C' of length at least q, then there exists H' such that (C,H',F) is C_k-greedy in c and all of $E_{H'}[C,F]$ is incident to C'.*

Proof. Let ux be an edge of H with $u \in V(C)$ and $x \in V(F) - V(C')$. Let P be a shortest path in F from x to $V(C')$. Since C' has length at least q, we can extend P along C' to obtain a path P' whose length is congruent to q modulo $k-1$. Let y be the endpoint of P' on C'. By Lemma 12, $c(ux) = c(u'y)$, where u' is a vertex at distance q from u in C. Replace ux with $u'y$. Replacing all of $E_H[C, F - V(C')]$ in this way yields the desired graph H'. $\square$

Our next lemma extends early results of Ore [11] in the theory of Hamiltonian graphs. For completeness, we give a short self-contained inductive proof. Stronger results are known about panconnected graphs, where an n-vertex graph G is *panconnected* if whenever $d \le l \le n-1$ for vertices u and v at distance d in G, there is a u,v-path of length l in G ([4] surveys certain types of sufficient conditions for this and many other related properties about paths and cycles in graphs).

Lemma 14 *Let G be an n-vertex graph. If $e(\overline{G}) \le n-4$, then G has a u,v-path of length l whenever $u,v \in V(G)$ and $2 \le l \le n-1$. If $e(\overline{G}) \le n-3$, then G has a spanning cycle.*

Proof. We use induction on n; the claim is immediate for $n = 4$.

For $n > 4$, suppose first that $e(\overline{G}) \le n - 4$. Since at most $n - 4$ edges are missing and K_n has $n - 2$ edge-disjoint u, v-paths of length 2, there is a u, v-path of length 2 in G.

If v is not isolated in $\overline{G}$, then $e(\overline{G - v}) \le n - 5$. Since $e(\overline{G}) \le n - 4$, in $G - u$ there is a neighbor x of v. By the induction hypothesis, for $2 \le l \le n - 2$ there is a u, x-path of length l in $G - v$. Append v to obtain the desired path in G.

If v is isolated in $\overline{G}$, then $e(\overline{G - v}) \le (n - 1) - 3$. By the induction hypothesis, $G - v$ has a spanning cycle. Append v to the end of a path of length $l - 1$ along the cycle from u.

Finally, we need a spanning cycle when $e(\overline{G}) = n - 3$. This yields $\delta(G) \ge 2$. Since the complement of a graph with maximum degree at most 1 has a spanning cycle, we may assume that some vertex x has degree at least 2 in $\overline{G}$. Select $y, z \in N_{\overline{G}}(x)$. Since $e(\overline{G - x}) \le n - 5$, we can add the path z, x, y to a spanning y, z-path in $G - x$ to complete a spanning cycle in G. □

To facilitate the inductive proof of our bound on $f(n, \mathcal{C}_k)$, we need a stronger bound in the case when a $\mathcal{C}_k$-greedy (C, H, F) has few edges in $H[V(C)]$. Recall that $q = \left\lfloor \frac{k-1}{2} \right\rfloor$.

Theorem 15 *If $k \ge 3$ and $n \ge 2$, then $f(n, \mathcal{C}_k) \le \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n - 1$. Furthermore, if the vertices of each polychromatic $(k-1)$-cycle in a $\mathcal{C}_k$-good coloring c of $E(K_n)$ induce edges with at most $\binom{k-1}{2} - q$ colors, then the number of colors used in c is at most $\left(\frac{k-2}{2} + \frac{1}{k-1}\right) n - q$.*

Proof. We use induction on $n + k$. Let $\alpha_k = \left(\frac{k-2}{2} + \frac{1}{k-1}\right)$.

When $2 \le n \le k - 1$, we have $f(n, \mathcal{C}_k) = \binom{n}{2}$. For $2 \le n \le k - 2$, we require the stronger bound, and $\binom{n}{2} \le \alpha_k n - q$ requires $\frac{n-1}{2} + \frac{q}{n} \le \frac{k-2}{2} + \frac{1}{k-1}$. Since $h_q(x) = \frac{x}{2} + \frac{q}{x+1}$ defines a convex function, it suffices to observe that the inequality holds for $n = 2$ and $n = k - 2$.

If $n = k - 1$, then $\binom{n}{2} = \alpha_k n - 1$, and the bound holds with equality. Furthermore, the stronger bound holds if the stronger condition holds. If $k \le 4$, then $q = 1$, and Theorem 4 yields both desired statements.

We may thus assume that $n \ge k \ge 5$. This yields $\alpha_{k-1} n - 1 \le \alpha_k n - q$, so it suffices to consider colorings with polychromatic $(k-1)$-cycles. Hence we may let (C, H, F) be $\mathcal{C}_k$-greedy in c. Also let $H' = H[V(C)]$, and let *long cycle* mean a cycle of length at least $q + 1$.

Case 1: F has a long cycle C'. By Lemma 13, we may assume that all of $E_H[C, F]$ is incident to $V(C')$. Fix an orientation of C'.

For $x \in V(C')$, let x' denote its successor on C', and let y be the vertex q steps after x on C'. If $x, x' \in N_H(u)$ for some $u \in V(C)$, then let u' denote a vertex at distance q from u on C. By Lemma 12, $c(u'y) = c(ux)$. Since H is a representing graph, the colors on $u'y$, ux', and the x', y-path of length $q - 1$ on C' are distinct and do not appear on C. Combining these edges with the u, u'-path of length $k - 1 - q$ on C yields a polychromatic k-cycle.

Hence we may assume that no two consecutive vertices on C' have a common neighbor in H on C. Applying Lemma 11(2) to a spanning path in C' yields $e_H(C', C) \le q$.

Let $F' = H - V(F)$. Since $n(F) \ge 2$ and $n(F') \ge 2$, we can apply the induction hypothesis to the colorings obtained by restricting c to $V(F)$ and to $V(F')$. If $e(H') \le \binom{k-1}{2} - q$, then greediness of (C, H, F) implies that both F and F' have no polychromatic $(k-1)$-cycle inducing more than $\binom{k-1}{2} - q$ edges. Hence we can apply the tighter bound in the induction hypothesis, obtaining

$$e(H) = e(F) + e(F') + e_H(C, F) \le \alpha_k n(F) - q + \alpha_k n(F') - q + e_H(C, C') \le \alpha_k n(H) - q.$$

If $e(H') > \binom{k-1}{2} - q$, then we obtain $e(F) \le \alpha_k n(F) - 1$ and $e(F') \le \alpha_k n(F') - 1$. If $e(F) \le \alpha_k n(F) - q$, then applying the induction hypothesis as above yields $e(H) \le \alpha_k n(H) - 1$, which suffices. If $e(F) > \alpha_k n(F) - q$ and $n(F) \ge k$, then the inequality $\alpha_{k-1} n(F) - 1 \le \alpha_k n(F) - q$ allows us to assume that F contains a $(k-1)$-cycle; we take this as C'. On the other hand, if $n(F) < k$, then certainly $n(C') \le k - 1$.

We are left with $e(F) \le \alpha_k n(F) - 1$ and $e(F') \le \alpha_k n(F') - 1$ and $n(C') \le k - 1$. To obtain the desired bound on $e(H)$, it suffices to show that $e_H(C, C') \le 1$. We show first that all of $E_H[C', C]$ is incident to one vertex of C.

We have $e(\overline{H'}) \le q-1$. If $q-1 \le (k-1)-4$, then Lemma 14 applies. This inequality holds when $k \ge 6$. Suppose that ux and vy are edges of $E_H[C, C']$, with $u \ne v$ and $x, y \in V(C')$. Let r be the length of an x,y-path in C'. By Lemma 14, there is an x,y-path through xu and $V(C)$ and vy that has length l, for each l in $\{4, 5, \ldots, k\}$. We have $r + l \equiv 2 \bmod (k-2)$ for some l in this set unless $r = k - 3$. In this case, setting $l = 4$ gives us a polychromatic cycle of length $k+1$, which is congruent to 2 modulo $k-1$. Since c is C_k-good and C_{k+1}-good, this violates Lemma 3.

When $k = 5$, the argument still applies unless H' consists of a 4-cycle plus one chord uv and the edges from C' arrive at u and v. Now H' has no u,v-path of length 3. However, the remaining chord, whose color appears in H', lies on two u,v-paths of length 3 sharing no other edge, and one of these paths can be used.

Hence all edges of $E_H[C, C']$ must be incident to a single vertex u in $V(C)$. If $e_H(u, C') > 1$, then choose $x, z \in N_H(u) \cap V(C')$. Let y be a vertex of C' reached by a path of length q from x along C'. By Lemma 12, $c(vy) = c(ux)$, where v has distance q from u along C. Now vy and uz have distinct colors that do not appear in H' or C'. Replacing ux with vy in H now contradicts the argument about edges arriving at a single vertex of C.

Case 2: F has no long cycle.

In this case, it suffices by the induction hypothesis to find a set $W \subseteq V(F)$ with $e_H(W) \le \alpha_k |W|$ (as in Theorem 10). This proves simultaneously both the overall bound and the stronger bound needed when $e(H')$ is small. For $W \subseteq V(F)$, let $\beta(W) = e_H(W)/|W|$.

Among the paths in F with maximum length, let P (a u,v-path) be chosen to lexicographically maximize the pair (a,d), where $a = e_H(v,C)$ and $d = d_F(u)$. Index the vertices of P as $x_1,\ldots,x_m$ in order, with $u = x_1$ and $v = x_m$.

If $d \ge q$, then F has a long cycle, so we may assume that $d \le q-1$. If $m > q$, then by Lemma 12 we can shift all of $E_H(u,C)$ away from u. Only the d incident edges in F remain incident to u. We obtain $\beta(\{u\}) = d \le q-1 \le \alpha_k$ for the new representing graph used in place of H. Hence we may assume that $m \le q$. The maximality of P also yields $m > d$.

We consider subcases depending on the value of d.

Subcase 2.1: $d \ge 3$ and $m = d+1$. Since $q \ge d+1 \ge 4$, this requires $k \ge 9$. Also, $m = d+1$ requires $v \in N_F(u)$, so $P + uv$ is a cycle. The cycle must span $V(F)$, since P has maximum length in F. Lemma 11(1) now yields $e_H(F,C) \le k-1$. If $W = V(F)$, then

$$\beta(W) \le \frac{\binom{d+1}{2} + k - 1}{|W|} = \frac{\binom{d+1}{2} + k - 1}{d+1} = \frac{d}{2} + \frac{k-1}{d+1}.$$

Consider again the function of d defined by $h_{k-1}(d) = \frac{d}{2} + \frac{k-1}{d+1}$. Over an interval, h_{k-1} is maximized at an endpoint. For $k \ge 9$, we compute that $h_{k-1}(3) \le \alpha_k$ and $h_{k-1}(q-1) \le \alpha_k$. Hence $\beta(W) \le h_{k-1}(d) \le \alpha_k$.

Subcase 2.2: $d \ge 3$ and $m > d+1$. Since $q \ge m \ge d+2 \ge 5$, this requires $k \ge 11$. Let $W = \{x_i : x_{i+1} \in N_F(x_1)\}$. We have $|W| = d$ and $u \in W$ and $v \notin W$. For $w \in W$, there is a w,v-path of length $m-1$ with vertex set $V(P)$ (see Figure 2). Our choice of u and v thus yields $e_H(w,C) \le e_H(v,C)$ and $d_F(w) \le d$ for all $w \in W$. As in Lemma 9, $N_F(W) \subseteq V(P)$.

Computations as in Lemma 9 now yield $e_F(W) \le dm/2$. By Lemma 11(1), $e_H(P,C) + e_H(v,C) \le k-1$. Now $v \notin W$ yields $e_H(W,C) + e_H(v,C) + e_H(v,C) \le k-1$. By the choice of v, this yields $e_H(W,C) \le \frac{d}{d+2}(k-1)$. Thus

$$\beta(W) = \frac{1}{d}\left[e_F(W) + e_H(W,C)\right] \le \frac{1}{d}\left[\frac{dq}{2} + \frac{d}{d+2}(k-1)\right] \le \frac{k-1}{4} + \frac{k-1}{5}.$$

In the last inequality, we used $d \ge 3$. This bound simplifies to $\frac{k-2}{2} + \frac{1}{2} - \frac{k-1}{20}$, which is bounded by $\frac{k-2}{2}$ when $k \ge 11$.

Subcase 2.3: $d = 2$. Let $W = \{x_i : x_{i+1} \in N_F(u)\}$; we have $W = \{u,w\}$ for some $w \in V(P) - \{u,v\}$. As before, there is a longest path in F from w to v, so $e_H(w,C) \le e_H(v,C)$.

If $w \ne x_2$, then $q \ge m \ge 4$, which requires $k \ge 9$. Lemma 11(3) yields $e_H(W,C) \le q$. Since $e_F(W) \le 4$, we have $\beta(W) \le (q+4)/2$. This is bounded by α_k when $k \ge 10$. If $k = 9$, then $m = 4$ and $uv \in E(F)$. Hence $V(F) = V(P)$, since otherwise we find a longer path in F. Since $d = 2$, $F \ne K_4$; hence $e(F) \le 5$. Also $e_H(F,C) \le k-1$, by Lemma 11(1). Thus $\beta(V(F)) \le (5+k-1)/4 \le \alpha_k$.

If $w = x_2$, then $q \ge m \ge 3$, which requires $k \ge 7$. Lemma 11(1) yields $e_H(P,C) + e_H(v,C) \le k-1$. Thus $e_H(u,C) + e_H(w,C) + 2e_H(v,C) \le k-1$. The choice of v yields $e_H(W,C) \le q$. Since $e_F(W) \le 3$, we have $\beta(W) \le (q+3)/2$. This is bounded by α_k when $k \ge 8$.

For $k = 7$, we have $m \le q = 3$ and thus $F = C_3$. By Lemma 11(1), we have $e_H(P,C) + e_H(v,C) \le k-1$. Our choice of v thus yields $e_H(F,C) \le \left\lfloor \frac{3}{4}(k-1) \right\rfloor$, and then $\beta(V(F)) \le (\left\lfloor \frac{3}{4}(k-1) \right\rfloor + 3)/3 \le \left(\frac{k-2}{2} + \frac{1}{k-1}\right)$.

Subcase 2.4: $d = 1$. If $m \ge 3$ (which by $q \ge m$ requires $k \ge 7$), then u and v are not consecutive on P. Let $W = \{u, v\}$. By Lemma 11(3), $e_H(W,C) \le q$. Since $e_H(u,C) \le e_H(v,C)$, we have $e_H(u,C) \le \lfloor q/2 \rfloor$. We have $\beta(\{u\}) \le \lfloor q/2 \rfloor + 1 \le \alpha_k$ (since $k \ge 7$).

If $m = 2$, then $V(F) = \{u, v\}$. By Lemma 11(1), $e_H(P,C) + e_H(v,C) \le k-1$. Again $e_H(u,C) \le e_H(v,C)$, so $e_H(P,C) \le \left\lfloor \frac{2}{3}(k-1) \right\rfloor$. Thus, $\beta(V(F)) \le (\left\lfloor \frac{2}{3}(k-1) \right\rfloor + 1)/2 \le \alpha_k$.

Subcase 2.5: $d = 0$. In this case, $H - V(C)$ has no edges, by the greedy choice of F. If $e(\overline{H'}) \le q-1$, then as in Case 1 we conclude that $E_H[F,C]$ is incident to a single vertex of C. Thus $e(H) \le \binom{k-1}{2} + (n-k+1) \le n + \binom{k-2}{2} - 1 \le \alpha_k n - 1$.

When $e(\overline{H'}) \ge q$, we need to prove the stronger bound. Since $H - V(C)$ has no edges, $d_H(u) = e_H(u,C)$ when $u \notin V(C)$. If $N_H(u)$ has two consecutive vertices on C, then we have a forbidden polychromatic k-cycle. Hence $d_H(u) \le q$, which yields

$$e(H) \le \binom{k-1}{2} - q + (n-k+1)q = nq - q + \frac{k-1}{2}(k-2-2q).$$

When k is even, $q = (k-2)/2$ and $\alpha_k = q + \frac{1}{k-1}$. The bound becomes $e(H) \le \alpha_k n - q - \frac{n}{k-1} < \alpha_k n - q$, as desired.

When k is odd, $q = (k-1)/2$ and $\alpha_k = q - \frac{1}{2} + \frac{1}{k-1}$. The bound becomes $e(H) \le \alpha_k n - q + \frac{n-k+1}{2} - \frac{n}{k-1}$, and we need to improve it. Since k is odd, $d_H(u) = q$ requires neighbors and nonneighbors of u to alternate on C. Suppose that this occurs for two vertices $u, v \notin V(C)$. If $N_H(u) \cup N_H(v) = V(C)$, then let w, x, y, z be successive on C with $w, y \in N_H(u)$; replacing $\{wx, yz\}$ with $\{wu, uy, xv, vz\}$ yields a polychromatic $(k+1)$-cycle. Hence $N_H(u) = N_H(v)$; now consider the edge uv. By greediness, $c(uv)$ appears on an edge yb of H with w, x, y, z, a successive on C. Regardless of whether $y \in N_H(u)$ and/or $b \in \{x, z, u, v\}$, in all cases we can replace $\{wx, xy\}$ or $\{xy, yz\}$ or $\{yz, za\}$ with a detour through $\{u, v\}$ to complete a polychromatic k-cycle.

Hence $d_H(u) \le q-1$ when $u \notin V(C)$, except for at most one vertex. This reduces the upper bound by $n-k$, which is sufficient. □

6 Concluding Remarks

Our results suggest several approaches to proving the full Erdős-Simonovits-Sós Conjecture. With Theorem 15, it suffices to show that an optimal C_k-good coloring also has no polychromatic $(k+1)$-cycle or $(k+2)$-cycle. This condition holds in the construction of Theorem 2.

Another approach is to study a greedy structure based on a polychromatic $(k-1)$-cycle as in Theorem 15. Again Lemma 14 makes it possible to bound $e_H(F, C)$ tightly when $e(\overline{H'}) \leq k-5$, but there remain many cases when $e(\overline{H'})$ is larger. This approach is used in [9] to prove the conjecture for $k \leq 7$. For larger k, stronger results about nearly panconnected graphs may make it possible to handle the cases.

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